Revista Colombiana de Estadística - Applied Statistics
July 2021, volume 44, issue 2, pp. 331–368
http://doi.org/10.15446/rce.v44n2.89320

The Gamma Odd Burr III-G Family of Distributions: Model, Properties and Applications

La familia de Gamma Odd Burr III-G de distribuciones: modelo, propiedades y aplicaciones

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Abstract

A new family of distributions called Ristić-Balakhrishnan Odd Burr III-G (RBOB III-G) distribution is proposed. We obtain some mathematical and statistical properties of this distribution such as hazard and reverse hazard functions, quantile function, moments and generating functions, conditional moments, Rényi entropy, order statistics, stochastic ordering and probability weighted moments. The model parameters are estimated using maximum likelihood estimation technique. Finally, the usefulness of this family of distributions is demonstrated via simulation experiments.

***Key words*:** Family of distributions; Moments; Maximum likelihood; Odd Burr-III; Ristić-Balakhrishnan.

Abstract

Se supone una nueva familia de distribuciones llamada distribución Ristić-Balakhrishnan Odd Burr III-G (RBOB III-G). Se obtienen algunas propiedades matemáticas y estadísticas de esta distribución, tales como funciones de riesgo y riesgo inverso, función de cuantiles, momentos y funciones generadoras, momentos condicionales, entropía de Rényi, estadísticas de orden, ordenamiento estocástico y momentos ponderados por probabilidad. Los parámetros del modelo se estiman utilizando la técnica de estimación de máxima verosimilitud. Finalmente, la utilidad de esta familia de distribuciones se demuestra mediante experimentos de simulación.

***Key words*:** Familia de distribuciones; Momentos; Máxima verosimilitud; Odd Burr-Balakhrishnan.

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1. Introduction

Several statistical distributions are very useful in describing and predicting real data in many areas such as economics, engineering and biological studies. However, in many situations, classical distributions do not provide adequate fits to real data. Thus, several new classes of probability distributions have been suggested by introducing one or more parameters to generate new distributions. Recent developments have been focused to define new families by adding shape parameters to control skewness, kurtosis and tail weights thus providing great flexibility in modelling skewed data in practice, including the two-piece approach introduced by Hansen (1994) and the generators pioneered by Eugene et al. (2002), Cordeiro & de Castro (2011), Alexander et al. (2012). Many subsequent articles apply these techniques to induce skewness into well-known symmetric distributions such as the symmetric Student t, see Aas & Haff (2006) for a detailed review. The recent perculiar works include the gamma generator by Ristić & Balakhrishnan (2012), Odd Burr-G (OB-G) family by Alizadeh et al. (2017), Gamma-Weibull-G family by Oluyede et al. (2018), Generalized odd-log-logistic family by Cordeiro et al. (2017), New Generalized odd log-logistic family by Haghbin et al. (2017), and Odd log-logistic Lindley Poisson family by Ozel et al. (2017).

Following the system of twelve cumulative distribution functions proposed by Burr (1942*a*), similar work on Burr-XII models and other related distributions have been widely studied by several authors, (see Tadikamalla (1980), Shao (2004), Zimmer et al. (1998), Shao (2004), Soliman (2005), Wu et al. (2007), Silva et al. (2008)). Afify et al. (2016) also introduced a new four-parameter lifetime model called the Weibull Burr-XII distribution capable of modelling various shapes of aging and failure rate criteria. Recent useful extensions of the Burr-XII models are also studied in detail by Altun et al. (2018), Cordeiro et al. (2018),Yousof, Altun, Ramires, Alizadeh & Rasekhi (2018) and Yousof, Rasekhi, Altun, Alizadeh, Hamedani & Ali (2018).

In this paper, we propose a new family of distributions called Ristić-Balakhrishnan Odd Burr III-G (RBOB III-G) distribution by combining the Gamma generator proposed by Ristić & Balakhrishnan (2012) with the Odd Burr-G family proposed by Alizadeh et al. (2017). Our aim is to develop and generate distributions that can handle different forms of data behaviour with great flexibility providing better fits and statistical properties compared to other competitive models. Furthermore, the general objectives for generalizing these new family of distributions is to generate the distributions which are skewed, symmetric, J-shaped or reverse J-shaped which can provide better fits than other generalized distributions having the same number of parameters.

This paper is organized in the following manner. Section 2 presents the generalized family of distributions, series expansion of the density function, sub-families, hazard, reverse hazard and quantile functions. Some special cases are discussed under Section 3. In Section 4, some of the mathematical and statistical properties namely; moments and generating functions, conditional moments, entropy, distribution of order statistics, stochastic ordering and probability weighted moments are derived. Section 5 presents the maximum likelihood estimates. Simulation studies are conducted to examine the bias and mean square error of the maximum likelihood estimators for each parameter in Section 6. Applications of the proposed model to real data are given in Section 7 followed by concluding remarks presented under Section 8.

2. The Model

Consider the cumulative distribution function (cdf) and probability density function (pdf) of Burr III distribution (see Burr, 1942*b*), given by

$$F(x;\alpha,\beta) = (1+x^{-\alpha})^{-\beta} \tag{1}$$

and

$$f(x;\alpha,\beta) = \alpha\beta x^{-\alpha-1}(1+x^{-\alpha})^{-\beta-1}, \qquad x>0, \qquad \alpha,\beta>0, \tag{2}$$

respectively, where α and β are both shape parameters. Alizadeh et al. (2017) derived a new family of distributions, namely, Odd Burr III-G by replacing x with the odds $G(x)/[1-G(x)]$ and defined its cdf by

$$F_{OB-G}(x;\alpha,\beta,\psi) = \int_0^{\frac{G(x)}{1-G(x)}} \alpha\beta t^{-\beta-1}(1+t^{-\alpha})^{-\beta-1}dt = \left\{1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\right\}^{-\beta} \tag{3}$$

and the corresponding pdf is given by

$$f_{OB-G}(x;\alpha,\beta,\psi) = \alpha\beta\frac{[1-G(x;\psi)]^{\alpha-1}}{G(x;\psi)^{\alpha+1}}\left[1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\right]^{-\beta-1} g(x;\psi), \tag{4}$$

where $G(x;\psi)$ and $g(x;\psi)$ is the cdf and pdf respectively of any baseline distribution, and ψ is the vector of parameters. In the remaining part of this section, we present the RBOB III-G model by combining the Odd Burr III-G distribution with the Gamma generator proposed by Ristić & Balakhrishnan (2012) having the cdf and pdf given by

$$F_{RB}(x;\delta) = 1-\frac{1}{\Gamma(\delta)}\int_0^{-\log\big(G(x)\big)} t^{\delta-1}e^{-t}dt, \qquad \delta>0 \tag{5}$$

and

$$f_{RB}(x;\delta) = \frac{1}{\Gamma(\delta)}\big[-\log\big(G(x)\big)\big]^{\delta-1}g(x), \qquad x\in\mathbb{R}, \tag{6}$$

respectively.

If we substitute $G(x) = F_{OB-G}(x;\alpha,\beta,\psi)$ in equations (5) and (6), then we get a new extended generator called Ristić-Balakhrishnan Odd Burr III-G (RBOB III-G) having the cdf and pdf given

$$F_{RBOBIII-G}(x;\alpha,\beta,\delta,\psi)=1-\frac{1}{\Gamma(\delta)}\int_0^{-\log\left(\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta}\right)} t^{\delta-1}e^{-t}dt$$
$$=1-\frac{\gamma\Big(\beta\log\Big(1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big),\delta\Big)}{\Gamma(\delta)} \tag{7}$$

and

$$f_{RBOBIII-G}(x;\alpha,\beta,\delta,\psi)=\frac{\alpha\beta^{\delta}}{\Gamma(\delta)}\frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}}\left[\log\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)\right]^{\delta-1}$$
$$\times\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1}g(x;\psi), \tag{8}$$

respectively, for $\alpha,\beta,\delta>0$ and parameter vector ψ. The parameters α and β are two positive shape parameters representing different patterns of the RBOB III-G family of distributions. The other additional parameters are from the baseline distribution and their role is to enhance flexibility of these newly family of distriutions in modelling any form of lifetime data.

2.1. Series Expansion for RBOB III-G Density Function

This section presents series expansion of the RBOB III-G density function. Let $y=1-\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta}$, then by series representations $-\log(1-y)=\sum_{i=0}^{\infty}\frac{y^{i+1}}{i+1}$, we have

$$\Big[-\log(1-y)\Big]^{\delta-1}=y^{\delta-1}\left[\sum_{m=0}^{\infty}\binom{\delta-1}{m}y^m\left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m\right],$$

and using the result on power series raised to a positive integer, with $a_s=(s+2)^{-1}$, that is,

$$\left(\sum_{s=0}^{\infty}a_s y^s\right)^m=\sum_{s=0}^{\infty}b_{s,m}y^s, \tag{9}$$

where $b_{s,m}=(sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m}=a_0^m$, (see Gradshteyn & Ryzhik, 2000), and applying the generalized binomial series representations,

$$(1+z)^{-\beta-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\beta+j+1)}{\Gamma(\beta+1)j!}z^j \tag{10}$$

and

$$(1-z)^{k-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(k)}{\Gamma(k-j)\Gamma(j+1)}z^j, \tag{11}$$

for $|z| < 1$, $\beta > 0$ and $k > 0$, we can write the RBOB III-G pdf as

$$
\begin{aligned}
f_{RBOBIII-G}(x;\alpha,\beta,\delta,\psi) &= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \\
&\quad \times \left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1} g(x;\psi) \\
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \\
&\quad \times \left(1-\left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta}\right)^{m+s+\delta-1} \\
&\quad \times \left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta-1} g(x;\psi) \\
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\
&\quad \times \left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta(j+1)-1} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} g(x;\psi) \\
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\
&\quad \times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \left[1-(1-G(x;\psi))\right]^{-\alpha(i+1)-1} \\
&\quad \times [1-G(x;\psi)]^{\alpha(i+1)-1} g(x;\psi) \\
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k=0}^{\infty} \binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\
&\quad \times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \\
&\quad \times [1-G(x;\psi)]^{k+\alpha(i+1)-1} g(x;\psi) \\
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k,p=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} b_{s,m} \\
&\quad \times \frac{(-1)^{j+i+p}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \frac{\Gamma(\beta(j+1)+i+1)i!}{\Gamma(\beta(i+1)+1)} \\
&\quad \times \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} [G(x;\psi)]^{p} g(x;\psi) \\
&= \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k,p=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} \\
&\quad \times b_{s,m} \frac{(-1)^{j+i+p}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} [G(x;\psi)]^{p+1-1} g(x;\psi) \\
&\quad \times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \frac{p+1}{(p+1)} \\
&= \sum_{p=0}^{\infty} C_{p+1} f_{E-G}(x;\psi,p+1),
\end{aligned} \tag{12}
$$

where

$$C_{p+1} = \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} \frac{(-1)^{j+i+p}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)}$$
$$\times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \frac{b_{s,m}}{(p+1)}, \quad (13)$$

and $f_{E-G}(x;\psi,p+1) = (p+1)[G(x;\psi)]^{p+1-1}g(x;\psi)$ is the Exponentiated-G (E-G) pdf with power parameter vector $p+1$. The mathematical and statistical properties of the RBOB III-G family of distributions follows directly from those of the E-G distribution.

2.2. Sub-families of Distributions

Sub-families of the RBOB III-G family of distributions are presented below. If we fix some of the parameter values then we obtain the reduced form of the RBOB III-G family of distributions as sub-families.

(a) If $\delta = 1$, we obtain the baseline OB-G family of distributions having the pdf

$$f(x;\alpha,\beta,\psi) = \frac{\alpha\beta(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta-1} g(x;\psi), \quad (14)$$

for $\alpha, \beta > 0$ (see Alizadeh et al. (2017) for details).

(b) If $\alpha = \beta = \delta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$f(x;\psi) = G(x;\psi)^{-2} \left[1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-1}\right]^{-2} g(x;\psi). \quad (15)$$

(c) If $\alpha = \beta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$f(x;\delta,\psi) = \frac{1}{\Gamma(\delta)G(x;\psi)^2} \left[-\log\left(1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-1}\right)^{-1}\right]^{\delta-1}$$
$$\times \left(1 + \left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-1}\right)^{-2} g(x;\psi), \quad (16)$$

for $\delta > 0$.

(d) If $\alpha = \delta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$f(x;\beta,\psi) = \beta G(x;\psi)^{-2}\Big(1+\Big(\frac{G(x;\psi)}{1-G(x;\psi)}\Big)^{-1}\Big)^{-\beta-1} g(x;\psi), \tag{17}$$

for $\beta > 0$.

(e) If $\beta = 1$, we obtain a reduced RBOB III-G family of distributions having the pdf

$$\begin{aligned} f(x;\alpha,\delta,\psi) = {} & \frac{\alpha}{\Gamma(\delta)}\frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}}\Bigg[-\log\Big(1+\Big(\frac{G(x;\psi)}{1-G(x;\psi)}\Big)^{-\alpha}\Big)^{-1}\Bigg]^{\delta-1} \\ & \times \Big(1+\Big(\frac{G(x;\psi)}{1-G(x;\psi)}\Big)^{-\alpha}\Big)^{-2} g(x;\psi), \end{aligned} \tag{18}$$

for $\alpha, \delta > 0$.

2.3. Hazard, Reverse Hazard and Quantile Functions

This section presents the results on the hazard, reverse hazard and quantile functions for the RBOB III-G distribution. The hazard and reverse hazard functions for the RBOB III-G distribution are given by

$$\begin{aligned} h_{RBOBIII-G}(x) = {} & \frac{\alpha\beta^{\delta} g(x;\psi)\frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}}\Big[\log\Big\{1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big\}\Big]^{\delta-1}}{\gamma\Big\{\beta\log\Big(1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big),\delta\Big\}} \\ & \times \Big\{1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big\}^{-\beta-1} \end{aligned} \tag{19}$$

and

$$\begin{aligned} \tau_{RBOBIII-G}(x) = {} & \frac{\alpha\beta^{\delta} g(x;\psi)\frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}}\Big[\log\Big\{1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big\}\Big]^{\delta-1}}{\Gamma(\delta)-\gamma\Big\{\beta\log\Big(1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big),\delta\Big\}} \\ & \times \Big\{1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big\}^{-\beta-1}, \end{aligned} \tag{20}$$

respectively. Furthermore, the quantile function for the RBOB III-G family of distributions is also derived by solving the non-linear equation

$$F(x;\alpha,\beta,\delta,\psi) = 1 - \frac{\gamma\Big\{\beta\log\Big(1+\Big(\frac{1-G(x;\psi)}{G(x;\psi)}\Big)^{\alpha}\Big),\delta\Big\}}{\Gamma(\delta)} = u, \tag{21}$$

for $0 \leq u \leq 1$, that is,

$$\log\left(1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right) = \frac{\gamma^{-1}\left[\Gamma(\delta)(1-u),\delta\right]}{\beta} \tag{22}$$

and

$$\frac{G(x;\psi)}{1-G(x;\psi)} = \left(e^{\frac{\gamma^{-1}\left[\Gamma(\delta)(1-u),\delta\right]}{\beta}}-1\right)^{\frac{-1}{\alpha}}. \tag{23}$$

We can finally write the quantile function for the RBOB III-G family of distributions as

$$Q_{RBOBIII-G}(u;\alpha,\beta,\delta,\psi) = G^{-1}\left[\frac{\left(e^{\frac{\gamma^{-1}\left[\Gamma(\delta)(1-u),\delta\right]}{\beta}}-1\right)^{\frac{-1}{\alpha}}}{1+\left(e^{\frac{\gamma^{-1}\left[\Gamma(\delta)(1-u),\delta\right]}{\beta}}\right)^{\frac{-1}{\alpha}}}\right]. \tag{24}$$

3. Some Special Cases

Some special cases of various sub-models are derived in this section. The baseline distribution function $G(x;\psi)$ is changed to other flexible distributions restricting the parameter vector space to atmost 2 component vector to avoid over parametrization.

3.1. RBOB III - Log Logistic (RBOB III-LLoG) Distribution

Consider the Log Logistic distribution as the baseline distribution with parameter $\lambda > 0$ having cdf and pdf $G(x;\lambda) = 1-(1+x^{\lambda})^{-1}$ and $g(x;\lambda) = \lambda x^{\lambda-1}(1+x^{\lambda})^{-2}$ respectively. The cdf and pdf of RBOB III-LLoG distribution are given by

$$F_{RBOBIII-LLoG}(x;\alpha,\beta,\delta,\lambda) = 1-\frac{\gamma\left\{\beta\log\left(1+\left(\frac{(1+x^{\lambda})^{-1}}{1-(1+x^{\lambda})^{-1}}\right)^{\alpha}\right),\delta\right\}}{\Gamma(\delta)} \tag{25}$$

and

$$\begin{aligned} f_{RBOBIII-LLoG}(x;\alpha,\beta,\delta,\lambda) = {} & \frac{\alpha\beta^{\delta}}{\Gamma(\delta)}\frac{((1+x^{\lambda})^{-1})^{\alpha-1}}{(1-(1+x^{\lambda})^{-1})^{\alpha+1}}\lambda x^{\lambda-1}(1+x^{\lambda})^{-2} \\ & \times\left[\log\left(1+\left(\frac{(1+x^{\lambda})^{-1}}{1-(1+x^{\lambda})^{-1}}\right)^{\alpha}\right)\right]^{\delta-1} \\ & \times\left(1+\left(\frac{(1+x^{\lambda})^{-1}}{1-(1+x^{\lambda})^{-1}}\right)^{\alpha}\right)^{-\beta-1}, \end{aligned} \tag{26}$$

respectively, for $\alpha, \beta, \delta, \lambda > 0$. The hazard rate function is given by

$$\begin{aligned} h_{RBOBIII-LLoG}(x) = & \frac{\alpha\beta^{\delta}}{\Gamma(\delta)} \frac{((1+x^{\lambda})^{-1})^{\alpha-1}}{(1-(1+x^{\lambda})^{-1})^{\alpha+1}} \lambda x^{\lambda-1}(1+x^{\lambda})^{-2} \\ & \times \left[\log\left(1+\left(\frac{(1+x^{\lambda})^{-1}}{1-(1+x^{\lambda})^{-1}}\right)^{\alpha}\right)\right]^{\delta-1} \\ & \times \left(1+\left(\frac{(1+x^{\lambda})^{-1}}{1-(1+x^{\lambda})^{-1}}\right)^{\alpha}\right)^{-\beta-1} \\ & \times \left(\gamma\left\{\beta\log\left(1+\left(\frac{(1+x^{\lambda})^{-1}}{1-(1+x^{\lambda})^{-1}}\right)^{\alpha}\right), \delta\right\}\right)^{-1}. \end{aligned} \tag{27}$$

Pdf and hrf plots for the RBOB III-LLoG distribution are given in Figures 1 and 2.

Figures 1 and 2 describe the flexibility of the RBOB III-LLoG distribution for selected parameter values. The pdfs of the RBOB III-LLoG distribution assume various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the RBOB III-LLoG distribution exhibit decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

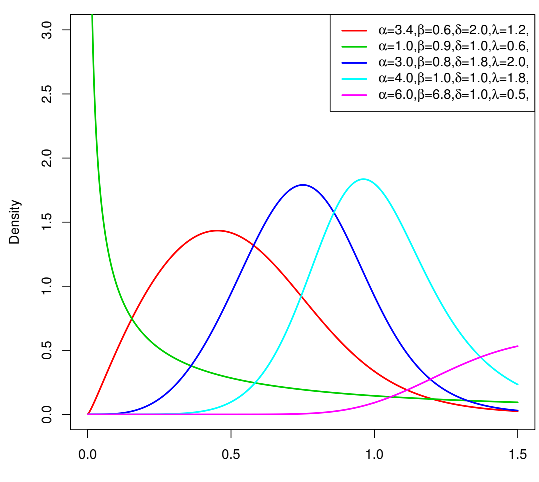

Figure 1: RBOB III-LLoG pdf plots.

Tables 1 and 2, give results on the quantiles and moments, respectively for the RBOB III-LLoG distribution for selected parameter values.

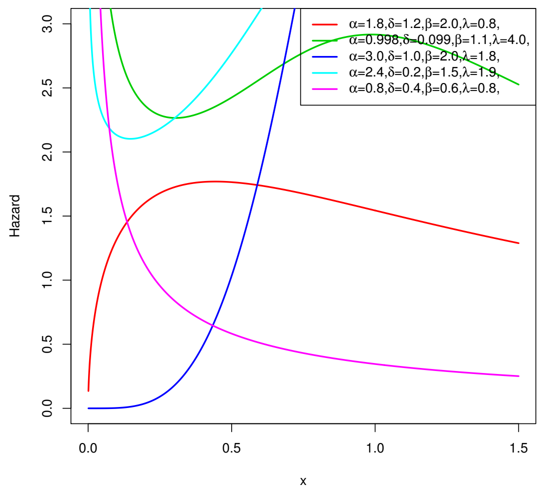

Figure 2: RBOB III-LLoG hrf plots.

Table 1: RBOB III-LLoG quantiles for selected values.

u	$(\alpha, \beta, \delta, \lambda)$				
	(0.2,1.2,2.1,1)	(0.1,1.2,2.2,1)	(0.2,1.2,2.3,1)	(0.2,1.2,2.4,1)	(0.1,1.2,2.5,1)
0.1	2.06150	2.18009	2.10302	2.11959	2.19788
0.2	2.24955	2.37087	2.29119	2.30802	2.39000
0.3	2.42861	2.55571	2.47151	2.48902	2.57667
0.4	2.60903	2.74396	2.65386	2.67232	2.76718
0.5	2.79908	2.94383	2.84642	2.86608	2.96978
0.6	3.00849	3.16533	3.05895	3.08010	3.19478
0.7	3.25245	3.42481	3.30692	3.32997	3.45879
0.8	3.56219	3.75578	3.62211	3.64773	3.79618
0.9	4.03111	4.25913	4.09970	4.12949	4.31049

Table 2: RBOB III-LLoG Moments for selected parameter values.

	(0.1,0.2,0.2,0.5)	(0.7,1.0,1.5,0.5)	(0.2,2.0,2.0,0.2)	(3.4,4.0,0.5,0.9)	(0.2,2.0,1.5,0.5)
$E(X)$	0.00216	0.04406	0.00434	0.02813	0.01167
$E(X^2)$	0.00118	0.03545	0.00345	0.02451	0.00882
$E(X^3)$	0.00081	0.02963	0.00286	0.02169	0.00708
$E(X^4)$	0.00062	0.02543	0.00245	0.01943	0.00592
$E(X^5)$	0.00050	0.02227	0.00213	0.01759	0.00508
$E(X^6)$	0.00042	0.01981	0.00189	0.01606	0.00445
SD	0.00036	0.01783	0.00170	0.01477	0.00396
CV	0.00032	0.01621	0.00154	0.01366	0.00357
CS	0.00028	0.01486	0.00141	0.01271	0.00324
CK	0.00025	0.01372	0.00130	0.01188	0.00297

3.2. RBOB III-Kumaraswamy (RBOB III-K) Distribution

Let the Kumaraswamy distribution be the baseline distribution with paramters $a, b > 0$ having cdf and pdf $G(x) = 1 - (1 - x^a)^b, 0 \leq x \leq 1$, and $g(x) = abx^{a-1}(1 - x^a)^{b-1}$, respectively. The cdf and pdf of RBOB III-K distribution are given by

$$F_{RBOBIII-K}(x; \alpha, \beta, \delta, a, b) = 1 - \frac{\gamma\left\{\beta \log\left(1 + \left(\frac{(1-x^a)^b}{1-(1-x^a)^b}\right)^{\alpha}\right), \delta\right\}}{\Gamma(\delta)}$$

and

$$\begin{aligned} f_{RBOBIII-K}(x; \alpha, \beta, \delta, a, b) = & \frac{\alpha\beta^{\delta}ab}{\Gamma(\delta)} \frac{((1 - x^a)^b)^{\alpha-1}}{(1 - (1 - x^a)^b)^{\alpha+1}} x^{a-1}(1 - x^a)^{b-1} \\ & \times \left\{1 + \left(\frac{(1 - x^a)^b}{1 - (1 - x^a)^b}\right)^{\alpha}\right\}^{-\beta-1} \\ & \times \left[\log\left(1 + \left(\frac{(1 - x^a)^b}{1 - (1 - x^a)^b}\right)^{\alpha}\right)\right]^{\delta-1}, \end{aligned} \tag{28}$$

respectively, for $\alpha, \beta, \delta, a, b > 0$. The hazard rate function is given by

$$\begin{aligned} h_{RBOBIII-K}(x) = & \frac{\alpha\beta^{\delta}ab}{\Gamma(\delta)} \frac{((1 - x^a)^b)^{\alpha-1}}{(1 - (1 - x^a)^b)^{\alpha+1}} \left[\log\left(1 + \left(\frac{(1 - x^a)^b}{1 - (1 - x^a)^b}\right)^{\alpha}\right)\right]^{\delta-1} \\ & \times x^{a-1}(1 - x^a)^{b-1} \left\{1 + \left(\frac{(1 - x^a)^b}{1 - (1 - x^a)^b}\right)^{\alpha}\right\}^{-\beta-1} \\ & \times \left(\gamma\left\{\beta \log\left(1 + \left(\frac{(1 - x^a)^b}{1 - (1 - x^a)^b}\right)^{\alpha}\right), \delta\right\}\right)^{-1}. \end{aligned} \tag{29}$$

Plots of the pdf and hrf for the RBOB III-K distribution are given in figures 3 and 4.

Figures 3 and 4 show the behaviour of RBOB III-K distribution plots for selected parameter values. The pdfs of the RBOB III-K distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. In addition, the plots of the hrf for the RBOB III-K distribution exhibit decreasing, increasing, bathtub, upside down bathtub and upside down bathtub followed by bathtub shapes.

Tables 3 and 4, presents results on the quantiles and moments, respectively for the RBOB III-K distribution on selected parameter values.

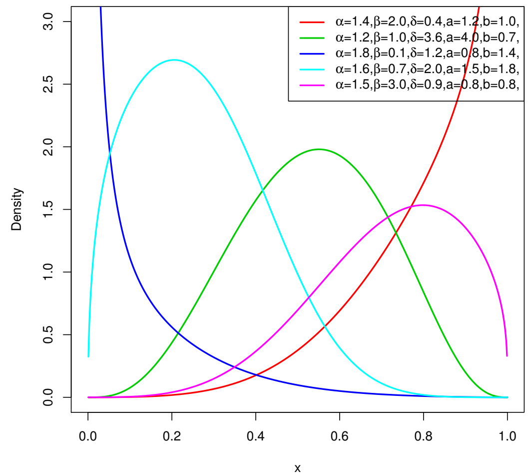

FIGURE 3: RBOB III-K pdf plots.

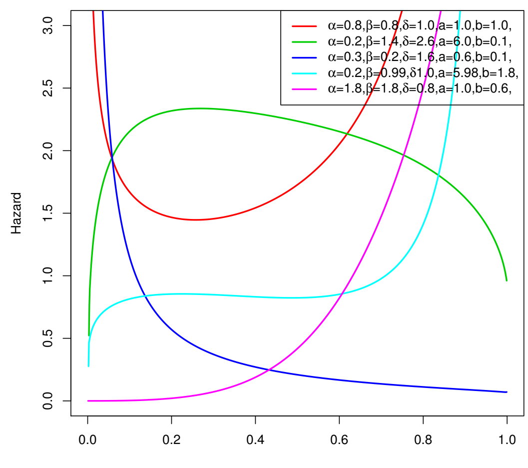

FIGURE 4: RBOB III-K hrf plots.

TABLE 3: RBOB III-K quantiles for selected values.

	$(\alpha,\beta,\delta,$a,b)				
u	(1.4,1.3,1.1,1.8,0.5)	(1.5,1.5,1.2,2,0.3)	(1.3,1.2,1,1,0.4)	(1.2,1.1,1.3,1.1,0.5)	(1.8,0.4,1.6,0.6,0.6)
0.1	0.98436	0.99806	0.99301	0.96344	0.50181
0.2	0.96354	0.99371	0.97525	0.91495	0.35243
0.3	0.93952	0.98718	0.94795	0.86016	0.25064
0.4	0.91213	0.97837	0.91067	0.79962	0.17432
0.5	0.88047	0.96673	0.86206	0.73312	0.11598
0.6	0.84323	0.95139	0.79916	0.65990	0.07205
0.7	0.79779	0.93076	0.71699	0.57851	0.04043
0.8	0.73910	0.90170	0.60505	0.48668	0.01931
0.9	0.65516	0.85701	0.43691	0.38025	0.00698

TABLE 4: RBOB III-K Moments for selected parameter values.

	(0.1 0.2 0.2 0.5 0.1)	(0.3 0.2 0.2 0.1 0.2)	(0.1 0.2 0.3 0.1 0.5)	(0.9 0.1 0.2 0.1 0.2)	(0.1 0.1 0.2 0.1 0.1)
E(X)	0.03430	0.00957	0.02966	0.01010	0.00726
$E(X^2)$	0.02639	0.00740	0.02089	0.00858	0.00561
$E(X^3)$	0.02159	0.00607	0.01617	0.00749	0.00460
$E(X^4)$	0.01831	0.00516	0.01321	0.00665	0.00391
$E(X^5)$	0.01592	0.00449	0.01118	0.00599	0.00341
$E(X^6)$	0.01410	0.00398	0.00969	0.00545	0.00302
SD	0.01265	0.00358	0.00855	0.00501	0.00271
CV	0.01148	0.00325	0.00765	0.00463	0.00246
CS	0.01051	0.00297	0.00693	0.00431	0.00226
CK	0.00969	0.00274	0.00633	0.00402	0.00208

3.3. RBOB III - Beta (RBOB III-B) Distribution

Consider the Beta distribution as the baseline distribution with parameters $a, b > 0$ having the cdf and pdf $G(x) = I_x(a,b) = \frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt$ and $g(x) = \frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$, for $0 < x < 1$, respectively. The cdf and pdf of RBOB III-B distribution are given by

$$F_{RBOBIII-B}(x;\alpha,\beta,\delta,a,b) = 1 - \frac{\gamma\left\{\beta\log\left(1+\left(\frac{(1-I_x(a,b)}{I_x(a,b)}\right)^{\alpha}\right),\delta\right\}}{\Gamma(\delta)}$$

and

$$f_{RBOBIII-B}(x;\alpha,\beta,\delta,a,b) = \frac{\alpha\beta^{\delta}}{\Gamma(\delta)B(a,b)}\frac{(1-I_x(a,b))^{\alpha-1}}{(I_x(a,b))^{\alpha+1}}x^{a}(1-x)^{b-1} \times\left\{1+\left(\frac{1-I_x(a,b)}{I_x(a,b)}\right)^{\alpha}\right\}^{-\beta-1} \times\left[\log\left(1+\left(\frac{1-I_x(a,b)}{I_x(a,b)}\right)^{\alpha}\right)\right]^{\delta-1}, \tag{30}$$

respectively, for $\alpha,\beta,\delta,a,b > 0$. The hrf is given by

$$h_{RBOBIII-B}(x) = \frac{\alpha\beta^{\delta}}{\Gamma(\delta)B(a,b)}\frac{(1-I_x(a,b))^{\alpha-1}}{(I_x(a,b))^{\alpha+1}}\left[\log\left(1+\left(\frac{1-I_x(a,b)}{I_x(a,b)}\right)^{\alpha}\right)\right]^{\delta-1} \times x^{a}(1-x)^{b-1}\left\{1+\left(\frac{1-I_x(a,b)}{I_x(a,b)}\right)^{\alpha}\right\}^{-\beta-1} \times\left(\gamma\left\{\beta\log\left(1+\left(\frac{1-I_x(a,b)}{I_x(a,b)}\right)^{\alpha}\right),\delta\right\}\right)^{-1}. \tag{31}$$

Plots of the pdf and hrf for the RBOB III-B distribution are given in Figures 5 and 6. Figures 5 and 6 gives good illustration on the flexible nature of the RBOB

III-B distribution for selected parameter values. The pdfs of the RBOB III-B distribution can take various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the RBOB III-B distribution exhibit decreasing, increasing, bathtub, upside down bathtub and upside down bathtub followed by bathtub shapes.

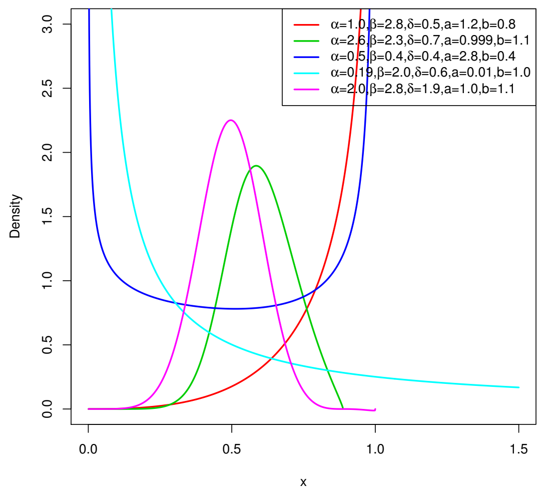

FIGURE 5: RBOB III-B pdf plots.

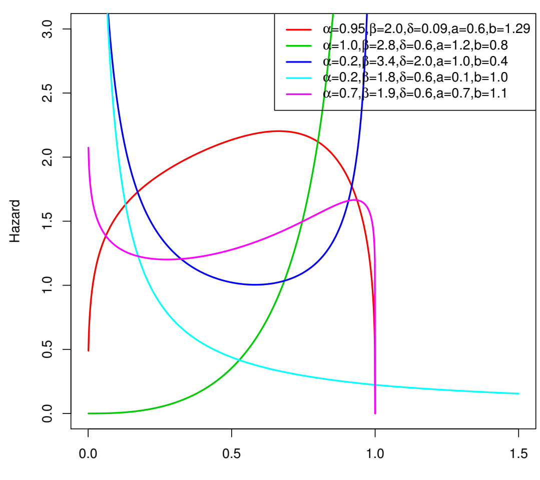

FIGURE 6: RBOB III-B hrf plots.

Tables 5 and 6, give results on the quantiles and moments, respectively for the RBOB III-B distribution with selected parameter values.

TABLE 5: RBOB III-B Quantiles for selected values.

u	$(\alpha,\beta,\delta$,a,b)				
	(1.6,0.7,0.9,0.5,1.9)	(1.6,0.7,0.9,0.6,1.9)	(1.5,0.7,0.9,0.5,1.9)	1.7,0.7,0.9,0.5,1.9)	(1.6,0.7,0.9,0.5,1.8)
0.1	0.56175	0.45911	0.62379	0.51395	0.60242
0.2	0.29265	0.31488	0.30989	0.27811	0.30853
0.3	0.18229	0.23067	0.18781	0.17757	0.19047
0.4	0.11812	0.17081	0.11837	0.11789	0.12253
0.5	0.07567	0.12411	0.07355	0.07759	0.07800
0.6	0.04582	0.08571	0.04295	0.04853	0.04695
0.7	0.02442	0.05315	0.02181	0.02694	0.02487
0.8	0.00955	0.02556	0.00794	0.01125	0.00966
0.9	0.00101	0.00419	0.00073	0.00137	0.00102

TABLE 6: RBOB III-B Moments for selected parameter values.

	(0.1 0.2 0.2 0.5 0.7)	(0.2 1.5 0.5 0.2 2.0)	(0.1 0.3 0.2 0.2 0.5)	(0.3 0.1 0.4 0.2 0.5)	(0.2 0.1 0.3 0.2 0.4)
$E(X)$	0.02499	0.03390	0.03628	0.23681	0.14712
$E(X^2)$	0.01525	0.01204	0.02387	0.15496	0.10258
$E(X^3)$	0.01145	0.00620	0.01893	0.12265	0.08433
$E(X^4)$	0.00936	0.00379	0.01613	0.10438	0.07374
$E(X^5)$	0.00800	0.00255	0.01427	0.09229	0.06660
$E(X^6)$	0.00704	0.00184	0.01293	0.08356	0.06135
SD	0.00632	0.00139	0.01190	0.07688	0.05728
CV	0.00575	0.00109	0.01108	0.07157	0.05400
CS	0.00529	0.00087	0.01041	0.06720	0.05128
CK	0.00492	0.00072	0.00984	0.06354	0.04898

4. Some Mathematical and Statistical Properties

This section presents some mathematical and statistical properties of the RBOB III-G family of distributions, namely; moments and generating functions, conditional moments, entropy, distribution of order statistics, stochastic ordering and probability weighted moments.

4.1. Moments and Generating Functions

Consider $Y_{p+1} \sim$ Exponentiated-G$(\psi; p+1)$, then the r^{th} moment can be derived using equation (12) as follows

$$E(X^r) = \sum_{p=0}^{\infty} C_{p+1} E(Y_{p+1}^r) = \sum_{p=0}^{\infty} C_{p+1} \int_0^{\infty} y^r f_{E-G}(y; \psi, p+1) dy$$
$$= \sum_{p=0}^{\infty} C_{p+1} (p+1) \int_0^1 [Q_F(u;\psi)]^r u^q du. \quad (32)$$

The moment generating function (mgf) for $|a| < 1$, is given by

$$M_X(a) = \sum_{p=0}^{\infty} C_{p+1} M_{Y_{p+1}}(a) = \sum_{p=0}^{\infty} \sum_{i=0}^{\infty} C_{p+1} \frac{a^i}{i!} E(Y_{p+1}^i), \quad (33)$$

where $M_{Y_{p+1}}(a)$ is the mgf of Y_{p+1}. The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), Standard deviation (SD=σ), CV, CS and CK are given by

$$\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1},$$

$$CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}$$

and

$$CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2},$$

respectively. The results of these coefficients and first six moments for various baseline distributions are presented under section 3. Figures 7 and 8 below presents the 3D plots of skewness and kurtosis for the RBOB III-LLoG distributions for some selected parameter values.

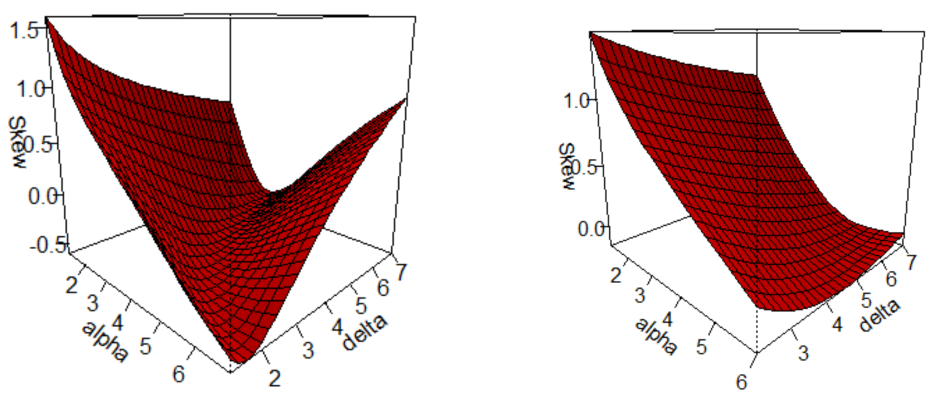

Figure 7: 3D plots of skewness for the RBOB III-LLoG distribution for some selected parameter values.

4.2. Conditional Moments

This subsection presents results of the conditional moments for the RBOB III-G distribution. The r^{th} conditional moment of the RBOB III-G distribution is given by

$$\begin{aligned} E(X^r|X \geq a) &= \frac{1}{\overline{F}(a;\alpha,\beta,\delta,\psi)} \int_t^{\infty} x^r f(x;\alpha,\beta,\delta,\psi)dx \\ &= \frac{1}{\overline{F}(a;\alpha,\beta,\delta,\psi)} \sum_{p=0}^{\infty} C_{p+1} E\left(Y_{p+1}^r I_{\{Y_{p+1}^r \geq t\}}\right), \end{aligned} \tag{34}$$

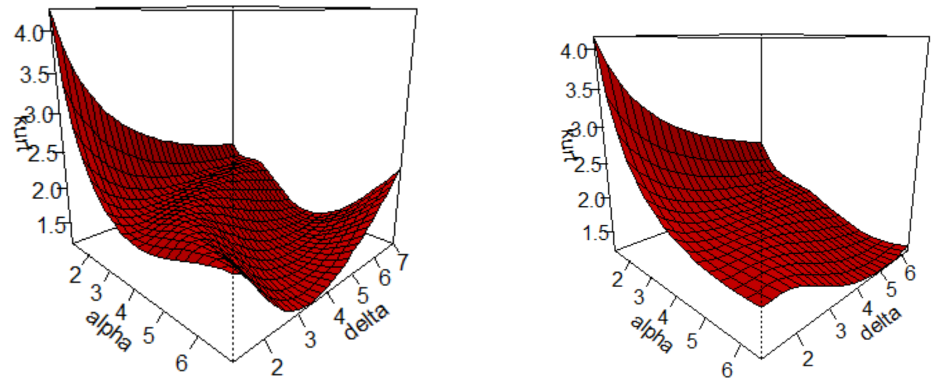

FIGURE 8: 3D plots of kurtosis for the RBOB III-LLoG distribution for some selected parameter values.

where

$$
\begin{aligned}
E\left(Y_{p+1}^{r} I_{\{Y_{p+1}^{r} \geq t\}}\right) &= \int_{t}^{\infty} y^{r} f_{E-G}(y;\psi,p+1)dy \\
&= (p+1)\int_{G(u;\xi)}^{1} [Q_{G}(u;\psi)]^{r} u^{q} du,
\end{aligned} \tag{35}
$$

for $\alpha, \beta, \delta > 0$, and parameter vector ψ. The mean residual life function is given by $E(X-a|X>a) = E(X|X>a) - a = V_{F}(a) - a$, where $V_{F}(a)$ is referred to as the vitality function of the distribution function F. Note that the mean deviations, Bonferroni and Lorenz curves can be readily obtained from the conditional and incomplete moments.

4.3. Rényi Entropy

Entropy measures variation of uncertainty for a random variable X having the probability distribution $f(x)$. The Rényi Entropy (see Rényi, 1961) which is an extension of Shannon Entropy, (see Shannon, 1951) can be defined as

$$
I_{R}(\nu) = \frac{1}{1-\nu}\log\Big(\int_{0}^{\infty}\Big[f(x;\alpha,\beta,\delta,\psi)\Big]^{\nu}dx\Big), \qquad \nu \neq 1, \nu > 0. \tag{36}
$$

Rényi Entropy tends to Shannon Entropy as $\nu \rightarrow 1$.

Let $y = 1 - \left\{1 + \left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right\}^{-\beta}$, then using the result on power series raised to a positive integer, with $c_s = (s+2)^{-1}$, that is,

$$
\left(\sum_{s=0}^{\infty} c_s y^s\right)^{m} = \sum_{s=0}^{\infty} d_{s,m} y^s, \tag{37}
$$

where $d_{s,m} = (sc_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]c_l d_{s-l,m}$, and $d_{0,m} = c_0^m$, (see Gradshteyn & Ryzhik, 2000), we have

$$\Big[-\log\Big(1-y\Big)\Big]^{\nu(\delta-1)} = y^{\delta-1}\Big[\sum_{m=0}^{\infty}\binom{\nu(\delta-1)}{m}y^m\Big(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\Big)^m\Big] = \sum_{m,s=0}^{\infty}\binom{\nu(\delta-1)}{m}d_{s,m}y^{m+s+\delta-1},$$

so that $[f(x;\alpha,\beta,\delta,\psi)]^\nu = f^\nu(x)$ can be written as

$$\begin{aligned}
f^\nu(x) &= \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu}\frac{(1-G(x;\psi))^{\nu(\alpha-1)}}{G(x;\psi)^{\nu(\alpha+1)}}\sum_{m,s=0}^{\infty}\binom{\nu(\delta-1)}{m}d_{s,m}\\
&\times\left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\nu(\beta+1)} y^{m+s+\delta-1}(g(x;\psi))^\nu\\
&= \sum_{m,s=0}^{\infty}\binom{\nu(\delta-1)}{m}d_{s,m}\frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu}\frac{(1-G(x;\psi))^{\nu(\alpha-1)}}{G(x;\psi)^{\nu(\alpha+1)}}\\
&\times\left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\nu(\beta+1)}\\
&\times\left[1-\left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta}\right]^{(m+s+\delta-1)} g(x;\psi)^\nu\\
&= \sum_{m,s=0}^{\infty}\binom{\nu(\delta-1)}{m}d_{s,m}\frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu}\frac{(1-G(x;\psi))^{\nu(\alpha-1)}}{G(x;\psi)^{\nu(\alpha+1)}}\\
&\times\sum_{k=0}^{\infty}\binom{(m+s+\delta-1)}{k}(-1)^k\\
&\times\left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta k+\nu(-\beta-1)} g(x;\psi)^\nu\\
&= \sum_{m,s=0}^{\infty}\binom{\nu(\delta-1)}{m}d_{s,m}\frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu}\sum_{k=0}^{\infty}\binom{(m+s+\delta-1)}{k}(-1)^k\\
&\times\sum_{i=0}^{\infty}\binom{-\beta k+\nu(-\beta-1)}{i}(1-G(x;\psi))^{\alpha(\nu+i)-\nu}\\
&\times G(x;\psi)^{-\alpha(i+\nu)-v}g(x;\psi)^\nu\\
&= \sum_{m,s,i,k=0}^{\infty}\binom{\nu(\delta-1)}{m}d_{s,m}\frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu}\sum_{k=0}^{\infty}\binom{(m+s+\delta-1)}{k}(-1)^k\\
&\times\binom{-\beta k+v(-\beta-1)}{i}\frac{\Gamma(\alpha(i+\nu)+\nu+k)}{\Gamma(\alpha(i+\nu)+\nu)k!}G(x;\psi))^k g(x;\psi)^\nu.
\end{aligned} \tag{38}$$

Now,

$$\int_0^\infty f^\nu(x;\alpha,\beta,\delta,\psi)dx = \sum_{m,s,i,k=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \times \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \binom{-\beta k+\nu(-\beta-1)}{i} \times \frac{\Gamma(\alpha(i+\nu)+\nu+k)}{\Gamma(\alpha(i+\nu)+\nu)k!} \Big[\int_0^\infty (G(x;\psi))^k g(x;\psi)^\nu dx\Big]. \quad (39)$$

Finally, the Rényi entropy for the RBOB III-G family of distribution is given by

$$I_R(\nu) = \frac{1}{1-\nu} \log \Bigg[\sum_{m,s,i,k=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \times \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \binom{-\beta k+\nu(-\beta-1)}{i} \times \frac{\Gamma(\alpha(i+\nu)+\nu+k)}{\Gamma(\alpha(i+\nu)+\nu)k!} \frac{1}{\left[\frac{k}{\nu}+1\right]^\nu} \int_0^\infty \left(\left[\frac{k}{\nu}+1\right] g(x;\psi) G^{\frac{k}{\nu}}(x;\psi) \right)^\nu dx \Bigg] \quad (40)$$

$$= (1-\nu)^{-1} \log \Bigg[\sum_{k=0}^{\infty} W_k e^{(1-\nu)I_{REG}} \Bigg],$$

where $I_{REG} = (1-\nu)^{-1} \log \int_0^\infty \left(\left[\frac{k}{\nu}+1\right] g(x;\psi) G^{\frac{k}{\nu}}(x;\psi) \right)^\nu dx$ is the Rényi entropy of exponentiated-G distribution with power parameter $\frac{k}{\nu}+1$ and

$$W_k = \sum_{m,s,i=0}^{\infty} \binom{\nu(\delta-1)}{m} d_{s,m} \frac{(\alpha\beta)^\nu}{(\Gamma(\delta))^\nu} \sum_{k=0}^{\infty} \binom{(m+s+\delta-1)}{k} (-1)^k \binom{-\beta k+\nu(-\beta-1)}{i} \times \frac{\Gamma(\alpha(i+\nu)+\nu+k)}{\Gamma(\alpha(i+\nu)+\nu)k!} \frac{1}{\left[\frac{k}{\nu}+1\right]^\nu}. \quad (41)$$

4.4. Distribution of Order Statistics

Consider the binomial expansion

$$(1-F(x))^{n-i} = \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j [F(x)]^j, \quad (42)$$

then we can write the i^{th} order statistic pdf from the RBOB III-G pdf $f(x;\alpha,\beta,\delta,\psi) = f(x)$ as

$$\begin{aligned} f_{i,n}(x) &= \frac{n!f(x)}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i} \\ &= \frac{n!f(x)}{(i-1)!(n-i)!}\sum_{m=0}^{n-i}(-1)^m\binom{n-i}{m}[F(x)]^{m+i-1}. \end{aligned} \tag{43}$$

From Hosseini et al. (2018), we the pdf of the i^{th} order statistic can also be written as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} m_{r,k}g^*_{r+k+1}(x), \tag{44}$$

where $g^*_{r+k+1}(x)$ is the exp-G density function with power parameter $r+k+1$,

$$m_{r,k} = \frac{n!(r+1)(i-1)!C_{r+1}}{r+k+1}\sum_{j=0}^{n-i}\frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!j!}, \tag{45}$$

and C_{r+1} is defined as C_{p+1} under equation (13). The quantities $f_{j+i-1,k}$ are recursively given by

$$f_{j+i-1,k} = (kC_0)^{-1}\sum_{m=1}^{k}[m(j+i)-k]C_m f_{j+i-1,k-m}. \tag{46}$$

4.5. Stochastic Ordering

In this section, we present the most commonly applied three orders for the RBOB III-G family of distributions namely; the usual stochastic order, the hazard rate order and the likelihood ratio order, (see Shaked & Shanthikumar (2007) for more details).

Consider the two random variables X and Y having the cdfs $F_X(t)$ and $F_Y(t)$, respectively, with $\overline{F}_X(t) = 1 - F_X(t)$ as the reliability or survival function. A random variable X is said to be stochastically smaller than the random variable Y if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t or $F_X(t) \geq F_Y(t)$ for all t. This is denoted by $X <_{st} Y$. The hazard rate order and likelihood ratio order are stronger and are given by $X <_{hr} Y$ if $h_X(t) \geq h_Y(t)$ for all t, and $X <_{\ell_r} Y$ if $\frac{f_x(t)}{f_y(t)}$ is decreasing in t. It holds that $X <_{\ell_r} Y \implies X <_{hr} Y \implies X <_{st} Y$.

Now, consider X_1 and X_2 as two independent random variables following $RBOBIII-G(\alpha,\beta,\delta_1,\psi)$ and $RBOBIII-G(\alpha,\beta,\delta_2,\psi)$ distributions, then the pdfs of X_1 and X_2 are

$$\begin{aligned} f_1(x) = \frac{\alpha\beta^{\delta_1}}{\Gamma(\delta_1)}\frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}}\left[\log\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)\right]^{\delta_1-1} \\ \times\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1}g(x;\psi) \end{aligned}$$

and

$$f_2(x) = \frac{\alpha\beta^{\delta_2}}{\Gamma(\delta_2)} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \left[\log\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)\right]^{\delta_2-1} \times \left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1} g(x;\psi).$$

The ratio, $\frac{f_1(x)}{f_2(x)}$ takes the form

$$\frac{f_1(x)}{f_2(x)} = \frac{\beta^{\delta_1}\Gamma(\delta_1)}{\beta^{\delta_2}\Gamma(\delta_2)} \left[\log\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)\right]^{\delta_1-\delta_2}. \tag{47}$$

If we differentiate equation (47) with respect to x, we get

$$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{\alpha\beta_1^{\delta}\Gamma(\delta_2)}{\beta_2^{\delta}\Gamma(\delta_1)} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \left[\log\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)\right]^{(\delta_1-\delta_2)-1} \times \frac{(\delta_2-\delta_1)g(x;\psi)}{\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)},$$

and finally $\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) < 0$, if $\delta_2 < \delta_1$, and therefore, the likelihood ratio order $X_1 <_{\ell_r} X_2$ exists. As a result, the random variables X_1 and X_2 are stochastically ordered.

4.6. Probability Weighted Moments (PWMs)

A distribution function $F \equiv F(x) = P(X \leq x)$ may be characterized by the probability weighted moments defined by

$$M(q,l,w) = E(X^q(F(X))^l(1-F(X))^w) = \int_{-\infty}^{\infty} x^q(F(x))^l(1-F(x))^w f(x)dx,$$

where q, l and w are real numbers. If $l = w = 0$ and q is a non-negative integer, then $M_{q,0,0}$ represents the conventional moment about the origin of order q. If $M_{q,0,0}$ exists and X is a continuous function of F, then $M(q,l,w)$ exists for all the non-negative real numbers l and w.

Now, by using equations 7 and 8, and applying the generalized binomial expansion, we can write

$$
\begin{aligned}
(F(x))^l(1-F(x))^w f(x) &= \sum_{z=0}^{\infty}\binom{w}{z}(-1)^z (F(x))^{l+z} f(x) = \sum_{z,r=0}^{\infty}\binom{l+z}{r}(-1)^{z+r}\binom{w}{z} f(x) \\
&\times \left[\frac{\gamma\left(\beta\log\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right),\delta\right)}{\Gamma(\delta)}\right]^r \\
&= \sum_{z,r,q=0}^{\infty}\binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\frac{\alpha\beta}{(\Gamma(\delta))^{r+1}}\frac{(1-G(x;\psi))^{\alpha-1}}{(G(x;\psi))^{\alpha+1}} \\
&\times \left[\left(-\log\left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)\right)^{-\beta}\right]^{q+2\delta-1} \\
&\times \left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1} g(x;\psi).
\end{aligned}
$$

Furthermore, by using the result on power series raised to a positive integer, with $a_s = (s+2)^{-1}$, that is,

$$
\left(\sum_{s=0}^{\infty} a_s y^s\right)^m = \sum_{s=0}^{\infty} b_{s,m} y^s, \tag{48}
$$

where $b_{s,m} = (sa_0)^{-1}\sum_{l=1}^{s}[m(l+1)-s]a_l b_{s-l,m}$, and $b_{0,m} = a_0^m$, Gradshteyn & Ryzhik (2000)see, and applying the generalized binomial series representations,

$$
(1+z)^{-\beta-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\beta+j+1)}{\Gamma(\beta+1)j!}z^j \tag{49}
$$

and

$$
(1-z)^{k-1} = \sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(k)}{\Gamma(k-j)\Gamma(j+1)}z^j, \tag{50}
$$

for $|z|<1$, $\beta>0$ and $k>0$, we can write

$$
\begin{aligned}
(F(x))^l(1-F(x))^w f(x) &= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}}\sum_{m,s=0}^{\infty}\sum_{z,r,q=0}^{\infty}\binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\binom{\delta-1}{m} b_{s,m} y^{m+s+\delta-1} \\
&\times \left(1+\left(\frac{1-G(x;\psi)}{G(x;\psi)}\right)^{\alpha}\right)^{-\beta-1} g(x;\psi)\frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} \\
&= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}}\sum_{m,s=0}^{\infty}\sum_{z,r,q=0}^{\infty}\binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\binom{\delta-1}{m} b_{s,m} \\
&\times \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}}\left(1-\left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta}\right)^{m+s+\delta-1} \\
&\times \left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta-1} g(x;\psi)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\
&\quad \times \left[1+\left(\frac{G(x;\psi)}{1-G(x;\psi)}\right)^{-\alpha}\right]^{-\beta(j+1)-1} \frac{(1-G(x;\psi))^{\alpha-1}}{G(x;\psi)^{\alpha+1}} g(x;\psi) \\
&= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\
&\quad \times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \left[1-(1-G(x;\psi))\right]^{-\alpha(i+1)-1} \\
&\quad \times [1-G(x;\psi)]^{\alpha(i+1)-1} g(x;\psi) \\
&= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i,k=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\binom{\delta-1}{m} b_{s,m} \frac{(-1)^{j+i}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\
&\quad \times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \\
&\quad \times [1-G(x;\psi)]^{k+\alpha(i+1)-1} g(x;\psi) \qquad (51) \\
&= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i,k,p=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\binom{\delta-1}{m}\binom{k+\alpha(i+1)-1}{p} b_{s,m} \\
&\quad \times \frac{(-1)^{j+i+p}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \frac{\Gamma(\beta(j+1)+i+1)i!}{\Gamma(\beta(i+1)+1)} \\
&\quad \times \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} [G(x;\psi)]^{p} g(x;\psi) \\
&= \frac{\alpha\beta}{(\Gamma(\delta))^{r+1}} \sum_{m,s,j,i,k,p=0}^{\infty} \sum_{z,r,q=0}^{\infty} \binom{w}{z}\binom{l+z}{r}(-1)^{z+r+q}\binom{\delta-1}{m}\binom{k+\alpha(i+1)-1}{p} \\
&\quad \times b_{s,m} \frac{(-1)^{j+i+p}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} [G(x;\psi)]^{p+1-1} g(x;\psi) \\
&\quad \times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \frac{p+1}{(p+1)} \\
&= \sum_{p=0}^{\infty} C^{**}_{p+1} f_{E-G}(x;\psi,p+1),
\end{aligned}
$$

where

$$C^{**}_{p+1} = \frac{\alpha\beta}{\Gamma(\delta)} \sum_{m,s,j,i,k=0}^{\infty} \binom{\delta-1}{m} \binom{k+\alpha(i+1)-1}{p} \frac{(-1)^{j+i+p}\Gamma(m+s+\delta)}{\Gamma(m+s+\delta-j)\Gamma(j+1)} \\ \times \frac{\Gamma(\beta(j+1)+i+1)}{\Gamma(\beta(i+1)+1)i!} \frac{\Gamma(\alpha(i+1)+1+k)}{\Gamma(\alpha(i+1)+1)k!} \frac{b_{s,m}}{(p+1)}, \tag{52}$$

and $f_{E-G}(x;\psi,p+1) = (p+1)[G(x;\psi)]^{p+1-1} f(x;\psi)$ is the Exponentiated-G (E-G) pdf with power parameter vector $p+1$. Finally, the PWMs of the RBOB III-G family of distributions can be written as

$$\begin{aligned} M(q,l,w) &= \int_{-\infty}^{\infty} x^q \sum_{p=0}^{\infty} C^{**}_{p+1} f_{E-G}(x;\psi,p+1)dx \\ &= \sum_{p=0}^{\infty} C^{**}_{p+1} \int_{-\infty}^{\infty} x^q f_{E-G}(x;\psi,p+1)dx, \end{aligned}$$

which shows that the $(q,l,w)^{th}$ PWMs of RBOB III-G family of distributions can be obtained from the moments of the Exp-G distribution.

5. Maximum Likelihood Estimation

Let $X \sim RBOBIII-G(\alpha,\beta,\delta,\psi)$ and $\mathbf{\Theta} = (\alpha,\beta,\delta,\psi)^T$ be the vector of model parameters. The log-likelihood function $\ell_n = \ell_n(\mathbf{\Theta})$ from a random sample of size n is given by

$$\begin{aligned} \ell_n(\mathbf{\Theta}) = {} & n\ln(\alpha) + n\ln(\beta) - n\ln(\Gamma(\delta)) + (\alpha-1)\sum_{i=1}^{n} \ln[1-G(x;\psi)] \\ & - (\alpha+1)\sum_{i=1}^{n} \ln[G(x;\psi)] + \sum_{i=1}^{n} \ln(g(x_i;\psi)) + (\delta-1) \\ & \times \sum_{i=1}^{n} \ln\left[-\log\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{-\beta}\right] \\ & - (\beta+1)\sum_{i=1}^{n} \ln\left[1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{\alpha}\right]. \end{aligned} \tag{53}$$

The elements of the score vector $U(\mathbf{\Theta})$ are given by

$$\frac{\partial \ell_n}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln[1 - G(x_i;\psi)] - \sum_{i=1}^{n} \ln[G(x_i;\psi)]$$
$$+ (\delta - 1) \sum_{i=1}^{n} \left\{ \frac{\beta\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{\beta-1}\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\ln\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)}{\left[-\log\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{-\beta}\right]\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{\beta}} \right\} \quad (54)$$
$$+ (\beta - 1) \sum_{i=1}^{n} \left\{ \frac{\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\ln\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)}{1+\left(\frac{G(x_i\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}} \right\},$$

$$\frac{\partial \ell_n}{\partial \beta} = \frac{n}{\beta} + (\delta - 1) \sum_{i=1}^{n} \left\{ \frac{\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{\beta}\ln\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{\beta}}{\left[-\log\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{-\beta}\right]\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{\beta}} \right\} \quad (55)$$
$$- \sum_{i=1}^{n} \ln\left[1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right],$$

$$\frac{\partial \ell_n}{\partial \delta} = \frac{n\Gamma'(\delta)}{\Gamma(\delta)} + \sum_{i=1}^{n} \ln\left[-\log\left(\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{-\beta}\right)\right] \quad (56)$$

and

$$\frac{\partial \ell_n}{\partial \psi_k} = -(\alpha-1)\sum_{i=1}^{n}\frac{\frac{\partial G(x,\psi_k)}{\partial \psi_k}}{1-G(x_i;\psi)} - (\alpha+1)\sum_{i=1}^{n}\frac{\frac{\partial G(x_i;\psi_k)}{\partial \psi_k}}{G(x_i;\psi)} + \sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\psi_k)}{\partial \psi_k}}{g(x_i;\psi)}$$
$$-(\delta-1)\sum_{i=1}^{n}\frac{\alpha\beta\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha-1}\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{\beta-1}}{\left[-\log\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{-\beta}\right]\left(1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}\right)^{\beta}}$$
$$\times\left[\frac{\frac{\partial G(x_i;\psi_k)}{\partial \psi_k}(1-G(x_i;\psi)) + \frac{\partial G(x_i;\psi_k)}{\partial \psi_k}G(x_i;\psi)}{(1-G(x_i;\psi))^2}\right] \quad (57)$$
$$-(-\beta-1)\sum_{i=1}^{n}\frac{\alpha\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha-1}\left[\frac{\frac{\partial G(x_i;\psi_k)}{\partial \psi_k}(1-G(x_i;\psi))+\frac{\partial G(x_i;\psi_k)}{\partial \psi_k}G(x_i;\psi)}{(1-G(x_i;\psi))^2}\right]}{1+\left(\frac{G(x_i;\psi)}{1-G(x_i;\psi)}\right)^{-\alpha}}.$$

Note that these functions are not in closed form and can only be solved using iterative methods from applicable softwares. The maximum likelihood estimates (MLEs) of the parameters, denoted by $\hat{\mathbf{\Theta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \delta}, \frac{\partial \ell_n}{\partial \psi})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. Note that from the multivariate normal distribution $N_{p+3}(\underline{0}, J(\hat{\mathbf{\Theta}})^{-1})$, where $\underline{0} = (0, 0, 0, \underline{0})^T$ is the mean vector and $J(\hat{\mathbf{\Theta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Theta}}$, we can construct confidence intervals and confidence regions for the individual model parameters. The $100(1-\varphi)\%$ two-sided confidence intervals for α, β, δ and ψ_k are as follows;

$$\hat{\alpha} \pm Z_{\frac{\varphi}{2}} \sqrt{I^{-1}_{\alpha\alpha}(\hat{\mathbf{\Theta}})}, \;\; \hat{\beta} \pm Z_{\frac{\varphi}{2}} \sqrt{I^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})}, \;\; \hat{\delta} \pm Z_{\frac{\varphi}{2}} \sqrt{I^{-1}_{\delta\delta}(\hat{\mathbf{\Theta}})}, \;\; \text{and} \;\; \hat{\psi}_k \pm Z_{\frac{\varphi}{2}} \sqrt{I^{-1}_{\psi_k\psi_k}(\hat{\mathbf{\Theta}})},$$

respectively, where $I^{-1}_{\alpha\alpha}(\hat{\mathbf{\Theta}})$, $I^{-1}_{\beta\beta}(\hat{\mathbf{\Theta}})$, $I^{-1}_{\delta\delta}(\hat{\mathbf{\Theta}})$ and $I^{-1}_{\psi_k\psi_k}(\hat{\mathbf{\Theta}})$ are the diagonal elements of $I_n^{-1}(\hat{\mathbf{\Theta}}) = (nI(\hat{\mathbf{\Theta}}))^{-1}$, and $Z_{\frac{\varphi}{2}}$ represents the standard normal $(\frac{\varphi}{2})^{th}$ percentile. Note that the asymptotic behaviour still holds when $I(\mathbf{\Theta})$ which is the expected Fisher information matrix is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Theta}}$.

6. Monte Carlo Simulation Study

In this section, we conduct a simulation study to evaluate consistency of the maximum likelihood estimators for the RBOB III-LLoG distribution using different parameter values. The simulation study is repeated for $N = 1000$ times with sample size $n = 35, 50, 100, 200, 400, 800$ and 1000. The simulation results (mean of the MLEs, average bias and root mean square error (RMSE)) are presented under Tables 9 and 10. The results shows that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, with Average Bias and RMSEs converging towards zero. The Average Bias and RMSEs are given by:

$$\text{Bias}(\hat{\theta}) = \frac{\sum_{i=1}^{n} \hat{\theta}_i}{n} - \theta \;\; \text{and} \;\; \text{RMSE}(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{n} (\hat{\theta}_i - \theta)^2}{n}},$$

respectively.

7. Applications

In this section, the RBOB III-LLoG distribution was applied to two real data sets to demonstrate its performance compared to other existing models. The first data set represents the lifetimes data relating to relief times (in minutes) for 20 patients receiving an analgesic as reported by Gross & Clark (1975) and is given by: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. The second data relates to the effects of variations in airborne exposure on concentration of urinary metabolites studied by Kumagai & Matsunaga (1995) and is given by: 1.5, 1.7, 2.1, 2.2, 2.4, 2.5, 2.6, 3.8, 3.8, 4.2, 4.3, 5.6, 6, 7, 7.5, 9.3, 9.9, 10.2, 10.6, 12.3, 12.9, 13.7, 14.1, 17.8, 27.6, 31, 42, 45.6, 51.9, 91.3, 131.8.

The RBOB III-LLoG distribution was compared to other existing models namely, the Burr-Weibull (BW) by Oluyede et al. (2019), Kumaraswamy Weibull (KW) by Cordeiro et al. (2010), Weibull-Lomax (WLx) by Tahir et al. (2015), Beta Generalized Lindley (BGL) by Oluyede & Yang (2015) and New Modified Weibull (NMW) b y Doostmoradi et al. (2014) distributions. The goodness-of-fit statistics was used for comparisons, namely, -2 log-likelihood ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$) and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters.

We obtained results on the Cramer̋-von Mises (W^*) and Anderson-Darling Statistics (A^*) described by Chen & Balakrishnan (1995), as well as Kolmogorov-Smirnov (KS) statistic and its p-value. Note that for the value of the log-likelihood function at its maximum (ℓ_n), the smaller value is preferred, and for AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and $K-S$, smaller values are also preferred. The results from two real lifetime data sets are presented under Tables 7 and 8. We used the R software to compute estimates for model parameters and run goodness-of-fit tests. The pdfs of the non-nested models (BW, KW, WLx, BGL and NMW) used for comparisons are

$$f(x;\alpha,\beta\, c,k) = e^{-\alpha x^{\beta}}[1+x^c]^{-k}\left\{\alpha x^{\beta-1}\beta + \frac{kcx^{c-1}}{(1+x^c)}\right\}, \tag{58}$$

for $\alpha,\beta,c,k > 0$ and $x > 0$,

$$\begin{aligned} f(x;a,b,c,\lambda) = {} & abc\lambda^c x^{c-1}\exp\{-(\lambda x)^c\}[1-\exp\{-(\lambda x)^c\}]^{a-1} \\ & \times \{1-[1-\exp\{-(\lambda x)^c\}]^a\}^{b-1}, \end{aligned} \tag{59}$$

for $a,b,c,\lambda > 0$ and $x > 0$,

$$\begin{aligned} f(x;a,b,\alpha,\beta) = {} & \frac{ab\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\alpha-1}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1} \\ & \times \exp\left\{-a\left\{1+\left(\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right\}, \end{aligned} \tag{60}$$

for $a,b,c,\lambda > 0$ and $x > 0$,

$$\begin{aligned} f(x;\alpha,\theta,a,b) = {} & \frac{\alpha\theta^2(1+x)e^{-\theta x}}{B(a,b)(1+\theta)}\left\{1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right\}^{a\alpha-1} \\ & \times \left\{1-\left\{1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right\}^{\alpha}\right\}^{b-1}, \end{aligned} \tag{61}$$

for $a,b,\alpha,\theta > 0$ and $x > 0$ and

$$f(x;\alpha,\beta,\lambda,\gamma)=(\alpha\gamma x^{\gamma-1}e^{\alpha x^{\gamma}}+\lambda\beta x^{\lambda-1}e^{-\beta x^{\gamma}})\times e^{-e^{\alpha x^{\lambda}}+e^{-\beta x^{\lambda}}}, \tag{62}$$

for $\alpha,\beta,\lambda,\gamma>0$ and $x>0$, respectively.

The TTT plots for relief times and urinary metobolites data sets are presented under Figures 9 and 10, respectively. These plots indicate that the empirical hazard rate functions of the data sets is increasing and therefore the RBOB III-LLoG distribution is appropriate to fit these data. The TTT plots are given by plotting $T(i/n)=[\sum_{k=1}^{n}y_{k:n}+(n-r)y_{i:n}]\,/\sum_{k=1}^{n}y_{k:n}$ against (i/n), where $i=1,\ldots,n$ and $y_{k:n}(k=1,\ldots,n)$ represent order statistics of the sample.

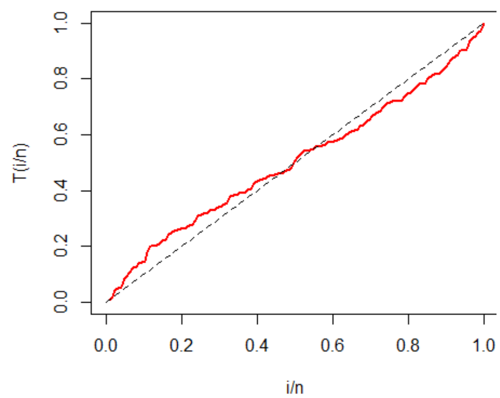

Figure 9: TTT plot for relief times dataset.

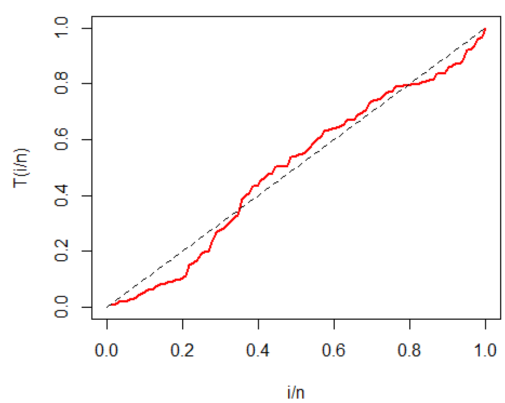

Figure 10: TTT plot for urinary metabolites dataset.

TABLE 7: Parameter estimates and goodness-of-fit statistics for different models fitted in dataset relating to relief times (in minutes) for patients receiving analgesic.

Model	Estimates				Statistics							
	α	δ	λ	β	$-2\log L$	AIC	CAIC	BIC	W^*	A^*	K-S	P-value
RBOB III-LLoG	2.2301 (0.6220)	0.8719 (0.8814)	2.0967 (0.6616)	7.4257 (2.8589)	30.8	38.8	41.5	42.8	0.03114	0.1956	0.1469	0.78
	a	b	c	λ								
KW	9.3780 (11.6562)	0.3073 (0.3462)	1.9780 (0.6181)	1.1543 (0.4181)	33.7	41.7	44.4	45.7	0.0743	0.4419	0.1795	0.54
	c	k	α	β								
BW	165.7300 (2.1×10^{-7})	0.0060 (0.0034)	0.0263 (0.0409)	3.4398 (0.9816)	37.8	45.8	48.4	49.7	0.1087	0.6373	0.1918	0.45
	α	γ	λ	β								
NMW	0.1255 (0.0612)	0.8021 (0.4008)	6.5457 (1.5533)	0.0149 (0.0152)	51.0	59.0	61.7	63.0	0.03811	0.3006	0.2286	0.25
	a	b	α	β								
WL	2806.0 (3.8×10^{-5})	5.8231 (4.0485)	0.1168 (0.0409)	0.3478 (0.9022)	38.9	46.9	49.5	50.9	0.1506	0.8897	0.1766	0.56
	α	θ	a	b								
BGL	0.1203 (0.0127)	1.5×10^{-7} (4.3×10^{-6})	0.3016 (0.0013)	10.1 (3.5×10^{-5})	162.5	170.5	173.2	174.5	0.0887	0.5230	0.5123	5.5×10^{-5}

Table 8: Parameter estimates and goodness-of-fit statistics for different models fitted to data relating to effects of variations in airborne exposure on concentration of urinary metabolites.

Model	Estimates				Statistics							
	α	δ	λ	β	$-2\log L$	AIC	CAIC	BIC	W^*	A^*	K-S	P-value
RBOB III-LLoG	1.9844 (0.1205)	0.9653 (1.3223)	0.5658 (0.4225)	6.7645 (3.0050)	233.0	241.0	242.5	246.7	0.0365	0.2754	0.0954	0.94
	a	b	c	λ								
KW	8.3694 (6.9477)	0.2140 (0.2990)	0.5854 (0.1549)	1.8193 (1.4783)	234.0	242.0	243.6	247.7	0.0488	0.3535	0.1188	0.77
	c	k	α	β								
BW	35.3400 (5.1×10^{-7})	0.0069 (0.0031)	0.0175 (0.0243)	1.0895 (0.2764)	236.5	244.5	246.0	250.2	0.0824	0.5288	0.1331	0.64
	α	γ	λ	β								
NMW	0.0907 (0.0899)	0.2647 (0.6904)	1.1684 (0.7031)	0.0356 (0.0506)	263.6	271.6	273.1	277.3	0.0633	0.5240	0.2488	0.04
	a	b	α	β								
WL	1339.4 (0.0059)	2.3647 (2.9203)	0.0173 (0.0343)	0.0065 (4.8302)	234.8	242.8	244.4	248.6	0.0506	0.3728	0.1038	0.89
	α	θ	a	b								
BGL	0.1089 (0.0087)	3.4×10^{-8} (1.79×10^{-6})	0.3007 (0.0006)	10.1000 (2.46×10^{-6})	347.7	355.7	357.4	361.6	0.0667	0.4774	0.4679	2.54×10^{-6}

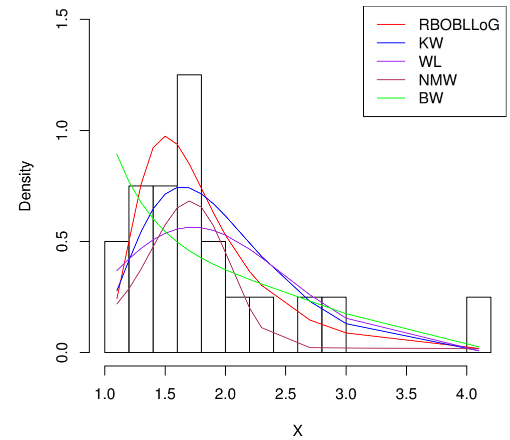

FIGURE 11: fitted density plots for relief times data.

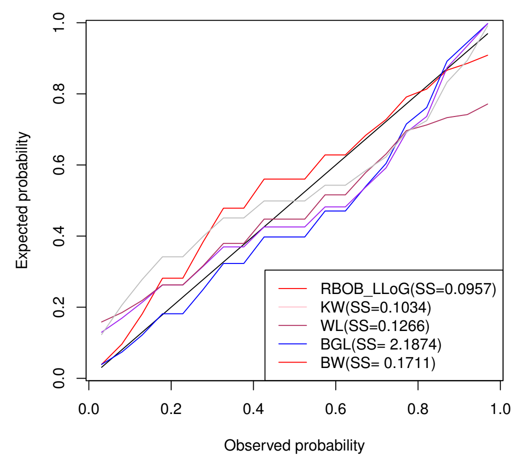

FIGURE 12: observed probability plots for relief times data.

The estimated covariance matrix is given by

$$\begin{bmatrix} 0.38695 & -0.52759 & 0.41157 & -0.57037 \\ -0.52759 & 0.77685 & -0.56115 & 1.26736 \\ 0.41157 & -0.56115 & 0.43775 & -0.60666 \\ -0.57037 & 1.26736 & -0.60666 & 8.17335 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [2.2301 \pm 1.21923]$, $\delta \in [0.8719 \pm 1.72752]$, $\lambda \in [2.0967 \pm 1.29679]$ and $\beta \in [7.4257 \pm 5.60346]$.

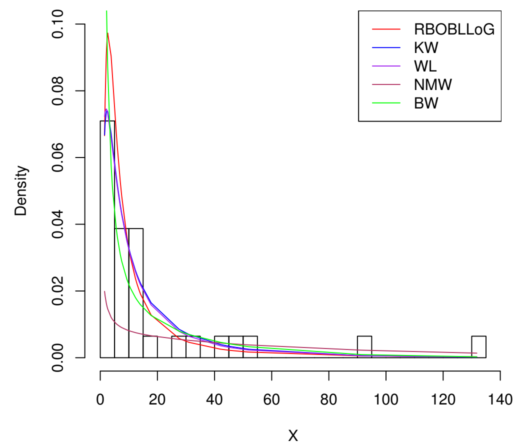

FIGURE 13: fitted density plots for urinary metabolites data.

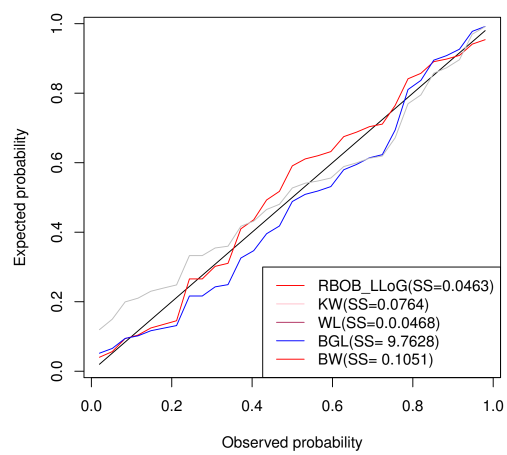

FIGURE 14: observed probability plots for urinary metabolites data.

The estimated covariance matrix is given by

$$\begin{bmatrix} 0.01450 & -0.15722 & 0.05089 & -0.27533 \\ -0.15722 & 1.75030 & -0.55154 & 3.30879 \\ 0.05089 & -0.55154 & 0.17852 & -0.96593 \\ -0.27533 & 3.30879 & -0.96593 & 9.02983 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [1.9844 \pm 0.23607]$, $\delta \in [0.9653 \pm 2.59306]$, $\lambda \in [0.5658 \pm 0.82813]$ and $\beta \in [6.7645 \pm 5.88973]$.

Table 9: Monte Carlo Simulation Results for RBOB III-LLoG Distribution; Mean, Average Bias and RMSE

Parameter	Sample Size	Set I: α=1.0, β=0.8 , δ=0.5 , λ=1.0			Set II: α=1.2, β=1.0, δ=0.6 , λ=1.5		
		Mean	Bias	RMSE	Mean	Bias	RMSE
α	35	0.95664	-0.04336	0.69133	1.01516	-0.18484	2.35483
	50	0.96133	-0.03867	0.58158	1.03503	-0.16497	0.67562
	100	0.96136	-0.03864	0.34073	1.09029	-0.10971	0.58593
	200	0.96517	-0.03483	0.27915	1.11648	-0.08352	0.51709
	400	0.96678	-0.03322	0.22092	1.11722	-0.08278	0.51518
	800	1.00089	0.00089	0.17599	1.21571	0.01571	0.41114
	1000	1.00012	0.00012	0.15728	1.20051	0.00051	0.28453
β	35	0.51213	-0.28787	0.57845	1.73592	0.73592	0.60866
	50	0.55566	-0.24434	0.54865	1.33972	0.33972	0.60192
	100	0.63118	-0.16882	0.51090	1.23692	0.23692	0.54608
	200	0.67801	-0.12199	0.50636	1.15581	0.15581	0.53129
	400	0.70886	-0.09114	0.50509	1.10924	0.10924	0.52400
	800	0.80053	0.00053	0.37358	1.08134	0.08134	0.50346
	1000	0.80036	0.00036	0.25251	1.00872	0.00872	0.46234
δ	35	0.71279	0.21279	0.43846	0.98798	0.38798	0.57907
	50	0.71109	0.21109	0.42965	0.84478	0.24478	0.56681
	100	0.65505	0.15505	0.33578	0.71697	0.11697	0.53755
	200	0.60439	0.10439	0.26315	0.67810	0.07810	0.49036
	400	0.56590	0.06590	0.20094	0.65597	0.05597	0.41630
	800	0.55032	0.05032	0.15559	0.63246	0.03246	0.34346
	1000	0.50053	0.00053	0.13370	0.60091	0.00091	0.25730
λ	35	0.88513	-0.11487	0.47263	1.75370	0.25370	0.78269
	50	0.88689	-0.11311	0.41593	1.74089	0.24089	0.77227
	100	0.89647	-0.10353	0.39038	1.72322	0.22322	0.75097
	200	0.95618	-0.04382	0.34119	1.63949	0.13949	0.69160
	400	1.00809	0.00809	0.32204	1.61459	0.11459	0.55876
	800	1.00061	0.00061	0.31592	1.51687	0.01687	0.28940
	1000	1.00004	0.00004	0.27577	1.50056	0.00056	0.24761

Table 10: Monte Carlo Simulation Results for RBOB III-LLoG Distribution; Mean, Average Bias and RMSE

Parameter	Sample Size	Set III: α=1.5, β=1.2 , δ=0.5 , λ=2.0			Set IV: α=0.6, β=1.0 , δ=1.5 , λ=1.4		
		Mean	Bias	RMSE	Mean	Bias	RMSE
α	35	1.65846	0.15846	0.72472	0.64134	0.04134	0.24185
	50	1.62834	0.12834	0.62353	0.62120	0.02120	0.20292
	100	1.53509	0.03509	0.46937	0.69012	0.09012	0.13789
	200	1.52670	0.02670	0.39081	0.68045	0.08045	0.10881
	400	1.51122	0.01122	0.30902	0.66008	0.06008	0.08265
	800	1.50808	0.00808	0.26033	0.61033	0.01033	0.06719
	1000	1.50052	0.00052	0.25740	0.60029	0.00029	0.06523
β	35	1.25877	0.05877	0.84088	1.25059	0.25059	0.79295
	50	1.24764	0.04764	0.73485	1.20947	0.20947	0.71019
	100	1.23983	0.03983	0.63747	1.16784	0.16784	0.64150
	200	1.23581	0.03581	0.50057	1.11054	0.11054	0.61578
	400	1.21107	0.01107	0.47207	1.07268	0.07268	0.57400
	800	1.20475	0.00475	0.42575	1.00823	0.00823	0.47615
	1000	1.20014	0.00014	0.14565	1.00106	0.00106	0.46641
δ	35	0.66112	0.16112	0.35865	1.66396	0.16396	0.59323
	50	0.63567	0.13567	0.33348	1.55225	0.05225	0.53846
	100	0.62631	0.12631	0.30234	1.52034	0.02034	0.47229
	200	0.56796	0.06796	0.23245	1.51497	0.01497	0.43555
	400	0.54348	0.04348	0.16918	1.50443	0.00443	0.42873
	800	0.52659	0.02659	0.12928	1.50259	0.00259	0.36730
	1000	0.50072	0.00072	0.11974	1.50028	0.00028	0.31811
λ	35	3.11606	1.11606	0.92648	1.55936	0.15936	0.58519
	50	2.87745	0.87745	0.92164	1.54008	0.14008	0.49292
	100	2.27689	0.27689	0.91941	1.40882	0.00882	0.44626
	200	2.16928	0.16928	0.45793	1.40580	0.00580	0.42997
	400	2.06812	0.06812	0.38234	1.40126	0.00126	0.37838
	800	2.00180	0.00180	0.25083	1.40116	0.00116	0.22376
	1000	2.00058	0.00058	0.20583	1.40065	0.00065	0.21932

8. Concluding Remarks

We developed and studied in detail a new family of generalized distributions called the Ristić-Balakhrishnan Odd Burr III - G (RBOB III-G) distribution. The new proposed distribution presents a flexible mechanism for fitting a wide spectrum of real world data sets. From the application results and comparison to several known models such as Burr-Weibull (BW) by Oluyede et al. (2019), Kumaraswamy Weibull (KW) by Cordeiro et al. (2010), Weibull-Lomax (WLx) by Tahir et al. (2015), Beta Generalized Lindley (BGL) by Oluyede & Yang (2015) and New Modified Weibull (NMW) by Doostmoradi et al. (2014), we confirm that it provides a better fit, see results under Tables 7 and 8.

Acknowledgement

The authors would like to thank the anonymous reviewers and Editors for the useful comments that greatly improved this article to its current version.

[Received: July 2020 — Accepted: April 2021]

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R-Code for Applications

```
RBOB IIILLoG_LL<- function(alpha,delta,lambda,beta) {
         -sum(log((alpha*beta/gamma(delta))*((((1+x^lambda)^(-1))^(alpha-1))
         /(1-((1+x^lambda)^(-1)))^(alpha+1))*lambda*(x^(lambda-1))*
         ((1+x^lambda)^(-2))*((-log((1+(((1+x^lambda)^(-1)/
         (1-(1+x^lambda)^(-1)))^alpha))^(-beta)))^(delta-1))*
         (1+(((1+x^lambda)^(-1)/(1-(1+x^lambda)^(-1)))^alpha))^(-beta-1)))
  }
  RBOB IIILLoG.result<-mle2(RBOB IIILLoG_LL,hessian = NULL,
  start=list(alpha=0.064885 ,delta=20.026  ,lambda=0.4885,beta=0.1243)
  ,optimizer="nlminb",lower=0)
  summary(RBOB IIILLoG.result)

  ####### Kumaraswamy Weibull
  KW<- function(a,b,c,lambda) {-sum(log(a*b*c*(lambda^c)*(x^(c-1))*
         exp(-(lambda*x)^c)*((1-exp(-(lambda*x)^c))^(a-1))*
         (1-(1-exp(-(lambda*x)^c))^a)^(b-1)))
  }
  KW.result<-mle2(KW,hessian = NULL,start=list(a=9.9180,b=0.07881,
  c=0.2097,lambda=0.0109),optimizer="nlminb",lower=0)
  summary(KW.result)

  ####### ####### Beta  GL
  BGL <- function(alpha,theta,a,b){-sum(log(
         (alpha*(theta^2)*(1+x)*(exp(-theta*x))/beta(a,b))*((1-((1+theta+theta
         *x)/(1+theta))*exp(-theta*x))^(a*alpha-1))*(1-(1-((1+theta+theta*x)/
         (1+theta))*exp(-theta*x))^alpha)^(b-1)))
  }
  BGL.result<-mle2(BGL,hessian = NULL,start=list(alpha=0.099,theta=0.03,a=0.3,
  b=10.0999),optimizer="nlminb",
  lower=0)
  summary(BGL.result)

  ####### BurrW
  BurrW<- function(c,k,alpha,beta) {-sum(log(exp(-alpha*x^beta)*
         (1+x^c)^(-k)*(alpha*(x^(beta-1))*beta+((k*c*x^(c-1))/(1+x^c)))))
  }
  BurrW.result<-mle2(BurrW,hessian = NULL,start=list(c=10.09,k=1.0,
  alpha=0.09,beta=0.871), optimizer="nlminb",lower=0)
  summary(BurrW.result)

  NMW <- function(alpha,gamma,lambda,beta){
         -sum(log(alpha*gamma*x^(gamma-1)+lambda*beta*(x^(lambda-1))*
         exp(-beta*x^lambda))-exp(alpha*x^gamma)+exp(-beta*x^lambda))
  }
  NMW.result<-mle2(NMW,hessian = NULL,start=list(alpha=1.009,gamma=1.0987,
  lambda=1.0,beta=1.0),optimizer="nlminb",lower=0)
  summary(NMW.result)
```