Revista Colombiana de Estadística - Applied Statistics
January 2022, volume 45, issue 1, pp. 209–229
http://doi.org/10.15446/rce.v45n1.90672

Some Inferential Problems from Log Student's T-distribution and its Multivariate Extension

Algunos problemas inferenciales a partir de la distribucion T de Student y su extension multivariante

Akinlolu Olosunde[a], Sylvester Olofintuade[b]

Department of Mathematics, Faculty of Sciences, Obafemi Awolowo University, Ile-Ife, Nigeria

Abstract

Assumption of normality in statistical analysis had been a common practice in many literature, but in the event where small sample is obtainable, then normality assumption will lead to erroneous conclusion in the statistical analysis. Taking a large sample had been a serious concern in practice due to various factors. In this paper, we further derived some inferential properties for log student's t-distribution (simply log-t distribution) which makes it more suitable as substitute to log-normal when carrying out analysis on right-skewed small sample data. Mathematical and Statistical properties such as the moments, cumulative distribution function, survival function, hazard function and log-concavity are derived. We further extend the results to case of multivariate log-t distribution; we obtained the marginal and conditional distributions. The parameters estimation was done via maximum likelihood estimation method, consequently its best critical region and information matrix were derived in order to obtain the asymptotic confidence interval. The applications of log-t distribution and goodness-of-fit test was carried out on two dataset from literature to show when the model is most appropriate.

***Key words*:** best critical region; log-t distribution; maximum likelihood estimation; Multivariate log-t distribution; Shannon entrop.

Resumen

La suposicion de normalidad en el analisis estadistico habia sido una pratica comun en mucha literatura, pero en el caso de que se pueda obtener una muestra pequena, la suposicion de normalidad conducira a conclusiones

[a]Ph.D. E-mail: aolosunde@oauife.edu.ng
[b]Ph.D. E-mail: sylvesterolalekan@gmail.com

erroneas en el analisis estadistico. En la practica, la toma de una muestra grande habia sido una gran preocupacion debido a varios factores. En este articulo, obtuvimos ademas algunas propiedades inferenciales para la distribucion t de log student (simplemente distribucion log-t) que la hace mas adecuada como sustituto de log-norma al realizar analisis en datos de muestras pequenas con sesgo a la derecha. Se derivan propiedades matematicas y estadisticas como los momentos, la funcion de supervivencia, la funcion de riesgo y la concavidad logaritmica. ampliamos aun mas el resultado al caso de distribucion log-t multivariante; obtuvimos las distribuciones marginales y condicionales. La estimacion de los parametros se realizo mediante el metodo de estimacion de maxima verosimilitud, por lo que se derivo su mejor region critica y matriz de informacion para obtener el intervalo de confianza asintotico. Las aplicaciones de la distribucion log-t y la prueba de bondad de ajuste se llevaron a cabo en dos conjuntos de datos de la literatura para mostrar cuando el modelo es mas apropiado.

***Palabras clave*:** distribucion log-t; distribucion log-t multivariante; estimacion de maxima verosimilitud; entropia de Shannon; mejor region critica.

1. Introduction

The student's t-distribution is a continuous probability distribution that arises when estimating the normally distributed population mean with unknown variance for a small sample size. It has the degree of freedom parameter which regulates it tails and generalizes Cauchy and normal distributions when varied, Gosset (1908) and Fisher (1925). Some univariate and multivariate extensions of the student's t-distribution have been studied by Lin (1972), Kotz & Nadarajah (2004), Kibria & Joarder (2006), Cassidy (2016) and Hassan & Assar (2016).

However, in lifetime data analysis, Saw et al. (2002) introduced the log-Exponential Inverse Gaussian distribution (log-EIG) by adopting the transformation that exist between the lognormal and normal distribution. The study showed that the log-EIG model outperformed other lifetime models such has the gamma, Weibull, lognormal and inverse Gaussian distributions that have been widely used. Mitzenmacher & Tworetzky (2003) also introduced the log-t distribution as a new model and method for file size distributions and it was remarked that the log-t was suitable just as the hybrid lognormal-pareto distribution owing to its fewer parameters and also outperformed the lognormal distribution.

Moreover, the log-t distribution is a positively skewed distribution derived from the transformation of the random variable that has a student's t-distribution. It possesses the same parameters as that of the student's t but on a positive real line like other lifetime distributions studied by Cassidy et al. (2013), Butt & Habibullah (2016). However, several distributions such as gamma, lognormal, Weibull, log-logistic and so on have been used in modelling lifetime data irrespective of the sample size. Whereas, the log-t distribution which tends to perform better than other distributions in this class due to its degree of freedom parameter which could be varied to generalize the log-Cauchy and lognormal distributions, and more importantly when analyzing relatively small sample size data ($n < 30$) has been under-utilized in literature as far as we know. It is on this basis, we examine some inferential statistics of the log-t distribution and its multivariate extension which may make it suitable in situations where researchers do not have large enough data to assume lognormality.

Finally, we applied the log-t distribution two dataset of different sample sizes: bladder cancer and acute leukamia data as presented by Lee & Wang (2003).

The paper is organized as follows: In section 1, the introduction of the study is presented. In section 2, the distribution is derived, some mathematical properties are obtained. In section 3, the moment of the distribution is investigated. In section 4, the Shannon entropy is obtained. In section 5, the log-concavity and monotonicity of the distribution are investigated. In section 6, the special cases of the distribution are studied. In section 7, the distribution parameters are estimated by the maximum likelihood estimation (MLE) method. In section 8, the information matrix and asymptotic confidence interval are obtained. In section 9, the distribution's multivariate version, marginal and conditional distributions, log-concavity, bivariate densities and contours are studied. Finally, in section 10, the application of the distribution is investigated on two datasets of different sample sizes.

2. Log Student's T-distribution

We derive the log-t distribution using a transformation of the student's t density function. This is presented in proposition 2.1.

Proposition 1. *Suppose that a random variable W follows the univariate student's t-distribution. Then it has a density function given as*

$$f(w;\mu,\sigma^2,k) = \frac{\Gamma(\frac{k+1}{2})\left[1+\frac{1}{k}\left(\frac{w-\mu}{\sigma}\right)^2\right]^{-\left(\frac{k+1}{2}\right)}}{\Gamma(\frac{k}{2})\sigma\sqrt{k\pi}}; \qquad w \in \mathbb{R} \tag{1}$$

where $\mu \in \mathbb{R}$ is the mean of the distribution, $\sigma^2 > 0$ is the variance and $k > 0$ is the degree of freedom parameter which regulates its tails.
A random variable U is said to have log-t distribution if

$$f(u;\mu,\sigma^2,k) = \frac{\Gamma(\frac{k+1}{2})\left[1+\frac{1}{k}\left(\frac{\ln(u)-\mu}{\sigma}\right)^2\right]^{-\left(\frac{k+1}{2}\right)}}{u\Gamma(\frac{k}{2})\sigma\sqrt{k\pi}}; \qquad u > 0 \tag{2}$$

where $\mu \in \mathbb{R}^+$, is the mean of the distribution, $\sigma^2 > 0$ is the variance and $k > 0$ is the degree of freedom.

Proof. The proof is obtained by simply applying the transformation $U = e^W$ $F_U(u) = Pr(U \leq u) = Pr(e^W \leq u), \implies Pr(W \leq \ln(u)) = F_W(\ln(u))$, $\frac{d}{du}F_W(\ln(u)) = \frac{1}{u}f_W(\ln(u))$. Then Equation (1) becomes Equation (2). □

Henceforth equation (2) will be referred to as the Log-t Distribution (LTD). We denote this distribution as U $\sim$ LTD (μ, σ, k). We shall proof in subsequent section that when $k = 1$ Equation (2) reduces to log-Cauchy distribution and approaches log-normal distribution as the degree of freedom parameter k grows large. The plot of the density function (2) is given in figure 1 for some values of the degree of freedom.

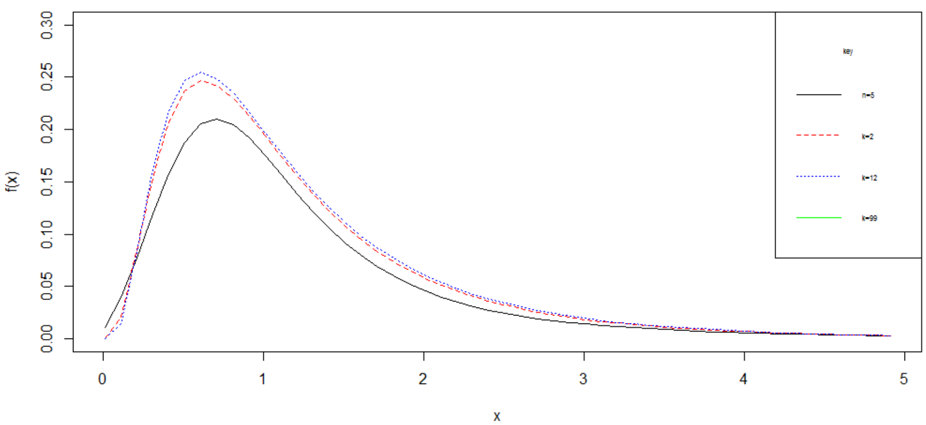

FIGURE 1: Log-T Distribution density plot for n=5, and k taking the values 2, 12 and 99.

Proposition 2. *Let U be a random variable having log-t distribution given in Equation (2). Then the cumulative distribution function (CDF), the survival and the hazard function are respectively;*

$$F(a;k) = \frac{1}{2} + \frac{1}{2} R_a\left(\frac{1}{2}, \frac{k}{2}\right); \tag{3}$$

$$S(t) = 1 - F(a) = \frac{1}{2} - \frac{1}{2} R\left(a; \frac{1}{2}, \frac{k}{2}\right) \tag{4}$$

and

$$h(t) = \frac{f(t)}{1 - F(a)} = \frac{t^{-\frac{1}{2}}[1+t]^{-\left(\frac{k+1}{2}\right)}}{B(\frac{1}{2}, \frac{k}{2})[1 - R(a; \frac{1}{2}, \frac{k}{2})]} \tag{5}$$

Where $B_a(\lambda, \tau)$ is the incomplete beta function and $R(a; \lambda, \tau)$ is the regularized incomplete beta function.

Proof. Let $t = \frac{(\ln(u)-\mu)^2}{k\sigma^2}$ in Equation (2) then $(\ln(u) - \mu)^2 = tk\sigma^2$ which implies that

$$u = e^{(\mu + \sigma\sqrt{tk})}$$

and

$$du = \frac{\sigma}{2}\sqrt{\frac{k}{t}} e^{(\mu+\sigma\sqrt{tk})} dt$$

The density function (2) can be written as

$$f(t;\mu,\sigma^2,k)=\frac{\Gamma(\frac{k+1}{2})[1+t]^{-\left(\frac{k+1}{2}\right)}}{e^{(\mu+\sigma\sqrt{tk})}\Gamma(\frac{k}{2})\sigma\sqrt{k\pi}}\frac{\sigma}{2}\sqrt{\frac{k}{t}}e^{(\mu+\sigma\sqrt{tk})}dt. \tag{6}$$

Equation (6) can be written as

$$f(t;k)=\frac{1}{2B(\frac{1}{2},\frac{k}{2})}\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{k+1}{2}}};\quad t>0 \tag{7}$$

where $B(.)$ is the beta function defined by $B(\lambda,\tau)=\frac{\Gamma(\lambda)\Gamma(\tau)}{\Gamma(\lambda+\tau)}$, $\lambda,\tau\in\mathbb{R}^+$ and $k>0$ is the degree of freedom parameter. □

Since the log-t distribution is a special case of the generalized beta distribution of the second kind (GB2) Hence, the CDF of the log-t distribution given as:

$$F(b;k)=\frac{1}{2B(\frac{1}{2},\frac{k}{2})}\int_0^b\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{k+1}{2}}}dt \tag{8}$$

where $b=\frac{a}{1-a}$, $t>0$, $k>0$

$$F(a;k)=\frac{1}{2B(\frac{1}{2},\frac{k}{2})}.B_a\left(\frac{1}{2},\frac{k}{2}\right) \tag{9}$$

$$F(a;k)=\frac{1}{2}+\frac{1}{2}R_a\left(\frac{1}{2},\frac{k}{2}\right) \tag{10}$$

where $B_a(\frac{1}{2},\frac{k}{2})$ is the incomplete beta function and $R_a\left(\frac{1}{2},\frac{k}{2}\right)$ is the regularized incomplete beta function.

The survival and the hazard follow by substituting (2) and (10) in each definition. Hence

$$S(t)=1-F(a)=\frac{1}{2}-\frac{1}{2}R\left(a;\frac{1}{2},\frac{k}{2}\right), \tag{11}$$

and

$$h(t)=\frac{f(t)}{1-F(a)}=\frac{f(t)}{S(t)}=\frac{t^{-\frac{1}{2}}[1+t]^{-\left(\frac{k+1}{2}\right)}}{B(\frac{1}{2},\frac{k}{2})[1-R(a;\frac{1}{2},\frac{k}{2})]} \tag{12}$$

Some of the mathematical properties of the log-t distribution are presented below.

2.1. The Moment

To investigate the finite moment of the log-t distribution, let $u\sim LTD(k)$ for $r=1,2,3,\ldots$ The r^{th} non-central moments is given as

$$\mu_r=E[u^r]=\int_0^\infty u^r f(u)du$$

For $r = 1$, the first moment is given as

$$E[u] = \int_0^\infty u \frac{\Gamma(\frac{k+1}{2}) \left[1 + \frac{(\ln(u))^2}{k}\right]^{-\left(\frac{k+1}{2}\right)}}{u\Gamma(\frac{k}{2})\sigma\sqrt{k\pi}} du \tag{13}$$

$$= \lim_{a\to\infty} \int_0^a \frac{\Gamma(\frac{k+1}{2}) \left[1 + \frac{(\ln(u))^2}{k}\right]^{-\left(\frac{k+1}{2}\right)}}{\Gamma(\frac{k}{2})\sigma\sqrt{k\pi}} du \tag{14}$$

$$= \frac{1}{\sqrt{k}B(\frac{1}{2}, \frac{k}{2})} \lim_{a\to\infty} \int_0^a \left[1 + \frac{(\ln u)^2}{k}\right]^{-\left(\frac{k+1}{2}\right)} du \tag{15}$$

The approximate solution of the integral is obtained by the series expansion given as

$$\left[1 + \frac{(\ln(u))^2}{k}\right]^{-\left(\frac{k+1}{2}\right)} = 1 + \left[-\left(\frac{k+1}{2}\right)\right] \left[\frac{(\ln x)^2}{k}\right] + \cdots \tag{16}$$

$$= 1 - \frac{(k+1)(\ln(u))^2}{2k} + \cdots$$

Integrating the result with respect to u gives,

$$\lim_{u\to\infty} \left[u - \frac{k+1}{2k}(2u - 2u\ln(u) + u(\ln(u))^2)\right] = \infty \tag{17}$$

Hence, $E[u] = \infty$

Remark: Since the mean of the log-t distribution is infinite, thus higher moments of the log-t distribution do not exist.

3. The Shannon Entropy

An entropy provides a superior tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution (population). A large value of entropy implies greater uncertainty in the data. The two most popular entropies are Renyi (1960) and **?** entropies measures. The Renyi entropy of a random variable U with density function $f(u)$ is defined by

$$I_R = \frac{1}{1-\alpha} \log(\int_0^\infty f^\alpha(u) du); \quad \alpha > 0, \quad \alpha \neq 1. \tag{18}$$

While the Shannon entropy of a random variable U is defined by:

$$h_u = E[-\ln(f_u(u))] = -\int_S f_u(u) \ln(f_u(u)) du$$

where $S = \{u : f_u(u) > 0\}$ It is a special case of Renyi entropy as $\alpha \uparrow 1$.

Now, from the transformation that resulted to equation (8), the Shannon entropy of the log-t distribution is given by

$$h_t = -\int_0^{\infty} f(t;\mu,\sigma^2,k)\ln f(t;\mu,\sigma^2,k)dt \tag{19}$$

to have

$$h(k) = -\frac{1}{2B(\frac{1}{2},\frac{k}{2})}\int_0^{\infty}\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{k+1}{2}}}\ln\left[\frac{t^{-\frac{1}{2}}}{2B(\frac{1}{2},\frac{k}{2})(1+t)^{\frac{k+1}{2}}}\right]dt \tag{20}$$

$$= -\ln\left[2B(\frac{1}{2},\frac{k}{2})\right]\int_0^{\infty}\frac{1}{2B(\frac{1}{2},\frac{k}{2})}\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{k+1}{2}}}dt + \frac{1}{4B(\frac{1}{2},\frac{k}{2})}\int_0^{\infty}\ln(t)\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{k+1}{2}}}dt$$
$$+\frac{k+1}{4B(\frac{1}{2},\frac{k}{2})}\int_0^{\infty}\ln(1+t)\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{k+1}{2}}}dt \tag{21}$$

the first integral gives by the property of PDF over entire domain gives

$$= \ln\left[2B(\frac{1}{2},\frac{k}{2})\right] +$$
$$\frac{k+1}{4B(\frac{1}{2},\frac{k}{2})}\left[\int_0^{\infty}\ln(t)t^{-\frac{1}{2}}(1+t)^{-\left(\frac{k+1}{2}\right)}dt + (k+1)\int_0^{\infty}\ln(1+t)t^{-\frac{1}{2}}(1+t)^{-\left(\frac{k+1}{2}\right)}dt\right] \tag{22}$$

setting $y = 1+t$, $t = y-1$, $dt = dy$ $1 < y < \infty$ in the third integral, we have

$$\int_1^{\infty}\frac{(y-1)^{-\frac{1}{2}}\ln(y)}{y^{(\frac{k+1}{2})}}dy \tag{23}$$

From Gradshteyn & Ryzhik (1965), formula 4.255(1) pp. 541.

$$\int_u^{\infty}\frac{(y-u)^{\theta-1}\ln(y)}{y^{\lambda}}dy = u^{\theta-\lambda}B(\lambda-\theta,\theta)[\ln(u)+\psi(\lambda)-\psi(\lambda-\theta)] \tag{24}$$

provided $0 < Re(\theta) < Re(\lambda)$, $B(\cdot,\cdot)$ is the beta function and $\psi(\cdot)$ is the digamma function. Hence,

$$\int_1^{\infty}\frac{(y-1)^{-\frac{1}{2}}\ln(y)}{y^{(\frac{k+1}{2})}}dy = B(\frac{k}{2})\left[\psi(\frac{k+1}{2})-\psi(\frac{k}{2})\right] \tag{25}$$

where $u = 1$, $\theta = \frac{1}{2}$, $\lambda = \frac{k+1}{2}$ and Gradshteyn & Ryzhik (1965), formula 4.253(1) pp 538 gives

$$\int_0^1 y^{\theta-1}(1-y)^{\lambda}\ln(y)dy = \frac{1}{s^2}B\left(\frac{\theta}{s},\lambda\right)\left[\psi(\frac{\theta}{s})-\psi(\frac{\theta}{s}+\lambda)\right] \tag{26}$$

$Re(\theta), Re(\lambda), s > 0$.

Hence the first integral gives

$$\int_0^\infty \ln(t)t^{-\frac{1}{2}}(1+t)^{-(\frac{k+1}{2})}dt = B\left(\frac{1}{2},\frac{k}{2}\right)\left[\psi(\frac{1}{2})-\psi(\frac{k+1}{2})\right] \tag{27}$$

where $\theta = \frac{1}{2}$, $\lambda = \frac{k}{2}$ and $s = 1$ The resultant equation gives

$$= \ln(2)+\ln B\left(\frac{1}{2},\frac{k}{2}\right)+\frac{1}{4}\psi(\frac{1}{2})-\frac{1}{4}\psi(\frac{k+1}{2})+\frac{(k+1)B(\frac{k}{2})}{4B\left(\frac{1}{2},\frac{k}{2}\right)}\psi(\frac{k+1}{2})-\frac{(k+1)B(\frac{k}{2})}{4B\left(\frac{1}{2},\frac{k}{2}\right)}\psi(\frac{k}{2}) \tag{28}$$

Hence, the Shannon entropy of the log-t distribution is given as

$$log_2\left\{\frac{2B\left(\frac{1}{2},\frac{k}{2}\right)e^{\frac{1}{4}\psi(\frac{1}{2})}e^{\frac{(k+1)B(\frac{k}{2})}{4B\left(\frac{1}{2},\frac{k}{2}\right)}\psi(\frac{k+1}{2})}}{e^{\frac{1}{4}\psi(\frac{1}{2})}e^{\frac{(k+1)B(\frac{k}{2})}{4B\left(\frac{1}{2},\frac{k}{2}\right)}\psi(\frac{k}{2})}}\right\}bits \tag{29}$$

4. Log-concavity and Monotonicity

Bagnoli & Bergstrom (2005) defined the log-concavity of twice-differentiable real-valued function, g whose domain is an interval on the extended real line as a function that satisfies the condition: $(\ln g(x))'' < 0$. Log-concavities of distributions have important properties in modelling. Based on this property, the log-concavity of the log-t distribution is presented in proposition 3.3.1.

Proposition 3. *The Log-t distribution with probability density function, $f(u)$ is either log-concave or log-convex on its entire domain. It depends on the values of the random variable U and the degree of freedom k.*

Proof. Given the density in equation (2), without loss of generality when $\mu = 0$ and $\sigma = 1$, the natural logarithm is given as

$$\ln f(u) = \ln\left[\Gamma\left(\frac{k+1}{2}\right]\right) - \ln\left[\Gamma(\frac{k}{2})\right] - \ln(\sqrt{k\pi}) - \left(\frac{k+1}{2}\right)\ln\left[\frac{k+(\ln(u))^2}{k}\right] - \ln(u) \tag{30}$$

The first order derivative of equation (30) with respect to u gives

$$\frac{d\ln f(u)}{du} = -\left(\frac{k+1}{2}\right)\left[\frac{2\ln(u)}{u[k+(\ln(u))^2]}\right] - \frac{1}{u} \tag{31}$$

The second order derivative of equation (30) with respect to u gives

$$\frac{d^2\ln f(u)}{du^2} = -\left(\frac{k+1}{2}\right)\left[\frac{2k(1-\ln(u))-2(\ln(u))^2-2(\ln(u))^3}{[u+k+(\ln(u))^2]^2}\right] + \frac{1}{u^2} \tag{32}$$

Hence, from equation (32), the log-concavity and log-convexity of $\frac{d^2[\ln f(u)]}{du^2}$ over positive real line depends on the values of random variable U and degree of freedom k. For example, suppose $u \in (0, 1.7]$ and $k > 1$ then, $[\ln f(u)]'' < 0$, this implies log-concavity and the log-t density function is monotonically increasing on the real line because it is easy to see that $f(u+\epsilon) - f(u) > 0$ for any $k > 0$. The case is reversed when on the interval $(1.7, \infty)$ for $k > 1$, it is log-convex and monotonically decreasing. Therefore, we conclude that equation (32) is neither strictly log-concave nor log-convex on its entire domain. □

5. The Special Cases of The Log-t Distribution

Proposition 4. *Suppose U is a random variable having equation (2) as its density function then;*

1. *for $k = 1$, we have the log-Cauchy distribution; and*
2. *as $k \to \infty$ we have the lognormal distribution*

Proof. The proof is trivial, by substituting $k = 1$ into the density function of log-t distribution in equation (2) and noting

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

We have,

$$f(u; \mu, \sigma) = \frac{1}{u\pi}\left[\frac{\sigma}{\sigma^2 + (\ln(u) - \mu)^2}\right]; \quad u > 0 \tag{33}$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ which is the density function of the log-Cauchy distribution. For the second part and without loss of generality from equation (2), let $\mu = 0$ and $\sigma = 1$. (2) becomes the standardized probability density function of the log-t distribution given by

$$f(u; k) = \frac{\Gamma(\frac{k+1}{2})\left[1 + \frac{(\ln(u))^2}{k}\right]^{-\left(\frac{k+1}{2}\right)}}{u\Gamma(\frac{k}{2})\sqrt{k\pi}}; 0 < u < \infty \tag{34}$$

Let $\frac{1}{p} = \frac{(\ln(u))^2}{k}$ and $k = p(\ln(u))^2$. The principal part of the numerator can be written as

$$\left[1 + \frac{(\ln(u))^2}{k}\right]^{-\left(\frac{k+1}{2}\right)} = \left[1 + \frac{1}{p}\right]^{-\left(\frac{p(\ln(u))^2+1}{2}\right)}$$

$$= \left[1 + \frac{1}{p}\right]^{-\frac{p(\ln(u))^2}{2}} \times \left[1 + \frac{1}{p}\right]^{-\frac{1}{2}}$$

Recall, $\lim_{p\to\infty}\left[1 + \frac{1}{p}\right]^p = e$

$$= \lim_{p\to\infty}\left[1 + \frac{1}{p}\right]^{p\frac{-(\ln(u))^2}{2}} \times \lim_{p\to\infty}\left[1 + \frac{1}{p}\right]^{\frac{-1}{2}}$$

which gives, $e^{\frac{-(\ln(u))^2}{2}} \times 1$. Taking the other part of (34)

$$\frac{\Gamma(\frac{k+1}{2})}{u\Gamma(\frac{k}{2})\sqrt{k\pi}} = \frac{\Gamma(\frac{k+1}{2})}{u\Gamma(\frac{k}{2})\sqrt{\frac{k2\pi}{2}}} = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{\frac{k}{2}}}\frac{1}{u\sqrt{2\pi}}$$

To show that $\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{\frac{k}{2}}} = 1$we apply the asymptotic formula given by

$$\Gamma(\lambda z + \tau) \sim \sqrt{2\pi} e^{-\lambda z} (\lambda z)^{\lambda z + \tau - \frac{1}{2}} \tag{35}$$

where $\lambda = \frac{1}{2}$, $\tau = \frac{1}{2}$ and $z = k$ respectively in the numerator while $\lambda = \frac{1}{2}$, $\tau = 0$ and $z = k$ respectively in the denominator.

Substituting this values we have

$$\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{\frac{k}{2}}} \times \frac{1}{u\sqrt{2\pi}} \sim \frac{\sqrt{2\pi} e^{-\frac{k}{2}} \frac{k}{2}^{\frac{k}{2}}}{\sqrt{2\pi} e^{-\frac{k}{2}} \frac{k}{2}^{\frac{k}{2} - \frac{1}{2} + \frac{1}{2}}} \times \frac{1}{u\sqrt{2\pi}}$$

$$\sim 1 \times \frac{1}{u\sqrt{2\pi}}$$

Combining these results we obtain the standardized lognormal distribution defined by

$$f(u) = \frac{1}{u\sqrt{2\pi}} e^{-\frac{1}{2}(\ln(u))^2}; 0 < u < \infty \tag{36}$$

Hence, the standardized log-t distribution generalizes the standardized lognormal distribution as the degree of freedom parameter $k \to \infty$. □

6. Maximum Likelihood Estimate of the Parameters

Given a random sample of n observations $u_1, u_2, \dots, u_n$ from log-t distribution, the likelihood function is given as:

$$L(u_i; \mu, \sigma, k) = \prod_{i=1}^{n} f(u_i; \mu, \sigma, k), \tag{37}$$

The log-likelihood function gives

$$\begin{aligned} \ell(u_i; \mu, \sigma, k) = {} & n \ln \Gamma\left(\frac{k+1}{2}\right) - n \ln \Gamma\left(\frac{k}{2}\right) - \sum_{i=1}^{n} \ln(u_i) \\ & - n \ln(\sigma) - \frac{n}{2} \ln(k) - \frac{n}{2} \ln(\pi) \\ & - \left(\frac{k+1}{2}\right) \sum_{i=1}^{n} \ln\left[\frac{k\sigma^2 + (\ln(u_i) - \mu)^2}{k\sigma^2}\right] \end{aligned} \tag{38}$$

Differentiating equation (38) with respect to each parameter to have

$$\frac{\partial}{\partial \mu} \ell(u_i; \mu, \sigma, k) = (k+1) \sum_{i=1}^{n} \left[\frac{\ln(u_i) - \mu}{k\sigma^2 + (\ln(u_i) - \mu)^2}\right] \tag{39}$$

$$\frac{\partial}{\partial \sigma}\ell(u_i;\mu,\sigma,k) = \frac{-n}{\sigma} + \left(\frac{k+1}{\sigma}\right)\sum_{i=1}^{n}\left[\frac{(\ln(u_i)-\mu)^2}{k\sigma^2+(\ln(u_i)-\mu)^2}\right] \tag{40}$$

$$\frac{\partial}{\partial k}\ell(u_i;\mu,\sigma,k) = \frac{n}{2}\left[\psi\left(\frac{k+1}{2}\right)-\frac{1}{k}\right] + \left(\frac{k+1}{2k}\right)\sum_{i=1}^{n}\left[\frac{(\ln(u_i)-\mu)^2}{k\sigma^2+(\ln(u_i)-\mu)^2}\right] - \frac{1}{2}\sum_{i=1}^{n}\ln\left[\frac{k\sigma^2+(\ln(u_i)-\mu)^2}{k\sigma^2}\right] \tag{41}$$

where $\frac{\partial \ln\Gamma(\frac{k+1}{2})}{\partial k} = \frac{1}{2}\psi\left(\frac{k+1}{2}\right)$ and $\frac{\partial \ln\Gamma(\frac{k}{2})}{\partial k} = \frac{1}{2}\psi\left(\frac{k}{2}\right)$ $\psi(,)$ is the digamma function.

Setting equations (39), (40) and (41) equals zero and solving simultaneously does not give a closed form solution. Hence, we adopt a numerical approach (Newton-Raphson Method) which gives the approximate value for each parameter estimated from the sample data.

7. Information Matrix and Asymptotic Confidence Interval

In statistical inference, the inverse of the Fishers information matrix is often used to construct the confidence interval and in testing hypotheses.

For the asymptotic inference of the parameter space $\mathbf{\Phi} = (\mu, \sigma, \mathbf{k})$, the Fisher information matrix $I(\mathbf{\Phi})$ is required such that its inverse is known to be the asymptotic variance matrix of the maximum likelihood estimators.

The Fisher information matrix for the log-t distribution is presented in the proposition given below

Proposition 5. *Given a random variable, u that follows log-t distribution, let* $\mathbf{\Phi}$ *be the parameter space* μ, σ *and* k, *then the second order partial derivatives of the log-likelihood function form the elements of the Fisher information matrix*

$$I(\mathbf{\Phi}) = -\begin{bmatrix} I_{\mu\mu} & I_{\mu\sigma} & I_{\mu k} \\ I_{\sigma\mu} & I_{\sigma\sigma} & I_{\sigma k} \\ I_{k\mu} & I_{k\sigma} & I_{kk} \end{bmatrix}$$

Proof. The elements of $I(\mathbf{\Phi})$ are

$$I_{\mu\mu} = \frac{\partial^2 l*}{\partial\hat{\mu}^2} = (k+1)\sum_{i=1}^{n}\left\{\frac{-k\sigma^2+[\ln(u_i)-\hat{\mu}]^2}{[k\sigma^2+[\ln(u_i)-\hat{\mu}]^2]^2}\right\} \tag{42}$$

$$I_{\mu\sigma} = \frac{\partial^2 l*}{\partial\hat{\mu}\partial\hat{\sigma}} = \frac{\partial^2 l*}{\partial\hat{\sigma}\partial\hat{\mu}} = (k+1)\sum_{i=1}^{n}\left\{\frac{-2k\hat{\sigma}[\ln(u_i)-\hat{\mu}]}{[k\hat{\sigma}^2+[\ln(u_i)-\hat{\mu}]^2]^2}\right\} \tag{43}$$

$$I_{\mu k} = \frac{\partial^2 l*}{\partial\hat{\mu}\partial\hat{k}} = \frac{\partial^2 l*}{\partial\hat{k}\partial\hat{\mu}} = (k+1)\sum_{i=1}^{n}\left\{\frac{-\sigma^2[\ln(u_i)-\hat{\mu}]}{\hat{k}\sigma^2+[\ln(u_i)-\hat{\mu}]^2}\right\} + \sum_{i=1}^{n}\left\{\frac{[\ln(u_i)-\hat{\mu}]}{\hat{k}\sigma^2+[\ln(u_i)-\hat{\mu}]^2}\right\} \quad (44)$$

$$I_{\sigma\sigma} = \frac{\partial^2 l*}{\partial\hat{\sigma}^2} = \left(\frac{k+1}{\hat{\sigma}}\right)\sum_{i=1}^{n}\left\{\frac{-2k\hat{\sigma}}{[k\hat{\sigma}^2+[\ln(u_i)-\hat{\mu}]^2]^2}\right\} - \left(\frac{k+1}{\hat{\sigma}^2}\right)\sum_{i=1}^{n}\left\{\frac{[\ln(u_i)-\hat{\mu}]}{\hat{k}\sigma^2+[\ln(u_i)-\hat{\mu}]^2}\right\} + \frac{n}{\hat{\sigma}^2} \quad (45)$$

$$I_{\sigma k} = \frac{\partial^2 l*}{\partial\hat{\sigma}\partial\hat{k}} = \frac{\partial^2 l*}{\partial\hat{k}\partial\hat{\sigma}} = -\left(\frac{k+1}{\hat{\sigma}}\right)\sum_{i=1}^{n}\left\{\frac{[\ln(u_i)-\hat{\mu}]^2}{[k\hat{\sigma}^2+[\ln(u_i)-\hat{\mu}]^2]^2}\right\} + \frac{1}{\hat{\sigma}}\sum_{i=1}^{n}\left\{\frac{[\ln(u_i)-\hat{\mu}]^2}{[k\hat{\sigma}^2+[\ln(u_i)-\hat{\mu}]^2]}\right\} \quad (46)$$

$$\begin{aligned} I_{kk} = \frac{\partial^2 l*}{\partial\hat{k}^2} &= \frac{n}{4}\left[\Psi'\left(\frac{\hat{k}+1}{2}\right) - \Psi'\left(\frac{\hat{k}}{2}\right) + \frac{2n}{\hat{k}^2}\right] - \left(\frac{k+1}{2\hat{k}}\right)\sum_{i=1}^{n}\left\{\frac{\hat{\sigma}[\ln(u_i)-\hat{\mu}]^2}{[k\hat{\sigma}^2+[\ln(u_i)-\hat{\mu}]^2]^2}\right\} \\ &- \frac{1}{2\hat{k}^2}\sum_{i=1}^{n}\left\{\frac{[\ln(u_i)-\hat{\mu}]^2}{[k\hat{\sigma}^2+[\ln(u_i)-\hat{\mu}]^2]}\right\} + \frac{1}{2\hat{k}}\sum_{i=1}^{n}\left\{\frac{[\ln(u_i)-\hat{\mu}]^2}{[k\hat{\sigma}^2+[\ln(u_i)-\hat{\mu}]^2]}\right\} \end{aligned} \quad (47)$$

Where $\Psi^{'}(\cdot)$ is the trigamma function. □

Consequently, let parameter vector $\mathbf{\Phi} = (\mu, \sigma, \mathbf{k})$ and the corresponding maximum likelihood estimate of ϕ as $\hat{\phi} = (\hat{\mu}, \hat{\sigma}, \hat{k})$, the asymptotic normality results can be written as

$$(\hat{\phi} - \phi) \to N_3(0, (I(\phi))^{-1}) \quad (48)$$

where $I(\phi)$ is the Fishers information matrix. Therefore, under certain regularity conditions of asymptotic properties of the maximum likelihood estimation ensure that

$$\sqrt{n}(\hat{\mathbf{\Phi}} - \mathbf{\Phi}) \to^{\mathbf{d}} \mathbf{N_3}(\mathbf{0}, \mathbf{I}(\mathbf{\Phi})^{\mathbf{-1}})$$

where $\to^{d}$ means the convergence in distribution, with mean $\mathbf{0} = (\mathbf{0}, \mathbf{0}, \mathbf{0},)^{\mathbf{T}}$ and $3X3$ variance covariance matrix $\mathbf{I}(\mathbf{\Phi})^{\mathbf{-1}}$.

Hence, the $100(1-\alpha)$ confidence interval for $\mathbf{\Phi} \equiv (\mu, \sigma, \mathbf{k})$ becomes:

$$\hat{\mathbf{\Phi}} \pm \mathbf{C}_{\frac{\alpha}{2}}\sqrt{\mathbf{var}(\hat{\mathbf{\Phi}})}$$

where $C_{\frac{\alpha}{2}}$ is the standard normal at the significance level $\frac{\alpha}{2}$ and $var(\cdot)$'s denote the diagonal elements of $\mathbf{I}(\mathbf{\Phi})^{\mathbf{-1}}$ corresponding to the model's parameters.

8. The Best Critical Region for the Mean μ Parameter

The best critical region is obtained by the Neymann Pearson lemma defined as follows:

Neyman-Pearson Lemma: Let $u_1, u_2, \ldots, u_n$ be a random sample from $f(u, \theta)$ where θ is one of the known values θ_0 and θ_1. Let $0 < \alpha < 1$ be fixed, q is a positive constant and A is a subset $\forall\ x \in \alpha$ that satisfy

(i) $Pr\left[u_1, u_2, \ldots, u_n \in A|H_0\right] = \alpha$

(ii) $\frac{L(u_1,u_2,\ldots,u_n\in A|H_0)}{L(u_1,u_2,\ldots,u_n\in A|H_1)} \leq q; \quad 0 < q < 1$

The Neyman-Pearson lemma demonstrates that the likelihood ratio test is the most powerful test. The likelihood function of the log-t distribution is given by

$$L(u;\mu,\sigma,k) = \left(\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\right)^n \left(\frac{1}{\sigma\sqrt{k\pi}}\right)^n . \frac{1}{\sum_{i=1}^n u_i} \prod_{i=1}^n \left[1 + \frac{1}{k}\left(\frac{\ln(u_i)-\mu}{\sigma}\right)^2\right]^{-\left(\frac{k+1}{2}\right)} \tag{49}$$

To test simple null hypothesis (H_0) against simple alternative hypothesis (H_1) denoted by

$$H_0 : \mu_0 = 0 \quad vs \quad H_1 : \mu_1 = 1$$

$$\frac{L(u;\mu_0=0)}{L(u;\mu_1=1)} = \frac{\left(\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\right)^n .(k\pi\sigma^2)^{-\frac{n}{2}} . \frac{1}{\sum_{i=1}^n u_i} . \prod_{i=1}^n \left[1 + \frac{1}{k}\left(\frac{\ln(u_i)-\mu_0}{\sigma}\right)^2\right]^{-\left(\frac{k+1}{2}\right)}}{\left(\frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})}\right)^n .(k\pi\sigma^2)^{-\frac{n}{2}} . \frac{1}{\sum_{i=1}^n u_i} . \prod_{i=1}^n \left[1 + \frac{1}{k}\left(\frac{\ln(u_i)-\mu_1}{\sigma}\right)^2\right]^{-\left(\frac{k+1}{2}\right)}} \tag{50}$$

$$= \frac{\prod_{i=1}^n \left[1 + \frac{1}{k}\left(\frac{\ln(u_i)}{\sigma}\right)^2\right]^{-\left(\frac{k+1}{2}\right)}}{\prod_{i=1}^n \left[1 + \frac{1}{k}\left(\frac{\ln(u_i)-1}{\sigma}\right)^2\right]^{-\left(\frac{k+1}{2}\right)}} \leq q; \quad for \quad 0 < q \leq 1 \tag{51}$$

Taking the natural logarithm of both sides of equation (51) to have

$$-\left(\frac{k+1}{2}\right)\sum_{i=1}^n \ln\left[1 + \frac{1}{k}\left(\frac{\ln(u_i)}{\sigma}\right)^2\right] + \left(\frac{k+1}{2}\right)\sum_{i=1}^n \ln\left[1 + \frac{1}{k}\left(\frac{\ln(u_i)-1}{\sigma}\right)^2\right] \leq \ln(q) \tag{52}$$

$$= -\left(\frac{k+1}{2}\right)\sum_{i=1}^n \left[\ln\left[1 + \frac{1}{k}\left(\frac{\ln(u_i)}{\sigma}\right)^2\right] - \ln\left[1 + \frac{1}{k}\left(\frac{\ln(u_i)-1}{\sigma}\right)^2\right]\right] \leq \ln(q) \tag{53}$$

$$= -\left(\frac{k+1}{2}\right)\sum_{i=1}^n \ln\left[\frac{k\sigma^2 + (\ln(u_i))^2}{k\sigma^2} \frac{k\sigma^2}{k\sigma^2 + (\ln(u_i)-1)^2}\right] \leq \ln(q) \tag{54}$$

$$= -\left(\frac{k+1}{2}\right)\sum_{i=1}^n \ln\left[\frac{k\sigma^2 + (\ln(u_i))^2}{k\sigma^2 + (\ln(u_i)-1)^2}\right] \leq \ln(q) \tag{55}$$

The exponential of both sides gives

$$\sum_{i=1}^n \left[\frac{k\sigma^2 + (\ln(u_i))^2}{k\sigma^2 + (\ln(u_i)-1)^2}\right] \geq qe^{-\left(\frac{2}{k+1}\right)} \tag{56}$$

Hence, we have a test whose critical point of acceptance or rejection of H_0 or H_1 is at q.

9. Multivariate Extension of Log-t Distribution

Proposition 6. *A random vector* $\mathbf{w} = (\mathbf{w_1}, \dots, \mathbf{w_p})'$ *with* $p \geq 1$ *has a* p*-dimensional log-t distribution with mean vector* $\mu = (\mu_\mathbf{1}, \dots, \mu_\mathbf{p})'$*, positive definite symmetric matrix* $\mathbf{\Sigma_{pxp}}$ *and degree of freedom parameter* $k \in (0, \infty)$ *if its density is*

$$f_w(\mathbf{w}; \mu, \mathbf{\Sigma}, \mathbf{k}) = \frac{\mathbf{\Gamma}(\frac{\mathbf{k+p}}{\mathbf{2}})\{\mathbf{1} + \frac{\mathbf{1}}{\mathbf{k}}[\ln(\mathbf{w}) - \mu]'\mathbf{\Sigma^{-1}}[\ln(\mathbf{w}) - \mu]\}^{-(\frac{\mathbf{k+p}}{\mathbf{2}})}}{\prod_{\mathbf{i=1}}^{\mathbf{p}} \mathbf{w_i} \mathbf{\Gamma}(\frac{\mathbf{k}}{\mathbf{2}}) \mathbf{k}^{\frac{\mathbf{p}}{\mathbf{2}}} \pi^{\frac{\mathbf{p}}{\mathbf{2}}} |\mathbf{\Sigma}|^{\frac{\mathbf{1}}{\mathbf{2}}}}$$

Proof. From the univariate log-t distribution we derive its multivariate counterpart by some transformations of random variable given as follows: Let $w = \Phi(u)$ where Φ is a smooth and bijective function.

$$P_w(w) = P_u(\Phi^{-1}(w))|detJ_{\Phi^{-1}}(w)|$$

where $J_{\Phi^{-1}}$ is the Jacobian matrix of the inverse transformation.

By definition, if $u = \ln(w) \sim t(\mu, \sigma, k)$ then $w = e^u \sim LTD(\mu, \sigma, k)$ $u = \Phi^{-1}(w) = \ln(w)$, $\frac{d\Phi^{-1}}{dw} = |\frac{1}{w}|$, $J_{\Phi^{-1}}(w) = diag(\frac{1}{w_1}, \dots, \frac{1}{w_n})$.

Since one of the properties of a diagonal matrix is $|D| = \prod_{i=1}^{n} d_i$. then,

$$|detJ_{\Phi^{-1}}(w)| = \prod_{i=1}^{n} w_i^{-1} = \frac{1}{w_1, w_2, \dots, w_n}$$

$$f_w(w) = f_u[\Phi^{-1}(w)].|\frac{d\Phi^{-1}(w)}{dw}|$$

Also, by the Mahalanobis distance approach,

$$\left(\frac{\ln(w) - \mu}{\sigma}\right)^2 = [\ln(w) - \mu]'(\sigma^2)^{-1}[\ln(w) - \mu]$$

$$[\ln(\mathbf{w}) - \mu]'\mathbf{\Sigma^{-1}}[\ln(\mathbf{w}) - \mu]$$

Thus, the density function of the multivariate log-t distribution is given by

$$f(\mathbf{w}; \mu, \mathbf{\Sigma}, \mathbf{k}) = \frac{\mathbf{\Gamma}(\frac{\mathbf{k+p}}{\mathbf{2}})\{\mathbf{1} + \frac{\mathbf{1}}{\mathbf{k}}(\mathbf{z}(\mathbf{w}) - \mu)'\mathbf{\Sigma^{-1}}(\mathbf{z}(\mathbf{w}) - \mu)\}^{-(\frac{\mathbf{k+p}}{\mathbf{2}})}}{\prod_{\mathbf{i=1}}^{\mathbf{p}} \mathbf{w_i} \mathbf{\Gamma}(\frac{\mathbf{k}}{\mathbf{2}}) \mathbf{k}^{\frac{\mathbf{p}}{\mathbf{2}}} \pi^{\frac{\mathbf{p}}{\mathbf{2}}} |\mathbf{\Sigma}|^{\frac{\mathbf{1}}{\mathbf{2}}}} \tag{57}$$

where $p \geq 1$, Σ is positive definite symmetric matrix, $k \in (0, \infty)$, $\mu_j > 0$, $j = 1, 2, 3, \dots, p$ and $\mathbf{z} = \mathbf{z}(\mathbf{w}) = [\ln(\mathbf{w_1}), \dots, \ln(\mathbf{w_p})]'$ and $\mu = [\mu_\mathbf{1}, \dots, \mu_\mathbf{p}]'$

The degree of freedom parameter k is also referred to as the shape parameter because the peakedness of equation (57) may be decrease, preserved or increased by varying k.

The distribution is said to be central if $\mu = \mathbf{0}$; otherwise it is non-central.

If $p = 1, \mu = 0$ and $\Sigma = 1$, then equation (57) is the density function of the univariate log-t distribution with degree of freedom k.

If $k = 1$, then equation (57) is the p-variate log-Cauchy distribution and the limiting of equation (57) as $k \to \infty$ is the joint PDF of the p-variate lognormal distribution with mean vector μ and covariance matrix $\mathbf{\Sigma}$.

□

9.1. Marginal Distributions

Let $\mathbf{w}$ be p-variate log-t distribution with degree of freedom k, mean vector μ, covariance matrix $\mathbf{\Sigma}$. Consider the partition using the notation in equation (57) $\mathbf{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $\mathbf{\Sigma} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$ where $\mathbf{z_1}$ is p_1x1 and Σ_{11} is p_1 x p_1. Then, $\mathbf{z_1}$ has p_1-variate log-t distribution with degree of freedom k, mean vector μ_1, covariance matrix Σ_{11} and with joint PDF given by

$$f(\mathbf{z_1}) = \frac{\mathbf{\Gamma}(\frac{\mathbf{k+p_1}}{\mathbf{2}})}{\mathbf{w_1\Gamma}(\frac{\mathbf{k}}{\mathbf{2}})(\mathbf{k}\pi)^{\frac{\mathbf{p_1}}{\mathbf{2}}}|\mathbf{\Sigma_{11}}|^{\frac{1}{2}}} \mathbf{X} \left[\mathbf{1} + \frac{\mathbf{1}}{\mathbf{k}}(\mathbf{z_1} - \mu_\mathbf{1})'\mathbf{\Sigma_{11}^{-1}}(\mathbf{z_1} - \mu_\mathbf{1})\right]^{-(\frac{\mathbf{k+p_1}}{\mathbf{2}})} \tag{58}$$

Similarly, $\mathbf{z_2}$ also has the $(p - p_1)$-variate log t distribution with degree of freedom k, mean vector $\mu_\mathbf{2}$, covariance matrix $\mathbf{\Sigma_{22}}$ and with joint PDF given by

$$f(\mathbf{z_2}) = \frac{\mathbf{\Gamma}(\frac{\mathbf{k+p-p_1}}{\mathbf{2}})}{\mathbf{w_2\Gamma}(\frac{\mathbf{k}}{\mathbf{2}})(\mathbf{k}\pi)^{\frac{\mathbf{p_1}}{\mathbf{2}}}|\mathbf{\Sigma_{22}}|^{\frac{1}{2}}} \left[\mathbf{1} + \frac{\mathbf{1}}{\mathbf{k}}(\mathbf{z_2} - \mu_\mathbf{2})'\mathbf{\Sigma_{22}^{-1}}(\mathbf{z_2} - \mu_\mathbf{2})\right]^{-(\frac{\mathbf{k+p-p_1}}{\mathbf{2}})} \tag{59}$$

9.2. Conditional Distributions

Considering central log-t such that $\mu = \mathbf{0}$, let $w = w_1 w_2$, $|\Sigma|^{\frac{1}{2}} = |\Sigma_{11}|^{\frac{1}{2}}|\Sigma_{22}|^{\frac{1}{2}}$

$$f(z_2|z_1) = \frac{f(z_1, z_2)}{f(z_1)}$$

$$f(\mathbf{z_2}|\mathbf{z_1}) = \frac{\mathbf{w_1\Gamma}(\frac{\mathbf{k+p}}{\mathbf{2}})}{\mathbf{w}(\mathbf{k}\pi)^{\frac{\mathbf{p_1}}{\mathbf{2}}}\mathbf{\Gamma}(\frac{\mathbf{k+p}}{\mathbf{2}})} \frac{|\mathbf{\Sigma_{11}}|^{\frac{1}{2}}}{|\mathbf{\Sigma}|^{\frac{1}{2}}} \frac{\left[\mathbf{1} + \frac{\mathbf{1}}{\mathbf{k}}\mathbf{z_1'\Sigma_{11}^{-1}z_1}\right]^{-(\frac{\mathbf{k+p_1}}{\mathbf{2}})}}{\left[\mathbf{1} + \frac{\mathbf{1}}{\mathbf{k}}\mathbf{z'\Sigma_{11}^{-1}z}\right]^{-(\frac{\mathbf{k+p}}{\mathbf{2}})}} \tag{60}$$

Since $|\mathbf{\Sigma}| = |\mathbf{\Sigma_{11}}||\mathbf{\Sigma_{22}} - \mathbf{\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}}|$ and

$$\mathbf{z'\Sigma^{-1}z} = \mathbf{z_1'\Sigma_{11}^{-1}z_1} + \mathbf{z_{2.1}'\Sigma_{22.1}^{-1}z_{2.1}}$$

9.3. Log-concavity of Multivariate Log-t Distribution

A non-negative function $\psi : \mathbb{R}^m \to \mathbb{R}$ is log-concave if for all $\mathbf{u,w} \in \mathbb{R}^m$ and $\beta \in (0, 1)$ we have

$$\psi(\beta\mathbf{u} + (1-\beta)\mathbf{w}) \geq [\psi(u)]^\beta[\psi(w)]^{1-\beta}$$

If $\psi(\mathbf{u}) > 0$ for all $\mathbf{u} \in \mathbb{R}^m$ then,

$$\ln\psi(\beta\mathbf{u} + (1-\beta)\mathbf{w}) \geq \beta\ln\psi(\mathbf{u}) + (1-\beta)\ln\psi(\mathbf{w})$$

Suppose $\Psi(\mathbf{u}) = -\ln\psi(\mathbf{u})$ then the definition is equivalent to $\bigtriangledown^2\Psi(\mathbf{u} \geq 0)$ provided $\Psi(\cdot)$ is twice differentiable and the Hessian elements given by

$$\bigtriangledown^2\Psi(\mathbf{u})_{i,j} = \frac{\partial^2\Psi(\mathbf{u})}{\partial u_i u_j}, \quad i, j = 1, \ldots, n$$

exist.

Proposition 7. *Let $\boldsymbol{U} = (U_1, \ldots, U_n)$ be a random vector with density function $\Psi(\boldsymbol{u}) = f(u_1, \ldots, u_n)$ of the multivariate log-t distribution given equation (57) then $\Psi(\boldsymbol{u})$ is neither log-concave nor log-convex in its entire domain.*

***Proof*.** The multivariate log-t distribution is

$$\psi(u;k) = C\frac{\left[1+\frac{(\ln(u))^2}{k}\right]^{-\left(\frac{k+1}{2}\right)}}{u}; \quad u > 0 \tag{61}$$

where $C = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k}{2})\sqrt{k\pi}}$

$$\Psi(u) = -\ln\Psi(u) = -\ln(C) + \left(\frac{k+1}{2}\right)\ln\left[1+\frac{(\ln(u))^2}{k}\right] + \ln(u) \tag{62}$$

The first order partial derivative of equation (62) with respect to u is

$$\Psi'(u) = \frac{(k+1)\ln(u)}{u[k+(\ln(u))^2]} + \frac{1}{u} \tag{63}$$

The second order partial derivative of equation (62) with respect to u is

$$\Psi''(u) = \frac{(k+1)[k - k\ln(u) - (\ln(u))^2 - (\ln(u))^3]}{[uk + u(\ln(u))^2]^2} - \frac{1}{u^2} \tag{64}$$

Thus,

$$\frac{\partial^2\Psi(\mathbf{u})}{\partial u_i u_j} = \frac{(k+p)[k - k\sqrt{z'z} - z'z - z'z\sqrt{z'z}]}{[ku_i + u_i z'z]^2} - \frac{1}{u_i^2} \tag{65}$$

for $i \neq j = 1, \ldots, n$. The elements of the Hessian matrix $\mathbf{A}$ are given by the above second order partial derivatives and the corresponding quadratic form becomes

$$\mathbf{Z'AZ} = \mathbf{Z'}\left[\frac{\boldsymbol{\partial^2\Psi(\mathbf{u})}}{\boldsymbol{\partial}\mathbf{u_i u_j}}\right]\mathbf{Z} \geq \mathbf{0} \tag{66}$$

Therefore, the multivariate log-t density is neither strictly log-concave nor log-convex on its entire domain, but its log-concavity or log-convexity depends on the random vectors U_i and their respective values of degree of freedom k_i. □

9.4. Bivariate Log-t Distribution

Considering the two random variables u_1 and u_2 in terms of their individual parameters: $\mu_{u1} = E(u_1)$, $\mu_{u2} = E(u_2)$, $\sigma_{11} = var(u_1)$, $\sigma_{22} = var(u_2)$, $\rho_{12} = \frac{\sigma_{12}}{\sqrt{\sigma_{11}\sigma_{22}}} = corr(z(u_1), z(u_2))$ and a positive semi-definite 2×2 variance-covariance matrix:

$$\mathbf{\Sigma} = \begin{pmatrix} var(u_1) & cov(u_1, u_2) \\ cov(u_1, u_2) & var(u_1) \end{pmatrix} = \begin{pmatrix} \sigma^2_{u_1} & \rho\sigma_{u_1}\sigma_{u_2} \\ \rho\sigma_{u_1}\sigma_{u_2} & \sigma^2_{u_2} \end{pmatrix}$$

where $\mathbf{z}(\mathbf{u}) = \begin{pmatrix} z(u_1) \\ z(u_2) \end{pmatrix} = \begin{pmatrix} \ln(u_1) \\ \ln(u_2) \end{pmatrix}$.

Then the joint PDF becomes

$$\frac{\Gamma(\frac{k+2}{2})\left\{1 + \frac{1}{k(1-\rho^2)}\left[\left(\frac{z(u_1)-\mu_{u1}}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{z(u_2)-\mu_{u2}}{\sqrt{\sigma_{22}}}\right)^2 - 2\rho\left(\frac{z(u_1)-\mu_{u1}}{\sqrt{\sigma_{11}}}\right)\left(\frac{z(u_2)-\mu_{u2}}{\sqrt{\sigma_{22}}}\right)\right]\right\}^{-\left(\frac{k+2}{2}\right)}}{u_1 u_2 \Gamma(\frac{k}{2})(k\pi)\sqrt{\sigma_{11}\sigma_{22}(1-\rho^2)}} \tag{67}$$

If the random variable u_1 and u_2 are uncorrelated so that $\rho = 0$, the joint density can be written as the product of two univariate log-t densities of the form $f(u_1, u_2) = f(u_1)f(u_2)$ where u_1 and u_2 are independent.

Thus, equation (67) becomes

$$f(u_1, u_2) = \frac{\left\{1 + \frac{1}{k}\left[\left(\frac{z(u_1)-\mu_{u1}}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{z(u_2)-\mu_{u2}}{\sqrt{\sigma_{22}}}\right)^2\right]\right\}^{-\left(\frac{k+2}{2}\right)}}{u_1 u_2 \Gamma(\frac{k}{2})(k\pi)\sqrt{\sigma_{11}\sigma_{22}}} \tag{68}$$

The densities and contour plots for different degrees of freedom parameter k are shown below It is observed that, as the degree of freedom increases the kurtosis of the densities increase. Also, increase in the value of k flattens the contours. The contour plots show the asymmetric nature of the bivariate log-t distribution. One desirable property of this distribution is its adaptivity to both peakedness and flatness in the dataset by varying the value of the degree of freedom k. Therefore, the distribution is flexible enough to capture the heavy-tail behaviour of large datasets.

10. Applications

In this section, we provide applications to two datasets presented by Lee & Wang (2003) to illustrate the performance of the lognormal and log-t distributions. The goodness-of-fit of statistics for these distributions are compared and the maximum likelihood estimations of their parameters are also provided. The log-likelihood and Akaike information criterion (AIC) are compared for the fitted distributions. However, the smaller these values the better the fit.

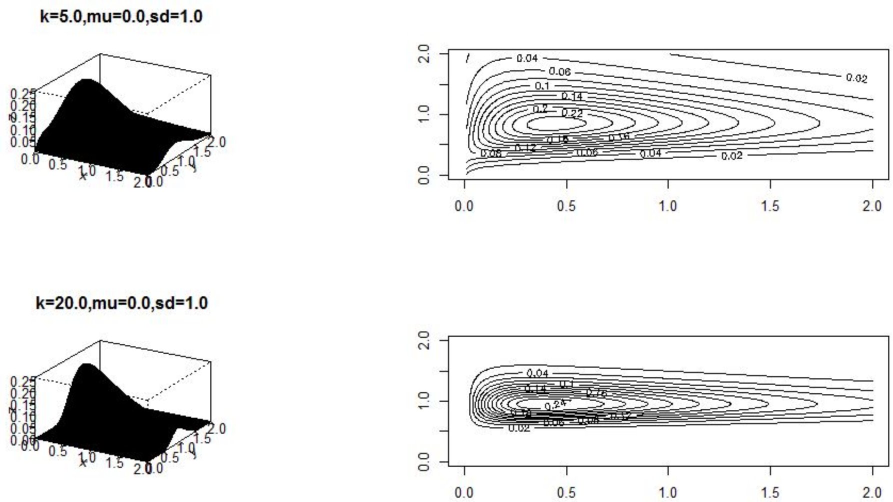

FIGURE 2: Densities and contour plots for the bivariate log-t distribution.

10.1. Remission Times of Bladder Cancer Patients

The first dataset is the remission times (months) of 128 bladder cancer patients as shown below: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 0.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 0.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 0.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 0.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 0.39, 10.34, 14.38, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 0.96, 3.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 0.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 0.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 0.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 0.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 0.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07,0.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 5.49

TABLE 1: Comparison of lognormal distribution with log-t distribution in the analysis of bladder cancer patents data.

Parameter	Lognormal	Log-T
$\hat{\mu}$	1.75345	1.76490
$\hat{\sigma}$	12.16384	0.37002
$\hat{k}$	******	3.80275
Log-likelihood	-228.3571	-228.3563
AIC	460.7142	462.7126

10.2. Remission Times of Acute Leukamia Patients

The second dataset is the remission times (in weeks) of 21 acute leukamia patients as shown below: 1, 1, 2, 2, 3, 4, 4, 5, 5, 6, 8, 8, 9, 10, 10, 12, 14, 16, 20, 24, 34.

Table 2: Comparison of lognormal distribution with log-t distribution in the analysis of acute leukamia patents data.

Parameter	Lognormal	Log-T
$\hat{\mu}$	1.84886	1.84903
$\hat{\sigma}$	4.36282	0.00605
$\hat{k}$	******	3.37290
Log-likelihood	-41.71817	-34.90832
AIC	87.43634	75.81664

10.3. Discussion

The probability density plot shown in Figure(1) reveals how increase in the degree of freedom parameter of the log-t distribution regulates its tails. Moreover, the flexibility and heaviness of its tails accommodate more data. Figure(2) shows the bivariate densities and contour plots. Table 1. shows the comparison of log-normal distribution with log-t distribution on bladder cancer patients data with relatively large sample size ($n = 128$). The comparison was done using the AIC values. Using this data, the maximum Likelihood estimate of parameters of lognormal distribution are $\hat{\mu} = 1.75345$ and $\hat{\sigma} = 12.16384$. While the parameters of the log-t distribution are $\hat{\mu} = 1.76490$, $\hat{\sigma} = 0.37002$ and $\hat{k} = 3.80275$. More so, the AIC values of the lognormal and log-t distributions are 460.7142 and 462.7126 respectively. The lognormal distribution which has the smaller value of AIC is considered to fit the data better. In the same vein,Table 2. shows the comparison of lognormal distribution with log-t distribution on acute leukamia patients data with relatively small sample size ($n = 21$). The comparison was done using the AIC values. Using this data, the maximum Likelihood estimate of parameters of lognormal distribution are $\hat{\mu} = 1.84886$ and $\hat{\sigma} = 4.36282$. While the parameters of the log-t distribution are $\hat{\mu} = 1.84903$, $\hat{\sigma} = 0.00605$ and $\hat{k} = 3.37290$. More so, the AIC values of the lognormal and log-t distributions are 87.43634 and 75.81664 respectively.The log-t distribution which has the smaller value of AIC is considered to fit the data better. Therefore, the log-t distribution outperforms the lognormal distribution for relatively small sample size ($n < 30$).

11. Conclusion

This study examined some inferential statistics of the log-t distribution which has degree of freedom parameter that regulates it tails. It generalizes both log-Cauchy and lognormal distributions. It fits better relatively small sample size data than the lognormal distribution. The multivariate log-t distribution is unique and generalizes the multivariate log-Cauchy and multivariate lognormal diatributions. Therefore, we advocate its application in survival analysis in situations where researchers could not get large enough data to assume lognormality.

[Received: September 2020 — Accepted: December 2021]

References

Bagnoli, M. & Bergstrom, T. (2005), 'Log-concave probability and its applications', *Economic theory* **26**(2), 445–469.

Butt, A. & Habibullah, S. (2016), On the derivation of the sia-log student's t distribution, *in* 'Proceedings of the 2nd International Multi-Disciplinary Conference', Gujrat, Pakistan.

Cassidy, D. T. (2016), 'A multivariate student's t-distribution', *Open Journal of Statistics* **6**(1), 443–450.

Cassidy, D. T., Hamp, M. J. & Ouyed, R. (2013), 'log student's t-distribution-based option sensitivities: greek for the gosset formulae', *Quantitative finance* **13**(1), 1289–1302.

Fisher, R. A. (1925), 'Application of "Students" distribution', *Metron* **5**(1), 90–104.

Gosset, W. S. (1908), 'The probable error of a mean', *Biometrika* **6**(1), 1–25.

Gradshteyn, I. S. & Ryzhik, I. M. (1965), *Tables of Integral Series and Products*, Academic Press, New York.

Hassan, A. & Assar, S. (2016), 'A compound class of lifetime distributions', *Institute of statistical Studies and Research* (doi:10.13140/RG.2.2.24870.93760).

Kibria, B. M. & Joarder, A. H. (2006), 'A short review of multivariate t-distribution', *Journal of Statistical Research* **40**(1), 59–72.

Kotz, S. & Nadarajah, S. (2004), *Multivariate T Distributions and their Applications*, University Press, Cambridge.

Lee, E. T. & Wang, J. W. (2003), *Statistical Methods for Survival Data Analysis*, 3 edn, John Wiley & Sons, Ltd, New York.

Lin, P.-E. (1972), 'Some characterizations of the multivariate t distribution', *Journal of Multivariate Analysis* **2**(3), 339–344.

Mitzenmacher, M. & Tworetzky, B. (2003), New models and methods for file size distributions, *in* 'Proceedings of the Annual Allerton Conference on Communication Control and Computing', Vol. 41, The University; 1998, pp. 603–612.

Renyi, A. (1960), 'On measures of entropy and information', *Berkeley Symposium on Mathematical Statistics and Probability* **1**(1), 547–561.

Saw, S. L., Balasooriya, U. & Tan, K. C. (2002), 'The log-eig distribution: A new probability model for lifetime data', *Communications in Statistics-Theory and Method* **31**(1), 11–23.