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# Finite Population Mixed Models for Pretest-Posttest Designs with Response Errors

## Modelos mixtos para estudios pretest-posttest en poblaciones finitas con error en la respuesta

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### Abstract

We consider a finite population mixed model that accommodates response errors and show how to obtain optimal estimators of the finite population parameters in a pretest-posttest context. We illustrate the method with the estimation of the difference in gain between two interventions and consider a simulation study to compare the empirical version of the proposed estimator (obtained by replacing variance components with estimates) with the estimator obtained via covariance analysis usually employed in such settings. The results indicate that in many instances, the proposed estimator has a smaller mean squared error than that obtained from the standard analysis of covariance model.

***Key words*:** analysis of covariance; BLUP; optimal estimator; random permutation model, .

### Resumen

Se considera un modelo mixto para población finita que tiene en cuenta el error de respuesta y que arroja estimadores óptimos de los parámetros de la población finita, para analizar datos de estudios con estructura del tipo pretest-posttest. Se ilustra el método estimando la diferencia en ganancia entre dos intervenciones y se considera un estudio de simulación para comparar la versión empírica del estimador propuesto (obtenido al reemplazar las componentes de varianza con sus estimativas) con el

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estimador obtenido vía análisis de covarianza, que es usualmente empleado en este tipo de estudios. Los resultados indican que en muchas circunstancias, el estimador propuesto tiene menor error cuadrático medio que el obtenido del análisis estándar usando el modelo de covarianza.

***Palabras clave*:** análisis de covarianza; BLUP; estimador óptimo; modelo de permutación aleatoria.

# 1. Introduction

Pretest-posttest studies are frequently used in areas such as Biology, Medicine, Psychology etc. The objective, in general, is to assess the effects of some kind of intervention (a treatment, for example) on some response distribution. The procedure consists of measuring the response variable before and after an intervention. We motivate our proposal via a study conducted at the Faculty of Medicine of the University of São Paulo, Brazil, designed to evaluate the homogeneous resistance of the respiratory system (HRRS) of mice under two experimental conditions. Each of 29 mice exposed to synthetic air (pretest) had the HRRS measured. A group of 15 mice was selected at random from the 29 and again submitted to synthetic air in the posttest period [control group ($\mathcal{C}$)]. The remaining 14 mice were exposed to a mixture of Helium-Oxygen [treatment group ($\mathcal{T}$)] in the posttest period. All mice had posttest HRRS measured. A profile plot of the data is shown in Figure 1.

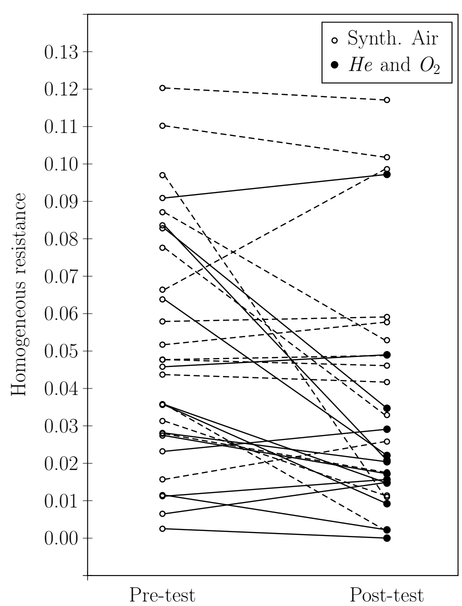


FIGURE 1: Profile plot of homogeneous resistance of the respiratory system (HRRS) of mice.

Statistical analyses of pretest-posttest studies have been addressed by numerous authors, among which we mention Brogan & Kutner (1980), Laird (1983), Stanek III (1988), Knoke (1991), Singer & Andrade (1997), Bonate (2000), Yang & Tsiatis (2001), Leon, Tsiatis & Davidian (2003), Aoki, Achcar, Bolfarine & Singer (2003) and Alencar, Singer & Rocha (2012).

In particular, the analysis of covariance model considered in Laird (1983) is suited for the setup described above. It assumes that the pretest HRRS, $Y_{i1}^*$, $i = 1, \ldots, n$, are fixed and that

$$Y_{i2}^{h*} = \mu_2 + \tau_h + \beta(Y_{i1}^* - \overline{Y}_1) + e_{hi}, \;\; i = 1, \ldots, n_h, \; h = \mathcal{C}, \mathcal{T}, \tag{1}$$

where $\overline{Y}_1 = n^{-1} \sum_{i=1}^{n} Y_{i1}^*$, $\tau_{\mathcal{C}} + \tau_{\mathcal{T}} = 0$, $e_{\mathcal{C}i}$ and $e_{\mathcal{T}i}$ are uncorrelated random measurement errors such that $\mathbb{E}(e_{\mathcal{C}i}) = \mathbb{E}(e_{\mathcal{T}i}) = 0$, $\mathbb{V}(e_{\mathcal{C}i}) = \mathbb{V}(e_{\mathcal{T}i}) = \sigma_e^2$. The term $\mu_2$ corresponds to the average posttest HRRS, $\beta$ represents the common linear regression coefficient relating the posttest HRRS ($Y_{i2}^{h*}$) to the pretest HRRS ($Y_{i1}^*$) and $\tau_{\mathcal{C}}$ ($\tau_{\mathcal{T}}$) corresponds to the effect of the ventilation with $\mathcal{C}$ ($\mathcal{T}$) on the average posttest HRRS.

A scatterplot of the HRRS along with the corresponding least squares regression line, $\widehat{\mathbb{E}(Y_{i2}^{h*})} = \widehat{\mu}_2 + \widehat{\tau}_h + \widehat{\beta}(Y_{i1}^* - \overline{Y}_1)$, $h = \mathcal{C}, \mathcal{T}$, is displayed in Figure 2.

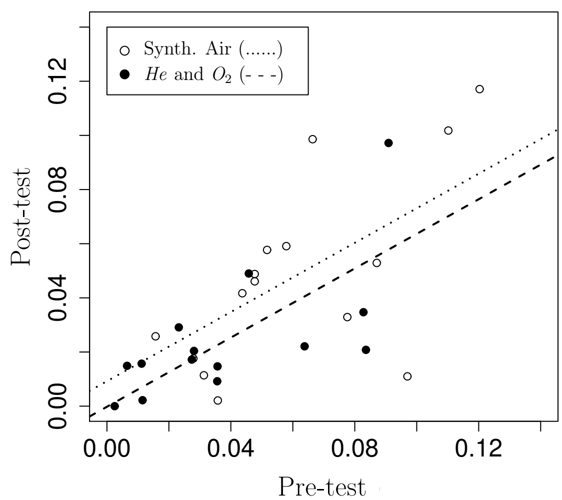


FIGURE 2: Scatter plot for the pretest and posttest measurements and expected curves under model (1).

In many practical situations, the assumptions underlying model (1) may not hold or may be difficult to verify as in Figure 2, where the straight lines generated by model (1) do not fit the data appropriately. To bypass this problem, we follow the ideas of Stanek III & Singer (2004) and propose an alternative model stemming directly from finite population sampling without additional assumptions and thus is design-based. We assume that response error may be present in the observed values.

In Section 2 we describe the finite population mixed model for the analysis of pretest-posttest designs. In Section 3 we present a simulation study designed to compare the performance of the proposed estimators to that of the standard analysis of covariance estimators. Section 4 is dedicated to the analysis of the motivating example. We conclude with a brief discussion in Section 5.

## 2. The Finite Population Mixed Model for Pretest-Posttest Designs With Response Error

We define a finite population as a collection of $N$ identifiable units, labeled $s$, $s = 1, \dots, N$, using a notation similar to that employed in Stanek III, Singer & Lencina (2004). In particular, we assume the existence of three quantities associated to unit $s$, namely, the pretest response $y_{s1}$, the posttest response for unit $s$ exposed to the control treatment, $y_{s2}^{\mathcal{C}}$ and the posttest response for unit $s$ exposed to active treatment, $y_{s2}^{\mathcal{T}}$. Also, we assume that if any unit could be submitted to both treatments, the three values would be observed (possibly) with response error. In practice, unit $s$ is randomly submitted to one treatment, $\mathcal{C}$ or $\mathcal{T}$, and therefore $y_{s2}^{\mathcal{C}}$ and $y_{s2}^{\mathcal{T}}$ are regarded as the *potential responses* of unit $s$ in the posttest. The potential response approach (or potential observation approach) has been considered by different authors such as Leon, Tsiatis & Davidian (2003) under a pretest-posttest context, Rubin (2005) under a different design of experiments context and Pfeffermann (2017) under an observational study context.

We formalize the process of simple random sampling without replacement by introducing a finite population model according to which any permutation of units in the population can be selected with equal probability $(N!)^{-1}$. We assign a new label, $i = 1, \dots, N$, to each unit according to its position in the permutation and represent the pretest and the potential posttest responses by a $N \times 3$ random matrix $[\boldsymbol{Y}_1, \boldsymbol{Y}_2^{\mathcal{C}}, \boldsymbol{Y}_2^{\mathcal{T}}]$ where $\boldsymbol{Y}_1 = (Y_{11}, \dots, Y_{i1}, \dots, Y_{N1})^\top$ and $\boldsymbol{Y}_2^h = (Y_{12}^h, \dots, Y_{i2}^h, \dots, Y_{N2}^h)^\top$, $h = \mathcal{C}, \mathcal{T}$.

To associate the values $\boldsymbol{y}_1 = (y_{11}, \dots, y_{s1}, \dots, y_{N1})^\top$ and $\boldsymbol{y}_2^h = (y_{12}^h, \dots, y_{s2}^h, \dots, y_{N2}^h)^\top$, $h = \mathcal{C}, \mathcal{T}$, to $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2^{\mathcal{C}}$ and $\boldsymbol{Y}_2^{\mathcal{T}}$, we consider

$$[\boldsymbol{Y}_1 \;\; \boldsymbol{Y}_2^{\mathcal{C}} \;\; \boldsymbol{Y}_2^{\mathcal{T}}] = \boldsymbol{U}[\boldsymbol{y}_1 \;\; \boldsymbol{y}_2^{\mathcal{C}} \;\; \boldsymbol{y}_2^{\mathcal{T}}] \tag{2}$$

where $\boldsymbol{U} = [\boldsymbol{U}_1, \dots, \boldsymbol{U}_N]^\top$ is a $N \times N$ matrix with $\boldsymbol{U}_i = (U_{i1}, \dots, U_{iN})^\top$ and $U_{is}$ is an indicator random variable that takes on a value 1 if unit $s$ is selected in position $i$ in the permutation and zero otherwise. Letting the subscript $S$ indicate expectation with respect to permutation of units, we have $\mathbb{E}_S(\boldsymbol{U}) = N^{-1}\boldsymbol{J}_N$ and $\mathbb{V}_S(vec(\boldsymbol{U})) = (N-1)^{-1}(\boldsymbol{P}_N \otimes \boldsymbol{P}_N)$ where $\boldsymbol{J}_a = \boldsymbol{1}_a\boldsymbol{1}_a^\top$, $\boldsymbol{1}_a$ is an $a \times 1$ column vector with all elements equal to 1, $\boldsymbol{P}_{a,b} = \boldsymbol{I}_a - b^{-1}\boldsymbol{J}_a$, $\boldsymbol{P}_a = \boldsymbol{P}_{a,a}$, $\boldsymbol{I}_a$ is an $a \times a$ identity matrix, and *vec* and $\otimes$ respectively denote the *vec* operator and the Kronecker product [see Harville (1997), for example].

To include response error both in the pretest and in the posttest response we consider the model

$$[\boldsymbol{Y}_1^* \;\; \boldsymbol{Y}_2^{\mathcal{C}*} \;\; \boldsymbol{Y}_2^{\mathcal{T}*}] = \boldsymbol{U}[(\boldsymbol{y}_1 \;\; \boldsymbol{y}_2^{\mathcal{C}} \;\; \boldsymbol{y}_2^{\mathcal{T}}) + (\boldsymbol{E}_1 \;\; \boldsymbol{E}_2^{\mathcal{C}} \;\; \boldsymbol{E}_2^{\mathcal{T}})] \tag{3}$$

where $\boldsymbol{Y}_1^* = (Y_{11}^*, \ldots, Y_{i1}^*, \ldots, Y_{N1}^*)^\top$, $\boldsymbol{Y}_2^{h*} = (Y_{12}^{h*}, \ldots, Y_{i2}^{h*}, \ldots, Y_{N2}^{h*})^\top$, $h = \mathcal{C}, \mathcal{T}$ and $[\boldsymbol{E}_1 \;\boldsymbol{E}_2^{\mathcal{C}} \;\boldsymbol{E}_2^{\mathcal{T}}]$ is a random matrix with $\boldsymbol{E}_1 = (E_{11}, \ldots, E_{s1}, \ldots, E_{N1})^\top$ and $\boldsymbol{E}_2^h = (E_{12}^h, \ldots E_{s2}^h, \ldots, E_{N2}^h)^\top$, $h = \mathcal{C}, \mathcal{T}$ for which $\mathbb{E}_R(E_{s1}) = \mathbb{E}_R(E_{s2}^h) = 0$, $\mathbb{V}_R(E_{s1}) = \sigma_{s1}^2$, $\mathbb{V}_R(E_{s2}^h) = (\sigma_{s2}^h)^2$. We also assume that the variables $E_{s1}$, $E_{s2}^{\mathcal{C}}$, $E_{s2}^{\mathcal{T}}$ and $U_{is}$, $s = 1, \ldots, N$, are uncorrelated. The subscript $R$ denotes expectation with respect to the response error distribution.

In particular, for position $i$ in a permutation,

$$\begin{aligned} Y_{i1}^* &= \sum_{s=1}^N U_{is}(y_{s1} + E_{s1}) = \mu_1 + d_{i1} + \sum_{s=1}^N U_{is}E_{s1}, \\ Y_{i2}^{h*} &= \sum_{s=1}^N U_{is}(y_{s2}^h + E_{s2}^h) = \mu_2^h + d_{i2}^h + \sum_{s=1}^N U_{is}E_{s2}^h, \end{aligned} \tag{4}$$

where $d_{i1} = \sum_{s=1}^N U_{is}\delta_{s1}$, with $\delta_{s1} = y_{s1} - \mu_1$, $\mu_1 = N^{-1}\sum_{s=1}^N y_{s1}$, $d_{i2}^h = \sum_{s=1}^N U_{is}\delta_{s2}^h$, with $\delta_{s2}^h = y_{s2}^h - \mu_2^h$ and $\mu_2^h = N^{-1}\sum_{s=1}^N y_{s2}^h$, $h = \mathcal{C}, \mathcal{T}$. Note that $\mu_2^h$ can be reparametrized as $\mu_2^h = \mu_1 + \gamma + \alpha_h$ where $\gamma = (2N)^{-1}\sum_{s=1}^N (y_{s2}^{\mathcal{C}} + y_{s2}^{\mathcal{T}})$ and $\alpha_h = \mu_2^h - \mu_1 - \gamma$.

Then, the mixed model for finite populations pretest-posttest designs with response error may be expressed as

$$\boldsymbol{Z}^* = vec[(\boldsymbol{Y}_1^* \;\; \boldsymbol{Y}_2^{\mathcal{C}*} \;\; \boldsymbol{Y}_2^{\mathcal{T}*})] = (\boldsymbol{I}_3 \otimes \boldsymbol{1}_N)\boldsymbol{\mu} + \boldsymbol{d} + \boldsymbol{E}^* \tag{5}$$

where $\boldsymbol{\mu} = (\mu_1 \;\; \mu_2^{\mathcal{C}} \;\; \mu_2^{\mathcal{T}})^\top$ corresponds to a vector of populational (fixed effects) pretest and posttest means, $\mu_1$, $\mu_2^{\mathcal{C}}$ and $\mu_2^{\mathcal{T}}$, $\boldsymbol{d} = [\boldsymbol{d}_1^\top \;\; (\boldsymbol{d}_2^{\mathcal{C}})^\top \;\; (\boldsymbol{d}_2^{\mathcal{T}})^\top]^\top$, with $\boldsymbol{d}_1 = (d_{11}, \ldots, d_{i1}, \ldots, d_{N1})^\top$ and $\boldsymbol{d}_2^h = (d_{i2}^h, \ldots, d_{i2}^h, \ldots, d_{N2}^h)^\top$, $h = \mathcal{C}, \mathcal{T}$, is a vector of random effects, and $\boldsymbol{E}^* = [\boldsymbol{E}_1^\top \;(\boldsymbol{E}_2^{\mathcal{C}})^\top \;(\boldsymbol{E}_2^{\mathcal{T}})^\top]^\top$, with $\boldsymbol{E}_1 = \boldsymbol{Y}_1^* - \boldsymbol{Y}_1$ and $\boldsymbol{E}_2^h = \boldsymbol{Y}_2^{h*} - \boldsymbol{Y}_2^h$, $h = \mathcal{C}, \mathcal{T}$, denoting the vector of the permuted response errors.

The vector of populational means and covariance matrix of $\boldsymbol{Z}^*$ are given respectively by

$$\mathbb{E}_{SR}(\boldsymbol{Z}^*) = (\boldsymbol{I}_3 \otimes \boldsymbol{1}_N)\boldsymbol{\mu},$$

and

$$\mathbb{V}_{SR}(\boldsymbol{Z}^*) = diag\{\overline{\sigma}_1^2\boldsymbol{I}_N, \; \overline{\sigma}_2^{\mathcal{C}2}\boldsymbol{I}_N, \; \overline{\sigma}_2^{\mathcal{T}2}\boldsymbol{I}_N\} + \boldsymbol{\Sigma} \otimes \boldsymbol{P}_N,$$

where $\boldsymbol{P}_a$ is defined in the sequence to expression (2) and

$$\begin{aligned}\overline{\sigma}_1^2 &= \mathbb{E}_S\left[\sum_{s=1}^{N}\sigma_{s1}^2 U_{is}\right] = N^{-1}\sum_{s=1}^{N}\sigma_{s1}^2,\\ \overline{\sigma}_2^{h2} &= \mathbb{E}_S\left[\sum_{s=1}^{N}(\sigma_{s2}^h)^2 U_{is}\right] = N^{-1}\sum_{s=1}^{N}\left(\sigma_{s2}^h\right)^2,\end{aligned} \tag{6}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix}\sigma_{11} & \sigma_{21}^{\mathcal{C}} & \sigma_{21}^{\mathcal{T}}\\ \sigma_{21}^{\mathcal{C}} & \sigma_{22}^{\mathcal{C}\mathcal{C}} & \sigma_{22}^{\mathcal{C}\mathcal{T}}\\ \sigma_{21}^{\mathcal{T}} & \sigma_{22}^{\mathcal{C}\mathcal{T}} & \sigma_{22}^{\mathcal{T}\mathcal{T}}\end{pmatrix} \tag{7}$$

with $\sigma_{11} = (N-1)^{-1}\sum_{s=1}^{N}(y_{s1}-\mu_1)^2$, $\sigma_{21}^h = (N-1)^{-1}\sum_{s=1}^{N}(y_{s2}^h-\mu_2^h)(y_{s1}-\mu_1)$, $\sigma_{22}^{hh'} = (N-1)^{-1}\sum_{s=1}^{N}(y_{s2}^h-\mu_2^h)(y_{s2}^{h'}-\mu_2^{h'})$, $h=\mathcal{C},\mathcal{T}$, $h'=\mathcal{C},\mathcal{T}$.

**Target random variables and optimal predictors.**

In general, our interest is to estimate (predict) linear combinations of the form

$$T = \sum_{i=1}^{N}\left(c_{i1}^{\mathcal{C}}Y_{i1} + c_{i2}^{\mathcal{C}}Y_{i2}^{\mathcal{C}}\right) + \sum_{i=1}^{N}\left(c_{i1}^{\mathcal{T}}Y_{i1} + c_{i2}^{\mathcal{T}}Y_{i2}^{\mathcal{T}}\right) \tag{8}$$

where the $c_{i1}^{\mathcal{C}}$, $c_{i2}^{\mathcal{C}}$, $c_{i1}^{\mathcal{T}}$, and $c_{i2}^{\mathcal{T}}$ are known constants attached to position $i$ in the permutation. Depending on the choice of $c_{i1}^{\mathcal{C}}$, $c_{i2}^{\mathcal{C}}$, $c_{i1}^{\mathcal{T}}$ and $c_{i2}^{\mathcal{T}}$ the quantity $T$ may represent a parameter or a random effect. Here we are concerned with the estimation of linear combinations of the parameters of model (5), *i.e.*, of the elements of the vector $\boldsymbol{\mu}$.

To specify the average response difference (posttest minus pretest) for the $\mathcal{C}$ intervention,

$$T_{\mathcal{C}} = N^{-1}\sum_{i=1}^{N}\left(Y_{i2}^{\mathcal{C}} - Y_{i1}\right),$$

we use $c_{i2}^{\mathcal{C}} = N^{-1}$, $c_{i1}^{\mathcal{C}} = -N^{-1}$ and $c_{i2}^{\mathcal{T}} = c_{i1}^{\mathcal{T}} = 0$, $i=1,\dots,N$.

To specify the average response difference (posttest minus pretest) for the $\mathcal{T}$ intervention,

$$T_{\mathcal{T}} = N^{-1}\sum_{i=1}^{N}\left(Y_{i2}^{\mathcal{T}} - Y_{i1}\right),$$

we use $c_{i2}^{\mathcal{T}} = N^{-1}$, $c_{i1}^{\mathcal{T}} = -N^{-1}$ and $c_{i2}^{\mathcal{C}} = c_{i1}^{\mathcal{C}} = 0$, $i=1,\dots,N$.

To specify the average response difference between the interventions $\mathcal{C}$ and $\mathcal{T}$,

$$T_D = N^{-1}\sum_{i=1}^{N}\left(Y_{i2}^{\mathcal{C}} - Y_{i1}\right) - N^{-1}\sum_{i=1}^{N}\left(Y_{i2}^{\mathcal{T}} - Y_{i1}\right), \tag{9}$$

we use $c_{i1}^{\mathcal{T}} = c_{i2}^{\mathcal{C}} = N^{-1}$ and $c_{i2}^{\mathcal{T}} = c_{i1}^{\mathcal{C}} = -N^{-1}$, $i=1,\dots,N$.

To estimate/predict $T$ in expression (8), under the finite population mixed model (5) based on a simple random sample obtained without replacement, we assume, without loss of generality, that

i) the first $n$ positions in the random permutation, *i.e.*, $i = 1, \dots, n$, correspond to the selected units in a simple random sample of size $n$ from the finite population,

ii) for the first $n_{\mathcal{C}}$ selected units, *i.e.*, for $i = 1, \dots, n_{\mathcal{C}}$ $(< n)$, we observe $Y_{i1}$ and $Y_{i2}^{\mathcal{C}}$ directly or indirectly (when there is response error), but we do not observe $Y_{i2}^{\mathcal{T}}$, because we suppose that the first $n_{\mathcal{C}}$ units are only exposed to the $\mathcal{C}$ intervention

iii) for the remaining $n - n_{\mathcal{C}}$ $(= n_{\mathcal{T}})$ selected units, *i.e.*, for $i = n_{\mathcal{C}} + 1, \dots, n$, we observe $Y_{i1}$ and $Y_{i2}^{\mathcal{T}}$ directly or indirectly (when there is response error), but we do not observe $Y_{i2}^{\mathcal{C}}$, because we assume that the last $n_{\mathcal{T}}$ selected units are only exposed to the $\mathcal{T}$ intervention.

As a result, $T$ in expression (8) can be rewritten as

$$T = T^{(1)} + T^{(2)} \tag{10}$$

with

$$T^{(1)} = \sum_{i=1}^{n_{\mathcal{C}}} \left( c_{i1}^{\mathcal{C}} Y_{i1} + c_{i2}^{\mathcal{C}} Y_{i2}^{\mathcal{C}} \right) + \sum_{i=n_{\mathcal{C}}+1}^{n} \left( c_{i1}^{\mathcal{T}} Y_{i1} + c_{i2}^{\mathcal{T}} Y_{i2}^{\mathcal{T}} \right)$$

and

$$T^{(2)} = \sum_{i=n_{\mathcal{C}}+1}^{N} \left( c_{i1}^{\mathcal{C}} Y_{i1} + c_{i2}^{\mathcal{C}} Y_{i2}^{\mathcal{C}} \right) + \sum_{i=1}^{n_{\mathcal{C}}} \left( c_{i1}^{\mathcal{T}} Y_{i1} + c_{i2}^{\mathcal{T}} Y_{i2}^{\mathcal{T}} \right) + \sum_{i=n_{\mathcal{C}}+1}^{N} \left( c_{i1}^{\mathcal{T}} Y_{i1} + c_{i2}^{\mathcal{T}} Y_{i2}^{\mathcal{T}} \right).$$

Given that $T^{(1)}$ is observed, to estimate $T$ in expression (8), we must predict $T^{(2)}$ and this may be accomplished via standard methods to obtain minimum variance unbiased linear predictors.

In the first step, we partition $\boldsymbol{Z}^*$ in expression (5) into the sample and the remainder via the pre-multiplication of $\boldsymbol{Z}^*$ by

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_I \\ \boldsymbol{K}_{II} \end{pmatrix} \tag{11}$$

where

$$\boldsymbol{K}_I = \begin{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} & \otimes & \begin{pmatrix} \boldsymbol{I}_{n_{\mathcal{C}}+n_{\mathcal{T}}} & \boldsymbol{0}_{(n_{\mathcal{C}}+n_{\mathcal{T}})\times(N-n_{\mathcal{C}}-n_{\mathcal{T}})} \end{pmatrix} \\ \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} & \otimes & \begin{pmatrix} \boldsymbol{I}_{n_{\mathcal{C}}} & \boldsymbol{0}_{n_{\mathcal{C}}\times(N-n_{\mathcal{C}})} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} & \otimes & \begin{pmatrix} \boldsymbol{0}_{n_{\mathcal{T}}\times n_{\mathcal{C}}} & \boldsymbol{I}_{n_{\mathcal{T}}} & \boldsymbol{0}_{n_{\mathcal{T}}\times(N-n_{\mathcal{C}}-n_{\mathcal{T}})} \end{pmatrix} \end{pmatrix}$$

and

$$\boldsymbol{K}_{II} = \begin{pmatrix} \boldsymbol{I}_3 & \otimes & \left(\mathbf{0}_{(N-n_{\mathcal{C}}-n_{\mathcal{T}})\times(n_{\mathcal{C}}+n_{\mathcal{T}})}\ \boldsymbol{I}_{N-n_{\mathcal{C}}-n_{\mathcal{T}}}\right) \\ \begin{pmatrix}0 & 1 & 0\end{pmatrix} & \otimes & \left(\mathbf{0}_{n_{\mathcal{T}}\times n_{\mathcal{C}}}\ \ \boldsymbol{I}_{n_{\mathcal{T}}}\ \mathbf{0}_{n_{\mathcal{T}}\times(N-n_{\mathcal{C}}-n_{\mathcal{T}})}\right) \\ \begin{pmatrix}0 & 0 & 1\end{pmatrix} & \otimes & \left(\boldsymbol{I}_{n_{\mathcal{C}}}\ \mathbf{0}_{n_{\mathcal{C}}\times(N-n_{\mathcal{C}})}\right) \end{pmatrix}.$$

In the case where $N = n_{\mathcal{C}} + n_{\mathcal{T}}$, the matrices $\boldsymbol{K}_I$ and $\boldsymbol{K}_{II}$ can be written as

$$\boldsymbol{K}_{I,N} = \begin{pmatrix} \begin{pmatrix}1 & 0 & 0\end{pmatrix} & \otimes & \boldsymbol{I}_N \\ \begin{pmatrix}0 & 1 & 0\end{pmatrix} & \otimes & \left(\boldsymbol{I}_{n_{\mathcal{C}}}\ \mathbf{0}_{n_{\mathcal{C}}\times n_{\mathcal{T}}}\right) \\ \begin{pmatrix}0 & 0 & 1\end{pmatrix} & \otimes & \left(\mathbf{0}_{n_{\mathcal{T}}\times n_{\mathcal{C}}}\ \boldsymbol{I}_{n_{\mathcal{T}}}\right) \end{pmatrix}$$

and

$$\boldsymbol{K}_{II,N} = \begin{pmatrix} \begin{pmatrix}0 & 1 & 0\end{pmatrix} & \otimes & \left(\mathbf{0}_{n_{\mathcal{T}}\times n_{\mathcal{C}}}\ \boldsymbol{I}_{n_{\mathcal{T}}}\right) \\ \begin{pmatrix}0 & 0 & 1\end{pmatrix} & \otimes & \left(\boldsymbol{I}_{n_{\mathcal{C}}}\ \mathbf{0}_{n_{\mathcal{C}}\times n_{\mathcal{T}}}\right) \end{pmatrix}.$$

Letting

$$\begin{pmatrix} \boldsymbol{Z}_I^* \\ \boldsymbol{Z}_{II}^* \end{pmatrix} = \begin{pmatrix} \boldsymbol{K}_I \boldsymbol{Z}^* \\ \boldsymbol{K}_{II} \boldsymbol{Z}^* \end{pmatrix},$$

it follows that

$$\boldsymbol{Z}_I^* = [Y_{1,1}^*, \ldots, Y_{(n_{\mathcal{C}}+n_{\mathcal{T}}),1}^*,\ Y_{1,2}^{\mathcal{C}*}, \ldots, Y_{n_{\mathcal{C}},2}^{\mathcal{C}*}, Y_{(n_{\mathcal{C}}+1),2}^{\mathcal{T}*}, \ldots, Y_{(n_{\mathcal{C}}+n_{\mathcal{T}}),2}^{\mathcal{T}*}]^\top$$

and

$$\begin{aligned}\boldsymbol{Z}_{II}^* = [&Y_{(n_{\mathcal{C}}+n_{\mathcal{T}}+1),1}^*, \ldots, Y_{N,1}^*,\ Y_{(n_{\mathcal{C}}+n_{\mathcal{T}}+1),2}^{\mathcal{C}*}, \ldots, Y_{N,2}^{\mathcal{C}*},\ Y_{(n_{\mathcal{C}}+n_{\mathcal{T}}+1),2}^{\mathcal{T}*}, \ldots, Y_{N,2}^{\mathcal{T}*} \\ &Y_{(n_{\mathcal{C}}+1),2}^{\mathcal{C}*}, \ldots, Y_{(n_{\mathcal{C}}+n_{\mathcal{T}}),2}^{\mathcal{C}*}, Y_{1,2}^{\mathcal{T}*},\ \ldots, Y_{n_{\mathcal{C}},2}^{\mathcal{T}*}]^\top.\end{aligned} \tag{12}$$

For the sake of simplicity and without loss of generality, we assume $n_{\mathcal{C}} = n_{\mathcal{T}} = n_0$ and $\sigma_{s1}^2 = \left(\sigma_{s2}^{\mathcal{C}}\right)^2 = \left(\sigma_{s2}^{\mathcal{T}}\right)^2$, $s = 1, \ldots, N$, under model (5). Then

$$\begin{aligned}
\mathbb{E}_{SR}(\boldsymbol{Z}_I^*) &= [\boldsymbol{H}_0 \otimes \mathbf{1}_{n_0}]\boldsymbol{\mu}, \\
\mathbb{E}_{SR}(\boldsymbol{Z}_{II}^*) &= \boldsymbol{H}_1\boldsymbol{\mu}, \\
\mathbb{V}_{SR}(\boldsymbol{Z}_I^*) &= \boldsymbol{V}_I^* = \overline{\sigma}_1^2\boldsymbol{I}_{4n_0} + (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0}) - N^{-1}(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})\boldsymbol{\Sigma}^b(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})^\top, \\
\mathbb{V}_{SR}(\boldsymbol{Z}_{II}^*) &= \boldsymbol{V}_{II}^* = \overline{\sigma}_1^2\boldsymbol{I}_{3N-4n_0} + n_0^{-1}\boldsymbol{H}_2^\top(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})\boldsymbol{\Sigma}^a(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})^\top\boldsymbol{H}_2 \\
&\quad + diag\{\boldsymbol{\Sigma} \otimes \boldsymbol{P}_{N-2n_0}, \begin{pmatrix}\sigma_{22}^{\mathcal{C}\mathcal{C}} & 0 \\ 0 & \sigma_{22}^{\mathcal{T}\mathcal{T}}\end{pmatrix} \otimes \boldsymbol{P}_{n_0}\} - N^{-1}\boldsymbol{H}_1\boldsymbol{\Sigma}\boldsymbol{H}_1^\top \\
&\quad + (N-2n_0)^{-1}\begin{pmatrix}\boldsymbol{I}_3 \otimes \mathbf{1}_{N-2n_0} \\ \mathbf{0}_{2n_0\times 3}\end{pmatrix}\boldsymbol{\Sigma}(\boldsymbol{I}_3 \otimes \mathbf{1}_{N-2n_0}^\top | \mathbf{0}_{3\times 2n_0}), \\
\mathbb{C}\mathrm{ov}_{SR}(\boldsymbol{Z}_I^*, \boldsymbol{Z}_{II}^*) &= \boldsymbol{V}_{I,II}^* = (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0})\boldsymbol{H}_3 - N^{-1}(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})\boldsymbol{H}_0\boldsymbol{\Sigma}\boldsymbol{H}_1^\top
\end{aligned} \tag{13}$$

with

$$\begin{aligned}
\boldsymbol{\Sigma}^a &= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\boldsymbol{\Sigma}\begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\otimes \boldsymbol{e}_1\boldsymbol{e}_1^\top + \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\boldsymbol{\Sigma}\begin{pmatrix}1 & 0\\ 0 & 0\\ 0 & 1\end{pmatrix}\otimes \boldsymbol{e}_2\boldsymbol{e}_2^\top,\\
\boldsymbol{\Sigma}^b &= \boldsymbol{H}_0\boldsymbol{\Sigma}\boldsymbol{H}_0^\top,\\
\boldsymbol{H}_0 &= \begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\\
\boldsymbol{H}_1 &= \begin{pmatrix}\boldsymbol{I}_3 & \otimes & \boldsymbol{1}_{N-2n_0}\\ \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} & \otimes & \boldsymbol{1}_{n_0}\end{pmatrix},\\
\boldsymbol{H}_2 &= \left[\boldsymbol{0}_{4n_0\times 3(N-2n_0)} \middle| \begin{pmatrix}0 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{pmatrix}\otimes \boldsymbol{I}_{n_0}\right],\\
\boldsymbol{H}_3 &= \left\{\left[(\boldsymbol{\Sigma}^a)^{-1}\boldsymbol{\Sigma}^b - \boldsymbol{I}_4\right]\otimes \boldsymbol{I}_{n_0}\right\}\boldsymbol{H}_2,
\end{aligned}\tag{14}$$

$\boldsymbol{e}_i$ denotes the *i-th* column of the matrix $\boldsymbol{I}_2$, and $\boldsymbol{P}_a$, $\boldsymbol{\mu}$, $\bar{\sigma}_1^2$ and $\boldsymbol{\Sigma}$ are defined in expressions (2), (5), (6) and (7), respectively.

An illustrative example designed to indicate the required elements for the model specification in a simple setup is presented in the Appendix.

Finally, we minimize the expected mean squared error of the linear predictor subject to an unbiased restriction, resulting in

$$\widehat{T}^* = [\boldsymbol{D}^\top(\boldsymbol{H}_4^\top)^{-1}\boldsymbol{H}_0^\top(\boldsymbol{\Sigma}^a)^{-1}(\boldsymbol{I}_4\otimes \boldsymbol{1}_{n_0})^\top + \boldsymbol{g}_I^\top + \boldsymbol{g}_{II}^\top\boldsymbol{H}_3^\top]\boldsymbol{A}^{-1}\boldsymbol{Z}_I^* \tag{15}$$

where

$$\boldsymbol{g}_I = [(c_{11}^{\mathcal{C}} + c_{11}^{\mathcal{T}}), \dots, (c_{2n_0,1}^{\mathcal{C}} + c_{2n_0,1}^{\mathcal{T}}), c_{12}^{\mathcal{C}}, \dots, c_{n_0,2}^{\mathcal{C}}, c_{n_0+1,2}^{\mathcal{T}}, \dots, c_{2n_0,2}^{\mathcal{T}}]^\top,$$

$$\begin{aligned}\boldsymbol{g}_{II} = [(c_{2n_0+1,1}^{\mathcal{C}} + c_{2n_0+1,1}^{\mathcal{T}}), \dots, (c_{N1}^{\mathcal{C}} + c_{N1}^{\mathcal{T}}), c_{2n_0+1,2}^{\mathcal{C}}, \dots, c_{N2}^{\mathcal{C}}, c_{2n_0+1,2}^{\mathcal{T}}, \dots,\\ c_{N2}^{\mathcal{T}}, c_{n_0+1,2}^{\mathcal{C}}, \dots, c_{2n_0,2}^{\mathcal{C}}, c_{12}^{\mathcal{T}}, \dots, c_{n_02}^{\mathcal{T}}]^\top,\end{aligned}$$

$\boldsymbol{A} = [\boldsymbol{I}_4 + \bar{\sigma}_1^2(\boldsymbol{\Sigma}^a)^{-1}]\otimes \boldsymbol{I}_{n_0}$ , with $\bar{\sigma}_1^2[= \bar{\sigma}_2^{\mathcal{C}2} = \bar{\sigma}_2^{\mathcal{T}2}]$, defined in expression (6),
$\boldsymbol{H}_4 = \boldsymbol{H}_0^\top(\boldsymbol{\Sigma}^a)^{-1}\boldsymbol{G}\boldsymbol{H}_0$,
$\boldsymbol{G} = (\boldsymbol{I}_4\otimes \boldsymbol{1}_{n_0})^\top\boldsymbol{A}^{-1}(\boldsymbol{I}_4\otimes \boldsymbol{1}_{n_0})$,
$\boldsymbol{D} = \boldsymbol{H}_0^\top(\boldsymbol{I}_4\otimes \boldsymbol{1}_{n_0})^\top[(\boldsymbol{I}_{4n_0} - \boldsymbol{A}^{-1})\boldsymbol{g}_I - \boldsymbol{A}^{-1}\boldsymbol{H}_3\boldsymbol{g}_{II}] + \boldsymbol{H}_1^\top\boldsymbol{g}_{II}$,
$\boldsymbol{Z}_I^*$ defined in expression (12) and $\boldsymbol{\Sigma}^a$, $\boldsymbol{H}_0$, $\boldsymbol{H}_1$ and $\boldsymbol{H}_3$ defined in expression (14).

The variance of the predictor $\widehat{T}^*$ is

$$
\begin{aligned}
\mathbb{V}_{SR}(\widehat{T}^* - T) = {} & \boldsymbol{D}^\top (\boldsymbol{H}_4^\top)^{-1} \boldsymbol{D} + \boldsymbol{g}_I^\top (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0})(\boldsymbol{I}_{4n_0} - \boldsymbol{A}^{-1})\boldsymbol{g}_I \\
& - \boldsymbol{g}_{II}^\top \boldsymbol{H}_3^\top \boldsymbol{A}^{-1} (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0}) \boldsymbol{H}_3 \boldsymbol{g}_{II} \\
& + 2\boldsymbol{g}_I^\top (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0}) \left(\boldsymbol{I}_{4n_0} - \boldsymbol{A}^{-1}\right) \boldsymbol{H}_3 \boldsymbol{g}_{II} \\
& + \boldsymbol{g}_{II}^\top diag \left\{ \boldsymbol{\Sigma} \otimes \boldsymbol{P}_{N-2n_0}, \begin{pmatrix} \sigma_{22}^{\mathcal{C}\mathcal{C}} & 0 \\ 0 & \sigma_{22}^{\mathcal{T}\mathcal{T}} \end{pmatrix} \otimes \boldsymbol{P}_{n_0} \right\} \boldsymbol{g}_{II} \\
& + (N - 2n_0)^{-1} \boldsymbol{g}_{II}^\top \begin{pmatrix} \boldsymbol{I}_3 \otimes \boldsymbol{1}_{N-2n_0} \\ \boldsymbol{0}_{2n_0 \times 3} \end{pmatrix} \boldsymbol{\Sigma} \left( \boldsymbol{I}_3 \otimes \boldsymbol{1}_{N-2n_0}^\top | \boldsymbol{0}_{3 \times 2n_0} \right) \boldsymbol{g}_{II} \\
& + n_0^{-1} \boldsymbol{g}_{II}^\top \boldsymbol{H}_2^\top \left(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0}\right) \boldsymbol{\Sigma}^a \left(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0}\right)^\top \boldsymbol{H}_2 \boldsymbol{g}_{II}
\end{aligned}
\tag{16}
$$

with $\boldsymbol{P}_a$ and $\boldsymbol{H}_2$ respectively defined in expressions (2) and (14) and $\sigma_{22}^{hh}, h = \mathcal{C}, \mathcal{T}$, and $\boldsymbol{\Sigma}$, in expression (7). Details are presented in the Appendix.

Given that $\boldsymbol{\Sigma}$, and $\overline{\sigma}_1^2$ [defined in expression (6)] are unknown, expressions (15) and (16) cannot be computed. Moreover, $\sigma_{22}^{\mathcal{C}\mathcal{T}}$ cannot be estimated since a unit only receives one of the treatments, $\mathcal{C}$ or $\mathcal{T}$. To bypass this problem, we propose to use an empirical version $\widehat{T}_e^*$ of $\widehat{T}^*$, where $\boldsymbol{\Sigma}$ and $\overline{\sigma}_1^2$ are replaced by the following estimates

$$
\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} \widehat{\sigma}_{11} & \widehat{\sigma}_{21}^{\mathcal{C}} & \widehat{\sigma}_{21}^{\mathcal{T}} \\ \widehat{\sigma}_{21}^{\mathcal{C}} & \widehat{\sigma}_{22}^{\mathcal{C}\mathcal{C}} & \widehat{\sigma}_{22}^{\mathcal{C}\mathcal{T}} \\ \widehat{\sigma}_{21}^{\mathcal{T}} & \widehat{\sigma}_{22}^{\mathcal{C}\mathcal{T}} & \widehat{\sigma}_{22}^{\mathcal{T}\mathcal{T}} \end{pmatrix} \tag{17}
$$

with

$$
\begin{aligned}
&\widehat{\sigma}_{11} = (n-1)^{-1} \sum_{i=1}^{n} (\overline{Y}_{i1}^* - \overline{Y}_1^*)^2, \ \overline{Y}_1^* = n^{-1} \sum_{i=1}^{n} \overline{Y}_{i1}^*, \ \overline{Y}_{i1}^* = m_i^{-1} \sum_{j=1}^{m_i} Y_{ij1}^*, \\
&\widehat{\sigma}_{22}^{\mathcal{C}\mathcal{C}} = (n_0-1)^{-1} \sum_{i=1}^{n_0} (\overline{Y}_{i2}^{\mathcal{C}*} - \overline{Y}_2^{\mathcal{C}*})^2, \ \overline{Y}_2^{\mathcal{C}*} = n_0^{-1} \sum_{i=1}^{n_0} \overline{Y}_{i2}^{\mathcal{C}*}, \ \overline{Y}_{i2}^{\mathcal{C}*} = m_i^{-1} \sum_{j=1}^{m_i} Y_{ij2}^{\mathcal{C}*}, \\
&\widehat{\sigma}_{22}^{\mathcal{T}\mathcal{T}} = (n_0-1)^{-1} \sum_{i=n_0+1}^{n} (\overline{Y}_{i2}^{\mathcal{T}*} - \overline{Y}_2^{\mathcal{T}*})^2, \overline{Y}_2^{\mathcal{T}*} = n_0^{-1} \sum_{i=n_0+1}^{n} \overline{Y}_{i2}^{\mathcal{T}*}, \overline{Y}_{i2}^{\mathcal{T}*} = m_i^{-1} \sum_{j=1}^{m_i} Y_{ij2}^{\mathcal{T}*}, \\
&\widehat{\sigma}_{21}^{\mathcal{C}} = (n_0-1)^{-1} \sum_{i=1}^{n_0} (\overline{Y}_{i1}^* - \overline{Y}_1^{\mathcal{C}*})(\overline{Y}_{i2}^{\mathcal{C}*} - \overline{Y}_2^{\mathcal{C}*}), \overline{Y}_1^{\mathcal{C}*} = n_0^{-1} \sum_{i=1}^{n_0} \overline{Y}_{i1}^*, \\
&\widehat{\sigma}_{21}^{\mathcal{T}} = (n_0-1)^{-1} \sum_{i=n_0+1}^{n} (\overline{Y}_{i1}^* - \overline{Y}_1^{\mathcal{T}*})(\overline{Y}_{i2}^{\mathcal{T}*} - \overline{Y}_2^{\mathcal{T}*}), \overline{Y}_1^{\mathcal{T}*} = n_0^{-1} \sum_{i=n_0+1}^{n} \overline{Y}_{i1}^*,
\end{aligned}
$$

$Y_{ij1}^*$ (respectively, $Y_{ij2}^{\mathcal{C}*}$ and $Y_{ij2}^{\mathcal{T}*}$), $j = 1, 2, \ldots, m_i$, denoting a set of pretest (respectively, posttest under intervention $\mathcal{C}$ and posttest under intervention $\mathcal{T}$) response measurements on the unit selected in position $i$. Here, $\widehat{\sigma}_{22}^{\mathcal{C}\mathcal{T}} = 0$ and

$$
\widehat{\overline{\sigma}}_1^2 = (2n)^{-1} \left( \sum_{i=1}^{n} \widehat{\sigma}_{i1}^2 + \sum_{i=1}^{n_0} \left(\widehat{\sigma}_{i2}^{\mathcal{C}}\right)^2 + \sum_{i=n_0+1}^{n} \left(\widehat{\sigma}_{i2}^{\mathcal{T}}\right)^2 \right),
$$

with $\widehat{\sigma}_{i1} = (m_i - 1)^{-1} \sum_{j=1}^{m_i} (Y_{ij1}^{*} - \overline{Y}_{i1}^{*})^2$, $(\widehat{\sigma}_{i2}^{\mathcal{C}})^2 = (m_i - 1)^{-1} \sum_{j=1}^{m_i} (Y_{ij2}^{\mathcal{C}*} - \overline{Y}_{i2}^{\mathcal{C}*})^2$ and $\left(\widehat{\sigma}_{i2}^{\mathcal{T}}\right)^2 = (m_i - 1)^{-1} \sum_{j=1}^{m_i} (Y_{ij2}^{\mathcal{T}*} - \overline{Y}_{i2}^{\mathcal{T}*})^2$.

# 3. Simulation Study

To compare the performance of the proposed estimator $\widehat{T}^*$ in expression (15) and its empirical version, $\widehat{T}_e^*$, with the estimator based on the analysis of covariance model $(2\widehat{\tau}_c)$, we conducted simulation studies.

First we generated finite populations of size $N = 300$ from each of 17 underlying distributions, namely

A) trivariate normal distributions of variables $X$ (pretest), $Y$ (posttest under $\mathcal{C}$ treatment) and $Z$ (posttest under $\mathcal{T}$ treatment) such that $\mathbb{E}(X) = 2$, $\mathbb{E}(Y) = 5$, $\mathbb{E}(Z) = 10$, $Var(X) = \mathbb{V}(Y) = \mathbb{V}(Z) = 1$, $\mathbb{Cov}(X, Y) = -0.3,\ 0.2$ or $0.5$, $\mathbb{Cov}(X, Z) = -0.4,\ 0.2$ or $0.4$ and $\mathbb{Cov}(Y, Z) = 0.2$,

B) trivariate Poisson distributions of variables $X$ (pretest), $Y$ (posttest under $\mathcal{C}$ treatment) and $Z$ (posttest under $\mathcal{T}$ treatment) such that $\mathbb{E}(X) = 1$, $\mathbb{E}(Y) = 0.9$, $\mathbb{E}(Z) = 1.1$, $\mathbb{V}(X) = \mathbb{V}(Y) = \mathbb{V}(Z) = 1$, $\mathbb{Cov}(X, Y) = \rho_{21}^{\mathcal{C}}$, $\mathbb{Cov}(X, Z) = \rho_{21}^{\mathcal{T}}$ and $\mathbb{Cov}(X, Y) = -0.3,\ 0.2$ or $0.5$, $\mathbb{Cov}(X, Z) = -0.6,\ 0.2$ or $0.4$ and $\mathbb{Cov}(Y, Z) = 0.2$, where $\rho_{21}^{\mathcal{C}}$ is the correlation between the pretest latent values and the posttest control latent values, and $\rho_{21}^{\mathcal{T}}$ is the correlation between the pretest latent values and the posttest treatment latent values. In this case, the correlation is equal to covariance because the variances are equal to one. We do not consider the pair $\mathbb{Cov}(X, Y) = 0.5$ and $\mathbb{Cov}(X, Z) = -0.6$ because for the sake of simplicity, the algorithm was not designed to simulate trivariate distributions.

For each of the 17 generated finite populations, we set the values $X$, $Y$ and $Z$, respectively, as the latent (potential) values $y_{s1}$, $y_{s2}^{\mathcal{C}}$, and $y_{s2}^{\mathcal{T}}$, $s = 1, \ldots, 300$ and computed the finite population parameters

$$\sigma_{11} = (N-1)^{-1} \sum_{i=1}^{N} (y_{i1} - \overline{y}_1)^2, \qquad \overline{y}_1 = N^{-1} \sum_{i=1}^{N} y_{i1},$$

$$\sigma_{22} = (N-1)^{-1} \sum_{i=1}^{N} (y_{i2} - \overline{y}_2)^2, \qquad \overline{y}_2 = N^{-1} \sum_{i=1}^{N} y_{i2},$$

$$\sigma_{21} = (N-1)^{-1} \sum_{i=1}^{N} (y_{i1} - \overline{y}_1)(y_{i2} - \overline{y}_2),$$

letting

$$\rho_{21}^{\mathcal{C}} = -0.3, 0.2 \text{ or } 0.5,$$
$$\rho_{21}^{\mathcal{T}} = -0.4, 0.2 \text{ or } 0.4 \text{ (trivariate normal distribution)},$$
$$\rho_{21}^{\mathcal{T}} = -0.6, 0.2 \text{ or } 0.4 \text{ (trivariate Poisson distribution)}.$$

We then selected $15,000$ simple random samples without replacement with size $n = 20$ from each finite population so that for each sample we obtained 20 vectors $(y_{i1}, y_{i2}^{\mathcal{C}}, y_{i2}^{\mathcal{T}})^\top$, $i = 1, \dots, n$ $(= 20)$, letting $y_{i1}$, $i = 1, \dots, n$, denote the 20 pretest latent values, $y_{i2}^{\mathcal{C}}$, $i = 1, \dots, n_0$ $(= 10)$ denote the 10 posttest control latent values and $y_{i2}^{\mathcal{T}}$, $i = n_0 + 1, \dots, n$, as the 10 posttest treatment latent values.

To each $y_{i1}$, $i = 1, \dots, n$, $y_{i2}^{\mathcal{C}}$, $i = 1, \dots, n_0$ $(= 10)$ and $y_{i2}^{\mathcal{T}}$, $i = n_0 + 1, \dots, n$ we added homoskedastic response errors, generated from a normal distribution with mean zero and variance $\overline{\sigma}_1 = \kappa V$, where $V = \sigma_{11} + \sigma_{22} - 2\sigma_{12}$ with $\kappa$ denoting a constant, obtaining the observed values $Y_{ij1}^{*}$, $i = 1, \dots, n$, $Y_{ij2}^{\mathcal{C}*}$, $i = 1, \dots, n_0$ and $Y_{ij2}^{\mathcal{T}*}$, $i = n_0 + 1, \dots, n$, $j = 1, \dots, 10$, *i.e.*, $m_i = 10$. We repeated this setup with $\kappa = 0.0, 0.05, 0.10, 0.15, 0.20$.

For each sample, the estimator in expression (15) and its empirical version, $\widehat{T}_e^*$ of expression (9), as well as the estimator based on the analysis of covariance model $(2\widehat{\tau}_c)$ were computed along with their means, the corresponding mean squared errors (MSE) and mean absolute error (MAE). We also evaluated the corresponding relative contribution of the bias to the MSE as [$= 100\times$(estimate-$T)^2$/MSE]. On the other hand, for each of the eight scenarios considered under the trivariate Poisson distribution, we used 100 simple random samples without replacement with size $n = 20$ to evaluate the coverage of the non-parametric confidence intervals (applied to $\widehat{T}_e^*$ and to $2\widehat{\tau}_c$), based on the $BC_a$ (bias corrected and accelerated) method with $\alpha = 0.05$ and 2000 bootstrap samples proposed by Efron & Tibshirani (1993).

All the simulations and the example were implemented using R Software, (R Core Team, 2021). The results are displayed in Figures 3, 4, 5, 6 and 7 and show that the empirical estimator $\widehat{T}_e^*$ is a good competitor to the analysis of covariance estimator, $2\hat{\tau}_c$, especially for non normal distributions. In general, the MSE and the MAE increase as $\bar{\sigma}_1$ increases, except when $\rho_{21}^{\mathcal{T}} = -0.6$ under the trivariate Poisson distribution. Coverage was shorter for $\alpha = 0.05$ and neither $\widehat{T}_e^*$ nor $2\widehat{\tau}_c$ had better performance with respect to the each other. Also, the MSE decreases as the sample size increases and the relative contribution of the bias to the MSE was less than 1% for all the simulations.

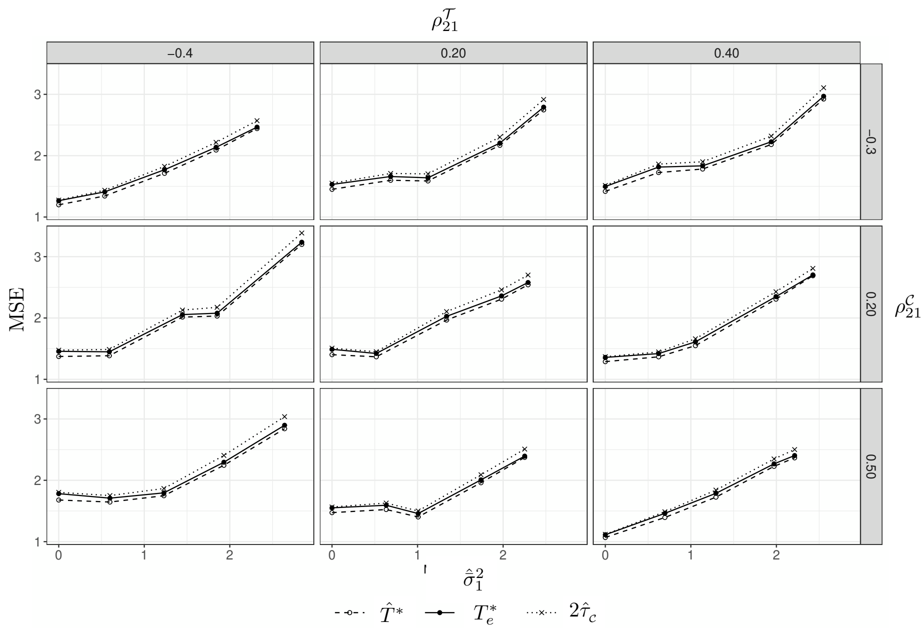


FIGURE 3: MSE of $\widehat{T}^*$, $\widehat{T}_e^*$ and $2\widehat{\tau}_{\mathcal{C}}$. Data generated from a trivariate normal distribution and $n_0 = 10$.

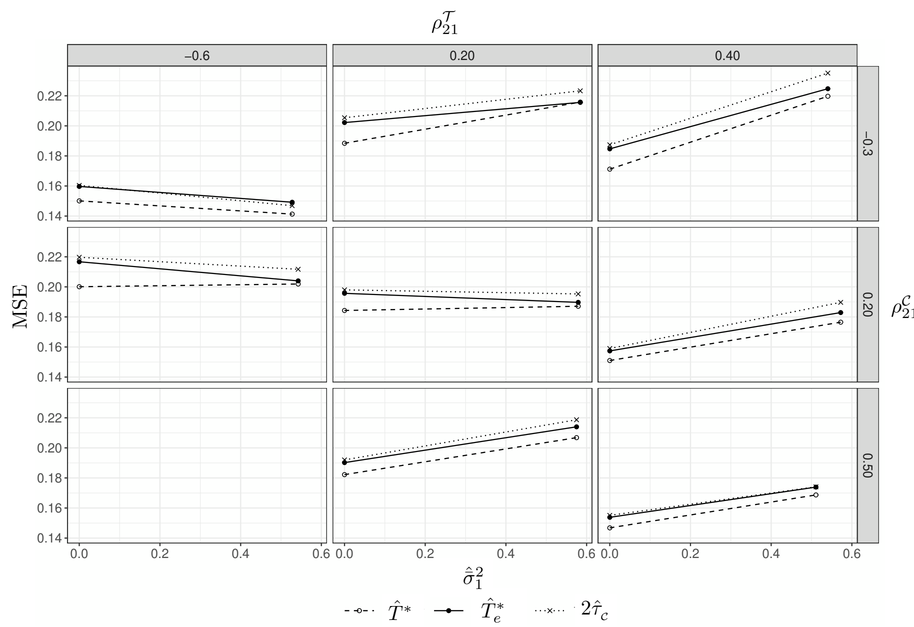


FIGURE 4: MSE of $\hat{T}^*$, $\hat{T}_e^*$ and $2\hat{\tau}_{\mathcal{C}}$. Data generated from a trivariate Poisson distribution and $n_0 = 10$.

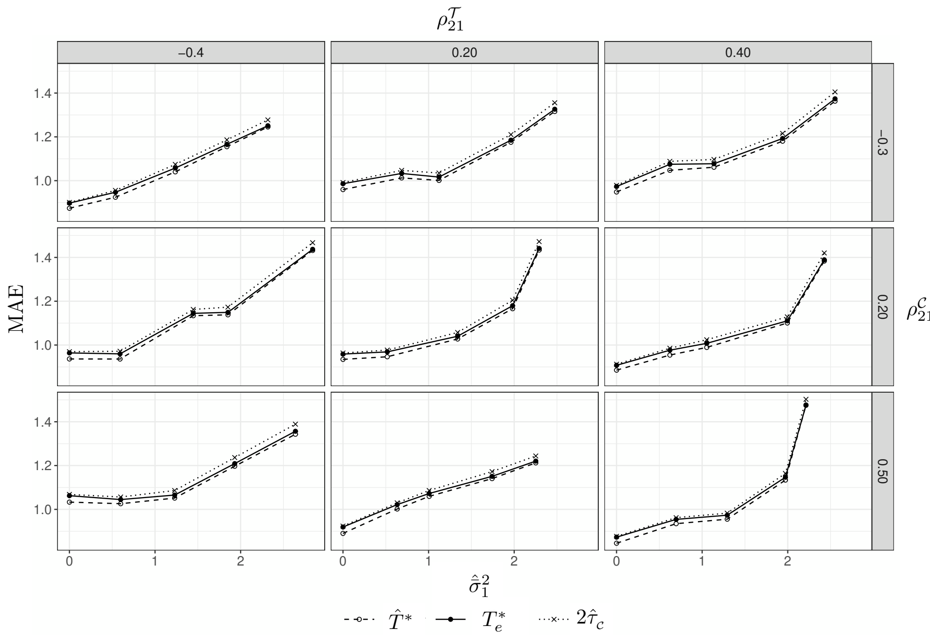


FIGURE 5: MAE of $\widehat{T}^*$, $\widehat{T}_e^*$ and $2\widehat{\tau}_{\mathcal{C}}$. Data generated from a trivariate normal distribution and $n_0 = 10$.

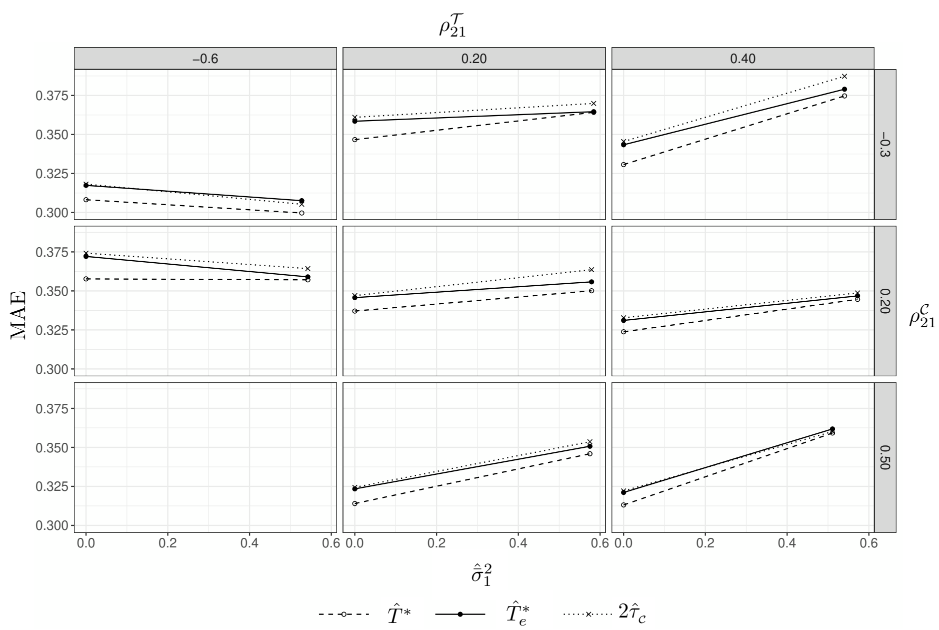


FIGURE 6: MAE of $\hat{T}^*$, $\hat{T}_e^*$ and $2\hat{\tau}_{\mathcal{C}}$. Data generated from a trivariate Poisson distribution and $n_0 = 10$.

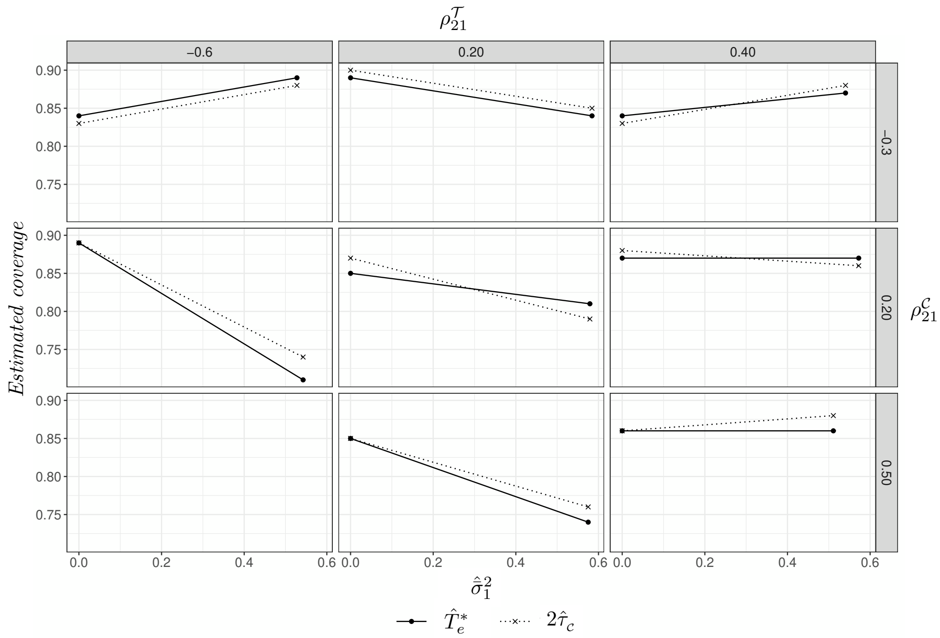


FIGURE 7: Estimated coverage of $\hat{T}_e^*$ and $2\hat{\tau}_{\mathcal{C}}$. Data generated from a trivariate Poisson distribution and $n_0 = 10$.

# 4. Example

Turning back to the example described in Section 1, we have $n_{\mathcal{C}} = 15$ and $n_{\mathcal{T}} = 14$. Since, for simplicity, we assumed that $n_{\mathcal{C}} = n_{\mathcal{T}} = n_0$ in Section 2, we deleted the last observation of the control group ($\mathcal{C}$). The simplicity of expression (15) depends on $n_{\mathcal{C}} = n_{\mathcal{T}} = n_0$. The proposed estimator, $\widehat{T}^*$, could be obtained when $n_{\mathcal{C}} \neq n_{\mathcal{T}}$ but require extensive modification to the notation and to the computations.

In this example, we are interested in the estimation of the difference between the average HRRS for interventions $\mathcal{C}$ and $\mathcal{T}$ . We have a sample of $n = 28$ animals and assume two values, namely $N = 100$ and $N = 500$ for the size of the corresponding finite population. We let

$$\widehat{\sigma}_{11} = (n-1)^{-1} \sum_{i=1}^{n} (Y_{i1}^* - \overline{Y}_1^*)^2$$

$$\widehat{\sigma}_{22} = (n-1)^{-1} \sum_{i=1}^{n} (Y_{i2}^* - \overline{Y}_2^*)^2$$

$$\widehat{\sigma}_{21} = (n-1)^{-1} \sum_{i=1}^{n} (Y_{i1}^* - \overline{Y}_1^*)(Y_{i2}^* - \overline{Y}_2^*)$$

with $\overline{Y}_1^* = n^{-1}\sum_{i=1}^{n} Y_{i1}^*$ and $\overline{Y}_2^* = n^{-1}\sum_{i=1}^{n} Y_{i2}^*$. We assume that the response error variance $\overline{\sigma}_1^2 = (\overline{\sigma}_2^{\mathcal{C}})^2 = (\overline{\sigma}_2^{\mathcal{T}})^2 = \kappa V$, where $\kappa = 0.0$ or $\kappa = 0.10$ and $V = \widehat{\sigma}_{11} + \widehat{\sigma}_{22} - 2\widehat{\sigma}_{12}$. Note that if there is no response error, $\kappa = 0$. Also, $m_i = 1$ for all $i$ because we have one response measurement per unit.

The proposed estimator $\widehat{T}_e^*$ and the covariance analysis model estimator in model (1), namely $2\widehat{\tau}_c$ along with their respective non-parametric confidence intervals based on the $BC_a$ (bias-corrected and accelerated) method with $\alpha = 0.05$ proposed by Efron & Tibshirani (1993) are shown in Table 1. The R code is available from `https://github.com/LuzM-GitHub/Pretest-Posttest`.

TABLE 1: Difference ($\times 10^{-3}$) in average HRRS between the interventions $\mathcal{C}$ and $\mathcal{T}$ along with their respective non-parametric confidence intervals ($\times 10^{-3}$).

| | | Confidence interval limits | | |
|---|---|---|---|---|
| | Estimate | Lower | Upper | Length |
| $\widehat{T}_e^*$ $(N = 100, \kappa = 0)$ | -12.2 | -26.1 | 4.7 | 30.9 |
| $2\widehat{\tau}_c$ | -10.5 | -24.8 | 5.0 | 29.8 |
| $\widehat{T}_e^*$ $(N = 500, \kappa = 0)$ | -12.2 | -25.9 | 5.2 | 31.1 |
| $2\widehat{\tau}_c$ | -10.5 | -24.9 | 5.8 | 30.7 |
| $\widehat{T}_e^*$ $(N = 100, \kappa = 0.10)$ | -12.2 | -26.7 | 6.0 | 32.7 |
| $\widehat{T}_e^*$ $(N = 500, \kappa = 0.10)$ | -12.2 | -26.4 | 5.0 | 31.4 |

From Table 1, we observe that as $N$ increases, the estimate $\widehat{T}_e^*$ remains the approximately the same, indicating that the size of the population exerts small effect on the corresponding confidence intervals. We also note that the confidence interval for the analysis of covariance model estimate ($2\widehat{\tau}_c$) is slightly shorter, but at the price of a less general model.

## 5. Conclusion

Although the only assumption required by the proposed finite population estimators is that the data be obtained by a possibly conceptual simple random sampling scheme, their performance is comparable to that of estimators based on the usual normal analysis of covariance model for a variety of problems where a finite population is exposed to two kinds of intervention. Target quantities may correspond to the average gain for the finite population units expressed as the linear combinations of their individual latent values. In fact, for asymmetrically distributed data and $n = 20$, the proposed estimator, $\widehat{T}_e^*$, has a smaller MSE than $2\hat{\tau}_c$, the estimator stemming from a standard analysis of covariance model.

In particular, when $\boldsymbol{\Sigma}$ in expression (7) is unknown, we can estimate $\widehat{T}^*$ with different pretest sample variances separately for each group ($\mathcal{C}$ or $\mathcal{T}$), *i.e.*, taking

$$\widehat{\boldsymbol{\Sigma}}^a = \begin{pmatrix} \widehat{\sigma}_{11}^{\mathcal{C}} & 0 & \widehat{\sigma}_{21}^{\mathcal{C}} & 0 \\ 0 & \widehat{\sigma}_{11}^{\mathcal{T}} & 0 & \widehat{\sigma}_{21}^{\mathcal{T}} \\ \widehat{\sigma}_{21}^{\mathcal{C}} & 0 & \widehat{\sigma}_{22}^{\mathcal{C}\mathcal{C}} & 0 \\ 0 & \widehat{\sigma}_{21}^{\mathcal{T}} & 0 & \widehat{\sigma}_{22}^{\mathcal{T}\mathcal{T}} \end{pmatrix} \text{ and } \widehat{\boldsymbol{\Sigma}}^b = \boldsymbol{H}_0 \widehat{\boldsymbol{\Sigma}} \boldsymbol{H}_0^\top$$

where $\widehat{\sigma}_{11}^{\mathcal{C}} = (n_0-1)^{-1}\sum_{i=1}^{n_0}(\overline{Y}_{i1}^* - \overline{Y}_1^{\mathcal{C}*})^2$, $\widehat{\sigma}_{11}^{\mathcal{T}} = (n_0-1)^{-1}\sum_{i=n_0+1}^{n}(\overline{Y}_{i1}^* - \overline{Y}_1^{\mathcal{T}*})^2$, and $\overline{Y}_{i1}^*$, $\overline{Y}_1^{\mathcal{C}*}$, $\overline{Y}_1^{\mathcal{T}*}$, $\widehat{\sigma}_{21}^{\mathcal{C}}$, $\widehat{\sigma}_{21}^{\mathcal{T}}$, $\widehat{\sigma}_{22}^{\mathcal{C}\mathcal{C}}$, $\widehat{\sigma}_{22}^{\mathcal{T}\mathcal{T}}$ and $\widehat{\boldsymbol{\Sigma}}$ are defined in expression (17). In this case, when there is no response error, the estimator $\widehat{T}^*$ in expression (15) is equal to the estimator obtained via standard analysis of covariance.

If the interest lies on the difference between the posttest latent value, $y_s^{\mathcal{C}}$ (or $y_s^{\mathcal{T}}$) and the pretest latent value $y_s$ for the $s$-th unit in the population, the model can be extended, but this requires the extended variable approach proposed in Stanek III & Singer (2004) and Stanek III, Singer & Lencina (2004).

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# Appendix. Illustrative Example with a Finite Population of Size 7

For illustrative purpose, we consider a hypothetical population of size $N = 7$ and assume that the response error in the pretest and posttest for unit $s$ can take only two possible equally likely values given by plus or minus $\sigma_{s1}$ $(= \sigma_{s2}^{\mathcal{C}} = \sigma_{s2}^{\mathcal{T}})$, with $\sigma_{11} = 1$, $\sigma_{21} = 3$, $\sigma_{31} = 10$, $\sigma_{41} = 7$, $\sigma_{51} = 6$, $\sigma_{61} = 5$ and $\sigma_{71} = 0.05$.

We consider the selection of a simple random sample of size $n = 6$ without replacement, letting the first $n_0 = 3$ selected units be exposed to intervention $\mathcal{C}$ and the remaining $n_0 = n - n_0 = 3$ units be exposed to intervention $\mathcal{T}$.

Let $\boldsymbol{y}_1 = \begin{pmatrix} 3.5 & 7 & 1.5 & 4 & 6 & 5.5 & 9 \end{pmatrix}^\top$,

$\boldsymbol{y}_2^{\mathcal{C}} = \begin{pmatrix} 12.25 & 13.5 & 7.75 & 10.5 & 15 & 12.25 & 18.5 \end{pmatrix}^\top$,

$\boldsymbol{y}_2^{\mathcal{T}} = \begin{pmatrix} 6.75 & 9.5 & 8.25 & 9.5 & 13 & 10.25 & 18.5 \end{pmatrix}^\top$,

$\boldsymbol{E}_1 = \begin{pmatrix} E_{11} & E_{21} & E_{31} & E_{41} & E_{51} & E_{61} & E_{71} \end{pmatrix}^\top$,

$\boldsymbol{E}_2^{\mathcal{C}} = \begin{pmatrix} E_{12}^{\mathcal{C}} & E_{22}^{\mathcal{C}} & E_{32}^{\mathcal{C}} & E_{42}^{\mathcal{C}} & E_{52}^{\mathcal{C}} & E_{62}^{\mathcal{C}} & E_{72}^{\mathcal{C}} \end{pmatrix}^\top$,

$\boldsymbol{E}_2^{\mathcal{T}} = \begin{pmatrix} E_{12}^{\mathcal{T}} & E_{22}^{\mathcal{T}} & E_{32}^{\mathcal{T}} & E_{42}^{\mathcal{T}} & E_{52}^{\mathcal{T}} & E_{62}^{\mathcal{T}} & E_{72}^{\mathcal{T}} \end{pmatrix}^\top$.

It follows that $\mu_1 = 5.214$, $\mu_2^{\mathcal{C}} = 12.821$, $\mu_2^{\mathcal{T}} = 10.821$, $\overline{\sigma}_1^2 = 31.429[= \overline{\sigma}_2^{\mathcal{C}2} = \overline{\sigma}_2^{\mathcal{T}2}]$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 6.071 & 7.815 & 7.732 \\ 7.815 & 11.536 & 11.036 \\ 7.732 & 11.036 & 15.119 \end{pmatrix}.$$

Then we have,

$$\mathbb{E}_{SR}(\boldsymbol{Z}_I^*) = [\boldsymbol{H}_0 \otimes \boldsymbol{1}_3] \begin{pmatrix} 5.214 \\ 12.821 \\ 10.821 \end{pmatrix}, \qquad \mathbb{E}_{SR}(\boldsymbol{Z}_{II}^*) = \boldsymbol{H}_1 \begin{pmatrix} 5.214 \\ 12.821 \\ 10.821 \end{pmatrix},$$

$$\boldsymbol{V}_I^* = 31.429\boldsymbol{I}_{12} + (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_3) - \frac{1}{7} (\boldsymbol{I}_4 \otimes \boldsymbol{1}_3) \boldsymbol{\Sigma}^b (\boldsymbol{I}_4 \otimes \boldsymbol{1}_3)^\top,$$

$$\begin{aligned} \boldsymbol{V}_{II}^* &= 31.429\boldsymbol{I}_9 + \frac{1}{3}\boldsymbol{H}_2^\top (\boldsymbol{I}_4 \otimes \boldsymbol{1}_3) \boldsymbol{\Sigma}^a (\boldsymbol{I}_4 \otimes \boldsymbol{1}_3)^\top \boldsymbol{H}_2 \\ &\quad + diag\left\{ \left(1 - \frac{1}{1}\right) \begin{pmatrix} 6.071 & 7.815 & 7.732 \\ 7.815 & 11.536 & 11.036 \\ 7.732 & 11.036 & 15.119 \end{pmatrix}, \begin{pmatrix} 11.536 & 0 \\ 0 & 15.119 \end{pmatrix} \otimes \boldsymbol{P}_3 \right\} \\ &\quad - \frac{1}{7}\boldsymbol{H}_1 \begin{pmatrix} 6.071 & 7.815 & 7.732 \\ 7.815 & 11.536 & 11.036 \\ 7.732 & 11.036 & 15.119 \end{pmatrix} \boldsymbol{H}_1^\top \\ &\quad + \begin{pmatrix} \boldsymbol{I}_3 \\ \boldsymbol{0}_{6\times 3} \end{pmatrix} \begin{pmatrix} 6.071 & 7.815 & 7.732 \\ 7.815 & 11.536 & 11.036 \\ 7.732 & 11.036 & 15.119 \end{pmatrix} (\boldsymbol{I}_3 \,|\boldsymbol{0}_{3\times 6}), \end{aligned}$$

$$\boldsymbol{V}_{I,II}^* = (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_3) \boldsymbol{H}_3 - \frac{1}{7} (\boldsymbol{I}_4 \otimes \boldsymbol{1}_3) \boldsymbol{H}_0 \begin{pmatrix} 6.071 & 7.815 & 7.732 \\ 7.815 & 11.536 & 11.036 \\ 7.732 & 11.036 & 15.119 \end{pmatrix} \boldsymbol{H}_1^\top$$

with

$$\boldsymbol{\Sigma}^a = \begin{pmatrix} 6.071 & 0 & 7.815 & 0 \\ 0 & 6.071 & 0 & 7.732 \\ 7.815 & 0 & 11.536 & 0 \\ 0 & 7.732 & 0 & 15.119 \end{pmatrix}, \qquad \boldsymbol{H}_1 = \begin{pmatrix} \boldsymbol{I}_3 \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \boldsymbol{1}_3 \end{pmatrix},$$

$$\boldsymbol{\Sigma}^b = \begin{pmatrix} 6.071 & 6.071 & 7.815 & 7.732 \\ 6.071 & 6.071 & 7.815 & 7.732 \\ 7.815 & 7.815 & 11.536 & 11.036 \\ 7.732 & 7.732 & 11.036 & 15.119 \end{pmatrix}, \quad \boldsymbol{H}_2 = \left[ \boldsymbol{0}_{12\times 3} \middle| \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \boldsymbol{I}_3 \right]$$

and $\boldsymbol{H}_3 = \left\{ \left[ (\boldsymbol{\Sigma}^a)^{-1} \boldsymbol{\Sigma}^b - \boldsymbol{I}_4 \right] \otimes \boldsymbol{I}_3 \right\} \boldsymbol{H}_2.$

If a realization of the random matrix $\boldsymbol{U}$ in expression (2) is

$$\boldsymbol{u} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

*i.e.*, the selected units are $s = 5$, $s = 7$ and $s = 2$ for the $\mathcal{C}$ treatment, $s = 3$, $s = 1$ and $s = 6$ for the $\mathcal{T}$ treatment so that $s = 4$ is not selected, we have

$$\boldsymbol{Y}^*_{1\{\boldsymbol{U}=\boldsymbol{u}\}} = (6 + E_{51}, 9 + E_{71}, 7 + E_{21}, 1.5 + E_{31}, 3.5 + E_{11}, 5.5 + E_{61}, 4 + E_{41})^\top,$$

$$\boldsymbol{Y}^{\mathcal{C}*}_{2\{\boldsymbol{U}=\boldsymbol{u}\}} = \big(15 + E^{\mathcal{C}}_{52}, 18.5 + E^{\mathcal{C}}_{72}, 13.5 + E^{\mathcal{C}}_{22}, 7.75 + E^{\mathcal{C}}_{32}, 12.25 + E^{\mathcal{C}}_{12}, 12.25 + E^{\mathcal{C}}_{62}, 10.5 + E^{\mathcal{C}}_{42}\big)^\top,$$

$$\boldsymbol{Y}^{\mathcal{T}*}_{2\{\boldsymbol{U}=\boldsymbol{u}\}} = \big(13 + E^{\mathcal{T}}_{52}, 18.5 + E^{\mathcal{T}}_{72}, 9.5 + E^{\mathcal{T}}_{22}, 8.25 + E^{\mathcal{T}}_{32}, 6.75 + E^{\mathcal{T}}_{12}, 10.25 + E^{\mathcal{T}}_{62}, 9.5 + E^{\mathcal{T}}_{42}\big)^\top,$$

$$\boldsymbol{Z}^*_{I\{\boldsymbol{U}=\boldsymbol{u}\}} = \big(6 + E_{51}, 9 + E_{71}, 7 + E_{21}, 1.5 + E_{31}, 3.5 + E_{11}, 5.5 + E_{61}, 15 + E^{\mathcal{C}}_{52}, 18.5 + E^{\mathcal{C}}_{72}, 13.5 + E^{\mathcal{C}}_{22}, 8.25 + E^{\mathcal{T}}_{32}, 6.75 + E^{\mathcal{T}}_{12}, 10.25 + E^{\mathcal{T}}_{62}\big)^\top,$$

$$\boldsymbol{Z}^*_{II\{\boldsymbol{U}=\boldsymbol{u}\}} = \big(4 + E_{41}, 10.5 + E^{\mathcal{C}}_{42}, 9.5 + E^{\mathcal{T}}_{42}, 7.75 + E^{\mathcal{C}}_{32}, 12.25 + E^{\mathcal{C}}_{12}, 12.25 + E^{\mathcal{C}}_{62}, 13 + E^{\mathcal{T}}_{52}, 18.5 + E^{\mathcal{T}}_{72}, 9.5 + E^{\mathcal{T}}_{22}\big)^\top.$$

## Derivation of $\mathbb{E}_{SR}(\boldsymbol{Z}^*)$ and $\mathbb{V}_{SR}(\boldsymbol{Z}^*)$ in Expression (5)

Given that $\mathbb{E}_S(\boldsymbol{U}) = N^{-1}\boldsymbol{J}_N$ and $\mathbb{V}_S[vec(\boldsymbol{U})] = (N-1)^{-1}(\boldsymbol{P}_N \otimes \boldsymbol{P}_N)$ we have

$$\begin{aligned}\mathbb{E}_{SR}(\boldsymbol{Z}^*) &= \mathbb{E}_S\left\{vec\left[\boldsymbol{U}(\boldsymbol{y}_1 \ \ \boldsymbol{y}^{\mathcal{C}}_2 \ \ \boldsymbol{y}^{\mathcal{T}}_2)\right]\right\} + \mathbb{E}_{SR}\left\{vec[\boldsymbol{U}(\boldsymbol{E}_1 \ \ \boldsymbol{E}^{\mathcal{C}}_2 \ \ \boldsymbol{E}^{\mathcal{T}}_2)]\right\} \\ &= (\boldsymbol{I}_3 \otimes \boldsymbol{1}_N)\left(\mu_1 \ \ \mu^{\mathcal{C}}_2 \ \ \mu^{\mathcal{T}}_2\right)^\top,\end{aligned}$$

because $\mathbb{E}_{SR}\left(U_{is}E_{s1}\right) = \mathbb{E}_S\left(U_{is}\right)\mathbb{E}_R\left(E_{s1}\right) = 0$, $\mathbb{E}_{SR}\left(U_{is}E_{s2}^{\mathcal{C}}\right) = \mathbb{E}_S\left(U_{is}\right)\times$ $\mathbb{E}_R\left(E_{s2}^{\mathcal{C}}\right) = 0$ and $\mathbb{E}_{SR}\left(U_{is}E_{s2}^{\mathcal{T}}\right) = \mathbb{E}_S\left(U_{is}\right)\mathbb{E}_R\left(E_{s2}^{\mathcal{T}}\right) = 0$, where $\mu_1$, $\mu_2^{\mathcal{C}}$ e $\mu_2^{\mathcal{T}}$ are defined in expression (4), and

$$\mathbb{V}_{SR}\left(\boldsymbol{Z}^*\right) = \mathbb{E}_S\left\{\mathbb{V}_{R|S}(\boldsymbol{Z}^*)\right\} + \mathbb{V}_S\left\{\mathbb{E}_{R|S}\left(\boldsymbol{Z}^*\right)\right\}$$

where

$$\mathbb{E}_{R|S}\left(\boldsymbol{Z}^*\right) = vec\left[\boldsymbol{U}\left(\boldsymbol{y}_1\ \ \boldsymbol{y}_2^{\mathcal{C}}\ \ \boldsymbol{y}_2^{\mathcal{T}}\right)\right]$$

and

$$\mathbb{V}_{R|S}\left(\boldsymbol{Z}^*\right) = \left(\boldsymbol{I}_3 \otimes \boldsymbol{U}\right)\mathbb{V}_{R|S}\left\{vec\left[\left(\boldsymbol{E}_1\ \ \boldsymbol{E}_2^{\mathcal{C}}\ \ \boldsymbol{E}_2^{\mathcal{T}}\right)\right]\right\}\left(\boldsymbol{I}_3 \otimes \boldsymbol{U}\right)^\top,$$

with

$$\mathbb{V}_{R|S}\left\{vec\left[\left(\boldsymbol{E}_1\ \ \boldsymbol{E}_2^{\mathcal{C}}\ \ \boldsymbol{E}_2^{\mathcal{T}}\ \right)\right]\right\} = diag\left\{\left[\bigoplus_{s=1}^{N}\sigma_{s1}^2\right], \left[\bigoplus_{s=1}^{N}\left(\sigma_{s2}^{\mathcal{C}}\right)^2\right], \left[\bigoplus_{s=1}^{N}\left(\sigma_{s2}^{\mathcal{T}}\right)^2\right]\right\}.$$

Then

$$\mathbb{V}_{SR}\left(\boldsymbol{Z}^*\right) = diag\left\{\overline{\sigma}_1^2\boldsymbol{I}_N, \overline{\sigma}_2^{\mathcal{C}2}\boldsymbol{I}_N, \overline{\sigma}_2^{\mathcal{T}2}\boldsymbol{I}_N\right\} + \boldsymbol{\Sigma}\otimes\boldsymbol{P}_N$$

with $\boldsymbol{P}_a = \boldsymbol{I}_a - a^{-1}\boldsymbol{J}_a$, $\boldsymbol{\Sigma}$ defined in expression (7), and $\overline{\sigma}_1^2$, $\overline{\sigma}_2^{\mathcal{C}2}$ and $\overline{\sigma}_2^{\mathcal{T}2}$ defined in expression (6).

## Development of the Linear Predictor of $T$ in Expression (8) and its Variance Under Model (5)

In order to obtain the best linear predictor of

$$T = \sum_{i=1}^{N}\left(c_{i1}^{\mathcal{C}}Y_{i1} + c_{i2}^{\mathcal{C}}Y_{i2}^{\mathcal{C}}\right) + \sum_{i=1}^{N}\left(c_{i1}^{\mathcal{T}}Y_{i1} + c_{i2}^{\mathcal{T}}Y_{i2}^{\mathcal{T}}\right),$$

under model (5), we consider the following three conditions:

a) it must be a linear function of the sample random variables, *i.e.*,

$$\widehat{T}^* = (\boldsymbol{g}_I^\top + \boldsymbol{a}^\top)\boldsymbol{Z}_I^*$$

where $\boldsymbol{Z}_I^*$ and $\boldsymbol{g}_I$ are defined in expressions (12) and (15), respectively,

b) it must be unbiased, *i.e.*,

$$\mathbb{E}_{SR}\left[\left(\boldsymbol{g}_I^\top + \boldsymbol{a}^\top\right)\boldsymbol{Z}_I^*\right] = \mathbb{E}_S\left(\boldsymbol{g}_I^\top\boldsymbol{Z}_I + \boldsymbol{g}_{II}^\top\boldsymbol{Z}_{II}\right)$$

where

$$\boldsymbol{Z}_I = \left(Y_{11}, \ldots, Y_{2n_0,1}, Y^{\mathcal{C}}_{12}, \ldots, Y^{\mathcal{C}}_{n_0 2}, Y^{\mathcal{T}}_{(n_0+1),2}, \ldots, Y^{\mathcal{T}}_{2n_0,2}\right)^\top,$$
$$\boldsymbol{Z}_{II} = \Big(Y_{(2n_0+1),1}, \ldots, Y_{N1}, Y^{\mathcal{C}}_{(2n_0+1),2}, \ldots, Y^{\mathcal{C}}_{N2}, Y^{\mathcal{T}}_{(2n_0+1),2}, \ldots, Y^{\mathcal{T}}_{N,2}, Y^{\mathcal{C}}_{(n_0+1),2}, \ldots, Y^{\mathcal{C}}_{2n_0,2}, Y^{\mathcal{T}}_{12}, \ldots, Y^{\mathcal{T}}_{n_0 2}\Big)^\top,$$

with $\boldsymbol{g}_{II}$ defined in expression (15) [recall that $\mathbb{E}_{SR}[\boldsymbol{Z}^*_I] = \mathbb{E}_S[\boldsymbol{Z}_I]$ and $\mathbb{E}_{SR}[\boldsymbol{Z}^*_{II}] = \mathbb{E}_S[\boldsymbol{Z}_{II}]$].

c) it must minimize $\mathbb{V}_{SR}(\widehat{T}^* - T)$, *i.e.*,

$$\begin{aligned}\mathbb{V}_{SR}(\widehat{T}^* - T) &= \mathbb{V}_{SR}\left[\left(\boldsymbol{g}_I^\top + \boldsymbol{a}^\top\right)\boldsymbol{Z}^*_I - \boldsymbol{g}_I^\top \boldsymbol{Z}_I - \boldsymbol{g}_{II}^\top \boldsymbol{Z}_{II}\right] \\ &= \left(\boldsymbol{g}_I^\top + \boldsymbol{a}^\top\right)\boldsymbol{V}^*_I\left(\boldsymbol{g}_I + \boldsymbol{a}\right) + \boldsymbol{g}_I^\top \boldsymbol{V}_I \boldsymbol{g}_I + \boldsymbol{g}_{II}^\top \boldsymbol{V}_{II}\boldsymbol{g}_{II} - 2\left(\boldsymbol{g}_I^\top + \boldsymbol{a}^\top\right)\boldsymbol{V}_I\boldsymbol{g}_I \\ &\quad - 2\left(\boldsymbol{g}_I^\top + \boldsymbol{a}^\top\right)\boldsymbol{V}_{I,II}\boldsymbol{g}_{II} - 2\boldsymbol{g}_I^\top \boldsymbol{V}_{I,II}\boldsymbol{g}_{II}.\end{aligned} \tag{A1}$$

where

$$\boldsymbol{V}_I = \mathbb{V}_S(\boldsymbol{Z}_I) = (\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0}) - N^{-1}(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0})\boldsymbol{\Sigma}^b(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0})^\top,$$
$$\boldsymbol{V}_{I,II} = \mathbb{C}\mathrm{ov}_S(\boldsymbol{Z}_I, \boldsymbol{Z}_{II}) = \mathbb{C}\mathrm{ov}_{SR}(\boldsymbol{Z}^*_I, \boldsymbol{Z}_{II}) = \mathbb{C}\mathrm{ov}_{SR}(\boldsymbol{Z}^*_I, \boldsymbol{Z}^*_{II}) = \boldsymbol{V}^*_{I,II},$$

subject to the constraint

$$\begin{aligned}\left(\boldsymbol{g}_I^\top + \boldsymbol{a}^\top\right)\mathbb{E}_{SR}[\boldsymbol{Z}^*_I] - \boldsymbol{g}_I^\top \mathbb{E}_S(\boldsymbol{Z}_I) - \boldsymbol{g}_{II}^\top \mathbb{E}_S(\boldsymbol{Z}_{II}) &= \\ \boldsymbol{a}^\top \mathbb{E}_{SR}[\boldsymbol{Z}^*_I] - \boldsymbol{g}_{II}^\top \mathbb{E}_S(\boldsymbol{Z}_{II}) &= 0.\end{aligned} \tag{A2}$$

Using Lagrangian multipliers along with expressions (A1) and (A2) we must solve the system

$$2[\boldsymbol{V}^*_I \widehat{\boldsymbol{a}} + (\boldsymbol{V}^*_I - \boldsymbol{V}_I)\boldsymbol{g}_I - \boldsymbol{V}_{I,II}\boldsymbol{g}_{II}] = [\boldsymbol{H}_0 \otimes \boldsymbol{1}_{n_0}]\boldsymbol{\lambda}_{3\times 1} \tag{A3}$$
$$[\boldsymbol{H}_0 \otimes \boldsymbol{1}_{n_0}]^\top \widehat{\boldsymbol{a}} - \boldsymbol{H}_1^\top \boldsymbol{g}_{II} = 0, \tag{A4}$$

where $\boldsymbol{V}^*_I$ and $\boldsymbol{V}^*_{I,II}$ are defined in expression (13), $\boldsymbol{g}_I$ and $\boldsymbol{g}_{II}$ are defined in expression (15), $\boldsymbol{\Sigma}^a$, $\boldsymbol{\Sigma}^b$, $\boldsymbol{H}_0$ and $\boldsymbol{H}_1$ are defined in expression (14) and $\boldsymbol{\lambda}$ is the Lagrangian multiplier. The result is

$$\begin{aligned}\widehat{\boldsymbol{a}} =& (\boldsymbol{V}^*_I)^{-1}[\boldsymbol{V}_{I,II}\boldsymbol{g}_{II} - (\boldsymbol{V}^*_I - \boldsymbol{V}_I)\boldsymbol{g}_I] - \\ & (\boldsymbol{V}^*_I)^{-1}(\boldsymbol{H}_0 \otimes \boldsymbol{1}_{n_0})\{(\boldsymbol{H}_0 \otimes \boldsymbol{1}_{n_0})^\top (\boldsymbol{V}^*_I)^{-1}(\boldsymbol{H}_0 \otimes \boldsymbol{1}_{n_0})\}^{-1} \times \\ & \{(\boldsymbol{H}_0 \otimes \boldsymbol{1}_{n_0})^\top (\boldsymbol{V}^*_I)^{-1}[\boldsymbol{V}_{I,II}\boldsymbol{g}_{II} - (\boldsymbol{V}^*_I - \boldsymbol{V}_I)\boldsymbol{g}_I] - \boldsymbol{H}_1^\top \boldsymbol{g}_{II}\}.\end{aligned} \tag{A5}$$

Now, observing that

i) $(\boldsymbol{V}^*_I)^{-1} = [(\boldsymbol{\Sigma}^a)^{-1} \otimes \boldsymbol{I}_{n_0}] \times$
$[\boldsymbol{A}^{-1} + N^{-1}\boldsymbol{A}^{-1}(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0})\boldsymbol{H}_0\boldsymbol{\Sigma}(\boldsymbol{I}_3 - N^{-1}\boldsymbol{H}_4\boldsymbol{\Sigma})^{-1}\boldsymbol{H}_0^\top(\boldsymbol{\Sigma}^a)^{-1}(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0}^\top)\boldsymbol{A}^{-1}]$

ii) $[(\boldsymbol{H}_0 \otimes \mathbf{1}_{n_0})^\top (\boldsymbol{V}_I^*)^{-1} (\boldsymbol{H}_0 \otimes \mathbf{1}_{n_0})]^{-1} = \boldsymbol{H}_4^{-1}(\boldsymbol{I}_3 - N^{-1}\boldsymbol{H}_4\boldsymbol{\Sigma})$

iii) $[(\boldsymbol{\Sigma}^a)^{-1} \otimes \boldsymbol{I}_n]\boldsymbol{A}^{-1}(\boldsymbol{V}_I^* - \boldsymbol{V}_I) = (\boldsymbol{I}_{4n_0} - \boldsymbol{A}^{-1}),$

iv) $[(\boldsymbol{\Sigma}^a)^{-1} \otimes \boldsymbol{I}_n]\boldsymbol{A}^{-1}[\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_n] = [\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_n]\boldsymbol{A}^{-1}[(\boldsymbol{\Sigma}^a)^{-1} \otimes \boldsymbol{I}_n] = \boldsymbol{A}^{-1}$, where $\boldsymbol{\Sigma}$ is defined in expression (7), $\boldsymbol{\Sigma}^a$ and $\boldsymbol{H}_0$ are defined in expression (14) and $\boldsymbol{A}$ and $\boldsymbol{H}_4$ are defined in expression (15).

it follows that the expression (A5) simplifies to

$$\widehat{\boldsymbol{a}} = \boldsymbol{A}^{-1}[(\boldsymbol{\Sigma}^a)^{-1}\boldsymbol{H}_0\boldsymbol{H}_4^{-1}\boldsymbol{D} \otimes \mathbf{1}_{n_0}] - [\boldsymbol{I}_{4n_0} - \boldsymbol{A}^{-1}]\boldsymbol{g}_I + \boldsymbol{A}^{-1}\boldsymbol{H}_3\boldsymbol{g}_{II}$$

We proceed to compute the variance of the proposed estimator, which may be expressed as

$$\begin{aligned}\mathrm{Var}_{SR}(\widehat{T}^* - T) =& \widehat{\boldsymbol{a}}^\top \boldsymbol{V}_I^* \widehat{\boldsymbol{a}} + \boldsymbol{g}_I^\top (\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I + 2\widehat{\boldsymbol{a}}^\top (\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I - 2\widehat{\boldsymbol{a}}^\top \boldsymbol{V}_{I,II}\boldsymbol{g}_{II} + \\ & \boldsymbol{g}_{II}^\top \boldsymbol{V}_{II}\boldsymbol{g}_{II}\end{aligned}$$

where

i)

$$\begin{aligned}\boldsymbol{V}_{II} = \mathbb{V}_S(\boldsymbol{Z}_{II}) =& +n_0^{-1}\boldsymbol{H}_2^\top(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})\boldsymbol{\Sigma}^a(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})^\top \boldsymbol{H}_2 \\ &+ diag\{\boldsymbol{\Sigma} \otimes \boldsymbol{P}_{N-2n_0}, \begin{pmatrix}\sigma_{22}^{\mathcal{CC}} & 0 \\ 0 & \sigma_{22}^{\mathcal{TT}}\end{pmatrix} \otimes \boldsymbol{P}_{n_0}\} - N^{-1}\boldsymbol{H}_1\boldsymbol{\Sigma}\boldsymbol{H}_1^\top \\ &+ (N-2n_0)^{-1}\begin{pmatrix}\boldsymbol{I}_3 \otimes \mathbf{1}_{N-2n_0} \\ \mathbf{0}_{2n_0 \times 3}\end{pmatrix}\boldsymbol{\Sigma}(\boldsymbol{I}_3 \otimes \mathbf{1}_{N-2n_0}^\top | \mathbf{0}_{3\times 2n_0}),\end{aligned}$$

ii)

$$\begin{aligned}\hat{\boldsymbol{a}}^\top \boldsymbol{V}_I^* \hat{\boldsymbol{a}} =& \boldsymbol{D}'\boldsymbol{H}_4^{-1^\top}\boldsymbol{D} + \boldsymbol{g}_I^\top(\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I - \boldsymbol{g}_I^\top(\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0})\left(\boldsymbol{I}_{4n_0} - \boldsymbol{A}^{-1}\right)\boldsymbol{g}_I \\ &- N^{-1}\boldsymbol{g}_{II}^\top \boldsymbol{H}_1\boldsymbol{\Sigma}\boldsymbol{H}_1^\top \boldsymbol{g}_{II} + \boldsymbol{g}_{II}^\top \boldsymbol{H}_3^\top \boldsymbol{A}^{-1}(\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0})\boldsymbol{H}_3\boldsymbol{g}_{II} \\ &- 2\boldsymbol{D}^\top\left(\boldsymbol{H}_4^\top\right)^{-1}\boldsymbol{H}_0^\top(\boldsymbol{\Sigma}^a)^{-1}(\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0})^\top \boldsymbol{A}^{-1}(\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I \\ &+ 2\boldsymbol{D}^\top\left[\boldsymbol{H}_4^\top\right]^{-1}\boldsymbol{H}_0^\top[\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0}]^\top \boldsymbol{A}^{-1}\boldsymbol{H}_3\boldsymbol{g}_{II} - 2\boldsymbol{g}_I^\top[\boldsymbol{V}_I^* - \boldsymbol{V}_I]\boldsymbol{A}^{-1}\boldsymbol{H}_3\boldsymbol{g}_{II} \\ &- 2\boldsymbol{g}_I^\top(\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0})\boldsymbol{A}^{-1}\boldsymbol{H}_3\boldsymbol{g}_{II} + 2\boldsymbol{g}_I^\top \boldsymbol{A}^{-1}(\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0})\boldsymbol{H}_3\boldsymbol{g}_{II},\end{aligned}$$

iii)

$$\begin{aligned}\widehat{\boldsymbol{a}}^\top(\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I =& \boldsymbol{D}^\top\left[\boldsymbol{H}_4^\top\right]^{-1}\boldsymbol{H}_0^\top[\boldsymbol{\Sigma}^a]^{-1}[\boldsymbol{I}_4 \otimes \mathbf{1}_{n_0}]^\top \boldsymbol{A}^{-1}[\boldsymbol{V}_I^* - \boldsymbol{V}_I]\boldsymbol{g}_I \\ &- \boldsymbol{g}_I^\top(\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I + \boldsymbol{g}_I^\top \boldsymbol{A}^{-1}(\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I \\ &+ \boldsymbol{g}_{II}^\top \boldsymbol{H}_3^\top \boldsymbol{A}^{-1}(\boldsymbol{V}_I^* - \boldsymbol{V}_I)\boldsymbol{g}_I,\end{aligned}$$

iv)

$$\begin{aligned}\widehat{\boldsymbol{a}}^\top \boldsymbol{V}_{I,II}\boldsymbol{g}_{II} &= \boldsymbol{D}^\top \left(\boldsymbol{H}_4^\top\right)^{-1} \boldsymbol{H}_0^\top \left(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0}\right)^\top \boldsymbol{A}^{-1}\boldsymbol{H}_3\boldsymbol{g}_{II} \\ &\quad - N^{-1}\boldsymbol{g}_{II}^\top \boldsymbol{H}_1 \boldsymbol{\Sigma} \boldsymbol{H}_1^\top \boldsymbol{g}_{II} - \boldsymbol{g}_I^\top \left(\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0}\right) \boldsymbol{H}_3 \boldsymbol{g}_{II} \\ &\quad + \boldsymbol{g}_I^\top \boldsymbol{A}^{-1} \left(\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0}\right) \boldsymbol{H}_3 \boldsymbol{g}_{II} + \boldsymbol{g}_{II}^\top \boldsymbol{H}_3^\top \boldsymbol{A}^{-1} \left(\boldsymbol{\Sigma}^a \otimes \boldsymbol{I}_{n_0}\right) \boldsymbol{H}_3 \boldsymbol{g}_{II},\end{aligned}$$

v)

$$\begin{aligned}\boldsymbol{g}_{II}^\top \boldsymbol{V}_{II}\boldsymbol{g}_{II} &= \boldsymbol{g}_{II}^\top diag\left\{\boldsymbol{\Sigma} \otimes \boldsymbol{P}_{N-2n_0}, \begin{pmatrix} \sigma_{22}^{\mathcal{CC}} & 0 \\ 0 & \sigma_{22}^{\mathcal{TT}} \end{pmatrix} \otimes \boldsymbol{P}_{n_0}\right\} \boldsymbol{g}_{II} \\ &\quad - N^{-1}\boldsymbol{g}_{II}^\top \boldsymbol{H}_1 \boldsymbol{\Sigma} \boldsymbol{H}_1^\top \boldsymbol{g}_{II} \\ &\quad + (N-2n_0)^{-1} \boldsymbol{g}_{II}^\top \begin{pmatrix} \boldsymbol{I}_3 \otimes \boldsymbol{1}_{N-2n_0} \\ \boldsymbol{0}_{2n_0 \times 3} \end{pmatrix} \boldsymbol{\Sigma} \left(\boldsymbol{I}_3 \otimes \boldsymbol{1}_{N-2n_0}^\top | \boldsymbol{0}_{3 \times 2n_0}\right) \boldsymbol{g}_{II} \\ &\quad + n_0^{-1} \boldsymbol{g}_{II}^\top \boldsymbol{H}_2^\top \left(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0}\right) \boldsymbol{\Sigma}^a \left(\boldsymbol{I}_4 \otimes \boldsymbol{1}_{n_0}\right)^\top \boldsymbol{H}_2 \boldsymbol{g}_{II},\end{aligned}$$

where $\sigma_{22}^{\mathcal{CC}}$ and $\sigma_{22}^{\mathcal{TT}}$ are defined in expression (7) and $\boldsymbol{H}_2$, in expression (14), respectively.

The resulting expression is given in expression (16).