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Cubic Rank Transmuted Lindley Distribution with Applications

Distribución Lindley transmutada de rango cúbico con aplicaciones

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Abstract

In this work, we propose a three-parameter generalized Lindley distribution using the cubic rank transmutation map approach by Granzotto et al. (2017). We derive expressions for several mathematical properties including moments and moment generating function, mean deviation, probability weighted moments, quantile function, reliability analysis, and order statistics. We conducted a simulation study to assess the performance of the maximum likelihood estimation procedure for estimating model parameters. The flexibility of the proposed model is illustrated by analyzing two real data sets.

***Key words*:** cubic rank transmutation map; Lindley distribution; reliability analysis; parameter estimation.

Resumen

En este trabajo, proponemos una distribución generalizada Lindley con tres parámetros utilizando el enfoque de mapa de transmutación de rango cúbico de Granzotto et al. (2017). Derivamos expresiones para varias propiedades matemáticas, incluyendo momentos y función generadora de momentos, desviación media, momentos ponderados por probabilidad, función cuantil, análisis de confiabilidad y estadísticas de orden. Se realizó un estudio de simulación para evaluar el rendimiento del procedimiento de estimación de máxima verosimilitud para estimar los parámetros del modelo. La flexibilidad del modelo propuesto se ilustra mediante el análisis de dos conjuntos de datos reales.

***Palabras clave*:** análisis de fiabilidad; distribución Lindley; estimación de parámetros; mapa de transmutación de rango cúbico.

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1. Introduction

The Lindley distribution was developed by Lindley (1958). It has many applications in modeling lifetime data. It is especially useful in modeling mortality studies although it has limitations in its hazard function which only has increasing, decreasing, unimodal, and bathtub shapes, Ghitany et al. (2011). The Lindley distribution can be written as a mixture of exponential and gamma distributions. However, it has superiority over the exponential distribution because of its unimodal and bathtub-shaped hazard rates, Ghitany et al. (2008). A study of its properties is presented by Ghitany et al. (2008). The probability density function (PDF) and cumulative distribution function (CDF) of the Lindley distribution are given by

$$g(x;\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}; \tag{1}$$

and

$$\begin{aligned} G(x;\theta) &= 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x} \\ &= 1 - e^{-\theta x} - \frac{\theta x}{\theta+1}e^{-\theta x}; \end{aligned} \tag{2}$$

respectively, where $x > 0$, and $\theta > 0$.

Extensions of the Lindley distribution have been developed in literature by adding more shape parameters to improve the fitting of various data sets. Earliest Lindley extensions include the discrete Poisson-Lindley distribution introduced by Sankaran (1970), which was used to model data from mistakes in copying groups of random digits with expected frequencies, and data from accidents to women working on high explosive shells. Nadarajah et al. (2011) introduced a generalized Lindley distribution which has better hazard rate properties than the gamma, lognormal and the Weibull distributions. Shanker et al. (2013) introduced a two-parameter Lindley distribution which was fitted to data on survival times of guinea pigs infected with virulent tubercle bacilli, and data for waiting times for bank customers. Other extensions include the extended Lindley Poisson distribution by Pararai et al. (2015), Weibull Lindley distribution by Asgharzadeh et al. (2018), truncated Lindley distribution by Zaninetti (2019), double Lindley distribution by Kumar & Jose (2018), and Zografos Balakrishnan Power Lindley Distriution by Khokhar et al. (2020), among others. Motivated by Shaw & Buckley (2007), Shaw & Buckley (2009), and Granzotto et al. (2017), we develop a cubic rank transmuted-type distribution using the Lindley distribution as the base distribution. This new distribution is called the cubic rank transmuted Lindley (CRTL) distribution.

Automation of data collection resulted in more complex data whose dynamics cannot be captured by common one or two parameter probability distributions which have been studied extensively in literature. Over the years, transformation methods for random variables have been developed to build probability models that reflect specific properties of the data being modeled, thus providing a better fit for the data, Gilchrist (2000). Considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remains many important problems where the real data does not follow any of the classical or standard probability models, Merovci & Sharma (2014). From this point of view, we have studied an extension of the classical Lindley distribution, Lindley (1958). The main contribution of this work is the introduction of a new distribution that outperforms its classical base distribution as well as other distributions in applications.

Shaw & Buckley (2007) and Shaw & Buckley (2009) proposed a new generalization method of statistical distributions called transmutation mapping, which is the functional composition of the CDF of one distribution with the inverse CDF of another. The purpose of this transformation was to discover families of parametric distributions with the ability to fit to levels of third and fourth moments that depart from those of a given base distribution. According to Shaw & Buckley (2007) and Shaw & Buckley (2009), a random variable X is said to have a transmuted probability distribution with the CDF $F_1(x)$ if

$$F_1(x) = (1+\lambda)G_1(x) - \lambda G_1(x)^2, \; |\lambda| \leq 1, \tag{3}$$

where $G_1(x)$ is the baseline distribution of $F_1(x)$. The corresponding PDF of the transmuted probability distribution is

$$f_1(x) = g_1(x)[1+\lambda - 2\lambda G_1(x)], \tag{4}$$

where $g_1(x)$ is the PDF of the sub-model $G_1(x)$.

Recent distributions in literature developed by applying these transformations include the Kumaraswamy transmuted Pareto distribution, Chhetri, Akinsete, Aryal & Long (2017), transmuted inverse Weibull distribution, Khan et al. (2014), a new generalized Cauchy distribution with an application to annual one day maximum rainfall data by Ball et al. (2021), and host of others. Many generalizations of the transmutation map have been developed over the years. Nofal et al. (2017) introduced the generalized transmuted-G family of distributions by incorporating two additional shape parameters to the transmuted mapping distribution. Alizadeh et al. (2017) also introduced a generalized transmuted family of distributions. According to Granzotto et al. (2017), the CDF and the PDF of a cubic rank transmuted distribution are given by

$$F(x) = \lambda_1 G(x) + (\lambda_2 - \lambda_1)[G(x)]^2 + (1-\lambda_2)[G(x)]^3, \tag{5}$$

$$f(x) = g(x)[\lambda_1 + 2(\lambda_2 - \lambda_1)G(x) + 3(1-\lambda_2)[G(x)]^2 \tag{6}$$

respectively, where $g(x)$ is the PDF and $G(x)$ is the CDF of the baseline distribution, $0 \leq \lambda_1 \leq 1$ and $-1 \leq \lambda_2 \leq 1$. A more recent approach is the higher rank transmuted (HRT-G) families of distributions developed by Riffi (2019). The HRT-G family generalized the transmutation map by using higher rank transmutation maps, which are useful in fitting real data from mixture distributions. Celik (2018) introduced the cubic rank transmuted

Frechet distribution, cubic rank transmuted Gumbel distribution and cubic rank transmuted Gompertz distribution. A notable recent work is cubic rank transmuted modified Burr III distribution by Bhatti et al. (2020).

The outline of this paper is as follows. Section 2 defines the CRTL distribution. Section 3 covers mathematical properties of the CRTL distribution. Section 4 covers parameter estimation using the maximum likelihood estimation (MLE) method. Section 5 covers a simulation study for the MLEs. For application purposes, the CRTL distribution is used to model datasets for survival times of infected guinea pigs, and failure times for the USS Halfbeak number 3 main propulsion diesel engine data in Section 6. Finally, we provide the concluding remarks in Section 7.

2. The Cubic Rank Transmutation Lindley Distribution

In this section, we define the cubic rank transmutation Lindley (CRTL) distribution. Substituting equations (1) and (2) into equations (5) and (6), we obtain the three parameter CRTL distribution with the CDF and PDF

$$\begin{aligned} F(x;\theta,\lambda_1,\lambda_2) = &\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)\Bigg[\lambda_1+(\lambda_2-\lambda_1) \\ &\times\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)+(1-\lambda_2)\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)^2\Bigg] \end{aligned} \tag{7}$$

and

$$\begin{aligned} f(x;\theta,\lambda_1,\lambda_2) = &\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\Bigg[\lambda_1+2(\lambda_2-\lambda_1)\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right) \\ &+3(1-\lambda_2)\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)^2\Bigg] \end{aligned} \tag{8}$$

respectively, where $0 \le \lambda_1 \le 1$, $-1 \le \lambda_2 \le 1$, $x > 0$ and $\theta > 0$. Graphs for the PDF and CDF of the CRTL distribution, CRTL $(\theta, \lambda_1, \lambda_2)$, for selected parameters θ, λ_1, and λ_2 are presented in Figure 1 and Figure 2, respectively.

3. Mathematical Properties

This section presents some structural and mathematical properties of the CRTL distribution including moments, mean deviation, probability weighted moments, quantile function, reliability analysis, order statistics and maximum likelihood estimation.

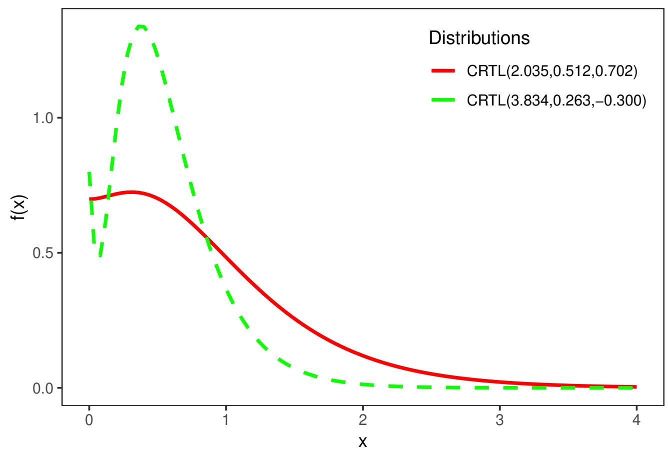

FIGURE 1: PDF Plots of CRTL for Selected Parameters.

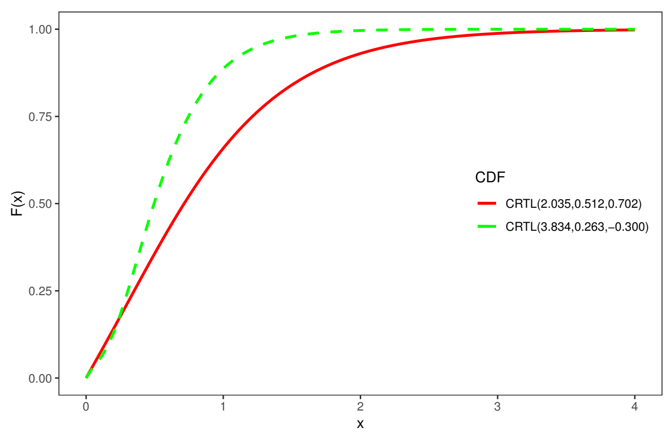

FIGURE 2: CDF Plots of CRTL for Selected Parameters.

3.1. Moments and Moment Generating Function

Moments are necessary and important in any statistical analysis, especially in applications. They can be used to study most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis), Chhetri, Long & Aryal (2017). The k^{th} order moment of the CRTL distribution is given by

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x;\theta,\lambda_1,\lambda_2)dx$$
$$= \int_0^{\infty} \frac{\theta^2}{\theta+1}(1+x)x^k e^{-\theta x}\Bigg[\lambda_1 + 2(\lambda_2-\lambda_1)\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)$$

$$
\begin{aligned}
&+3(1-\lambda_2)\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)^2\Bigg]dx\\
&=\frac{k!}{(\theta+1)\theta^{k-1}}\left(\frac{(3-\lambda_1-\lambda_2)(\theta+k+1)}{\theta}\right.\\
&+\frac{(\lambda_1+2\lambda_2-3)}{2^k}\left[1+\frac{(k+1)(4\theta+k+4)}{4\theta(\theta+1)}\right]\\
&+\frac{(1-\lambda_2)}{3^k}\left\{1+\frac{(k+1)}{27\theta(\theta+1)^2}\left[9(\theta+1)(3\theta+1)+3(3\theta+2)(k+2)\right]\right\}\\
&\left.+\frac{(1-\lambda_2)}{3^{k+3}}\frac{(k+1)}{\theta(\theta+1)^2}(k+2)(k+3)\right)
\end{aligned}
\tag{9}
$$

where $\theta > 0$, $0 \leq \lambda_1 \leq 1$, $-1 \leq \lambda_2 \leq 1$, $x > 0$ and $t > 0$.

The moment generating function of X may be obtained as follows.

$$
\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} f(x;\theta,\lambda_1,\lambda_2)dx\\
&=\int_0^{\infty}\frac{\theta^2}{\theta+1}(1+x)e^{(t-\theta)x}\left[\lambda_1+2(\lambda_2-\lambda_1)\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)\right.\\
&\left.+3(1-\lambda_2)\left(1-e^{-\theta x}-\frac{\theta x}{\theta+1}e^{-\theta x}\right)^2\right]dx\\
&=\frac{\theta^2}{\theta+1}\left\{\frac{(3-\lambda_1-\lambda_2)}{(\theta-t)}\left(1+\frac{1}{\theta-t}\right)\right.\\
&+\frac{2(\lambda_1+2\lambda_2-3)}{(2\theta-t)}\left[1+\frac{2\theta+1}{(\theta+1)(2\theta-t)}+\frac{2\theta}{(\theta+1)(2\theta-t)^2}\right]\\
&+\frac{3(1-\lambda_2)}{(3\theta-t)}\left[1+\frac{3\theta+1}{(\theta+1)(3\theta-t)}+\frac{2\theta(3\theta+2)}{(\theta+1)^2(3\theta-t)^2}\right]\\
&\left.+\frac{18\theta^2(1-\lambda_2)}{(\theta+1)^2(3\theta-t)^4}\right\}
\end{aligned}
\tag{10}
$$

where $\theta > 0$, $0 \leq \lambda_1 \leq 1$, and $-1 \leq \lambda_2 \leq 1$.

3.2. Mean Deviation

Let X be a CRTL random variable with mean $\mu = E(X)$ and median M. The mean deviation from the mean and the mean deviation from the median can be expressed as

$$\begin{aligned} D_1(x) &= \int_{\theta}^{\infty} |x-\mu| f(x) dx \\ &= \int_{\theta}^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\infty} (x-\mu) f(x) dx \\ &= 2[\mu F(\mu) - J(\mu)], \end{aligned} \tag{11}$$

and

$$\begin{aligned} D_2(x) &= \int_{\theta}^{\infty} |x-M| f(x) dx \\ &= \int_{\theta}^{M} (M - x) f(x) dx + \int_{M}^{\infty} (x-M) f(x) dx \\ &= \mu - 2J(M), \end{aligned} \tag{12}$$

where $F(.)$ and $f(.)$ are the CDF and PDF of the CRTL distribution, respectively, and $J(.)$ can be obtained by computing the integral $J(t) = \int_{\theta}^{t} x f(x) dx$.

3.3. Probability Weighted Moments

The probability weighted moments (PWMs) which were first introduced by Greenwood et al. (1979), are the expectations of certain functions of a random variable and can be defined for any random variable whose ordinary moments exist. Let $H(x; \lambda_1, \lambda_2, \theta)$ be the CDF of a CRTL random variable Y. Then the $(s,k)^{th}$ PWM of Y is defined by

$$\rho_{s,k} = E[Y^s H^k(Y)] = \int_{-\infty}^{\infty} y^s H^k(y; \lambda_1, \lambda_2, \theta) h(y; \lambda_1, \lambda_2, \theta) dy.$$

Note that

$$\begin{aligned} H^k(y)h(y) &= \sum_{r_1=0}^{k} \sum_{r_2=0}^{r_1} \sum_{r_3=0}^{k} \sum_{r_4=0}^{r_3} \sum_{r_5=0}^{r_3-r_4} \sum_{r_6=0}^{r_5} \sum_{r_7=0}^{2r_4} \sum_{r_8=0}^{r_7} A \left(\frac{\theta^2}{\theta+1}\right) y^B (y+1) \\ &\times \Bigg[(3-\lambda_1-\lambda_2) e^{-(C+1)\theta y} - 2(3-\lambda_1-2\lambda_2)\left(1+\frac{\theta y}{\theta+1}\right) \\ &\times e^{-(C+2)\theta y} + 3(1-\lambda_2)\left\{1+\frac{2\theta y}{\theta+1}+\left(\frac{\theta y}{\theta+1}\right)^2\right\} e^{-(C+3)\theta y}\Bigg] \end{aligned}$$

where

$$\begin{aligned} A &= \binom{k}{r_1}\binom{r_1}{r_2}\binom{k}{r_3}\binom{r_3}{r_4}\binom{r_3-r_4}{r_5}\binom{r_5}{r_6}\binom{2r_4}{r_7}\binom{r_7}{r_8}\left(\frac{\theta}{\theta+1}\right)^B \\ &\times \lambda_1^{k-r_3}(\lambda_2-\lambda_1)^{r_3-r_4}(1-\lambda_2)^{r_4}(-1)^C, \\ B &= k - r_1 + r_3 + r_4 - r_5 - r_7, \quad \text{and} \\ C &= k - r_1 + r_2 + r_3 + r_4 - r_5 + r_6 - r_7 + r_8. \end{aligned}$$

Thus,

$$
\begin{aligned}
\rho_{s,k} &= E[Y^s H^k(Y)] \\
&= \int_{-\infty}^{\infty} y^s H^k(y;\lambda_1,\lambda_2,\theta) h(y;\lambda_1,\lambda_2,\theta) dy \\
&= \sum_{r_1=0}^{k}\sum_{r_2=0}^{r_1}\sum_{r_3=0}^{k}\sum_{r_4=0}^{r_3}\sum_{r_5=0}^{r_3-r_4}\sum_{r_6=0}^{r_5}\sum_{r_7=0}^{2r_4}\sum_{r_8=0}^{r_7} A\left(\frac{\theta^2}{\theta+1}\right) \\
&\times \Bigg(\frac{(3-\lambda_1-\lambda_2)(B+s)!}{\{\theta(C+1)\}^{(B+s+2)}}[\theta(C+1)+B+s+1] \\
&- \frac{2(3-\lambda_1-2\lambda_2)(B+s)!}{\{\theta(C+2)\}^{(B+s+1)}}\Bigg[1+\frac{B+s+1}{\theta(\theta+1)(C+2)}\Bigg\{2\theta+1+\frac{B+s+1}{C+2}\Bigg\}\Bigg] \\
&+ \frac{3(1-\lambda_2)(B+s)!}{\{\theta(C+3)\}^{(B+s+1)}}\Bigg[1+\frac{(3\theta+1)(B+s+1)}{\theta(\theta+1)(C+3)} \\
&+ \frac{\theta(B+s+1)(B+s+2)}{[\theta(\theta+1)(C+3)]^2}\left(3\theta+2+\frac{B+s+3}{C+3}\right)\Bigg]\Bigg),
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
A &= \binom{k}{r_1}\binom{r_1}{r_2}\binom{k}{r_3}\binom{r_3}{r_4}\binom{r_3-r_4}{r_5}\binom{r_5}{r_6}\binom{2r_4}{r_7}\binom{r_7}{r_8}\left(\frac{\theta}{\theta+1}\right)^B \\
&\times \lambda_1^{k-r_3}(\lambda_2-\lambda_1)^{r_3-r_4}(1-\lambda_2)^{r_4}(-1)^C, \\
B &= k-r_1+r_3+r_4-r_5-r_7, \quad \text{and} \\
C &= k-r_1+r_2+r_3+r_4-r_5+r_6-r_7+r_8.
\end{aligned}
$$

3.4. Quantiles and Random Number Generator

We use the inverse transformation method to compute an expression for the quantile function of the proposed model. Quantiles are the points in a distribution that relate to the rank order of values. The q^{th} quantile x_q, $0 \le q \le 1$, of the CRTL distribution is defined by

$$
\begin{aligned}
q &= Pr(x \le x_q) = F(x_q) \\
&= G(x_q;\theta)\Bigg[\lambda_1 + (\lambda_2-\lambda_1)G(x_q;\theta) + (1-\lambda_2)G^2(x_q;\theta)\Bigg]
\end{aligned}
\tag{14}
$$

where $G(x_q;\theta) = 1 - e^{-\theta x_q} - \frac{\theta x_q}{\theta+1}e^{-\theta x_q}$.

Solving the non-linear equation (14), we obtain

$$
x_q = \frac{-W\left((A-1)(\theta+1)e^{-\theta-1}\right) - \theta - 1}{\theta} \tag{15}
$$

where

$$A = \frac{K}{3(2)^{\frac{1}{3}}(1-\lambda_2)} - \frac{(2)^{\frac{1}{3}}B}{3K(1-\lambda_2)} - \frac{\lambda_2-\lambda_1}{3(1-\lambda_2)} \tag{16}$$

$$\begin{aligned}
K &= C + \sqrt{((C+D)^2 + 4B^3 + D)^{\frac{1}{3}}}\\
B &= -\lambda_1^2 - \lambda_1\lambda_2 + 3\lambda_1 - \lambda_2^2\\
C &= 2\lambda_1^3 + 3\lambda_1^2\lambda_2 - 9\lambda_1^2\\
D &= -3\lambda_1\lambda_2^2 + 9\lambda_1\lambda_2 - 2\lambda_2^3 + 27\lambda_2^2 q - 54\lambda_2 q + 27q,
\end{aligned}$$

and $W(.)$ is known as the product log-function or the omega function.

We can choose specific values of parameters $\theta, \lambda_1,\ \lambda_2$, and $q \in (0,1)$, and use the expression (15) to generate a random variable X having the CRTL distribution (7). Furthermore, we can obtain the first quartile, median, and the third quartile by setting $q = 0.25, 0.50$ and 0.75 respectively in equation (15).

3.5. Reliability Analysis

The reliability is the probability of an item not failing prior to some time t which is defined by the formula $R(t) = 1 - F(t)$, where $F(t)$ is the CDF of the CRTL distribution (7).

The hazard rate function of the CRTL can be expressed as

$$\begin{aligned}
h(t) &= \frac{f(t)}{1-F(t)}\\
&= \frac{g(t,\theta)[\lambda_1 + 2(\lambda_2-\lambda_1)G(t,\theta) + 3(1-\lambda_2)[G(t,\theta)^2]}{1-\lambda_1 G(t,\theta) - (\lambda_2-\lambda_1)[G(t,\theta)]^2 + (1-\lambda_2)[G(t,\theta)]^3}
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
g(t;\theta) &= \frac{\theta^2}{\theta+1}(1+t)e^{-\theta t} \quad \text{and}\\
G(t;\theta) &= 1 - e^{-\theta t} - \frac{\theta t}{\theta+1}e^{-\theta t},\ t>0,\ \theta>0.
\end{aligned}$$

The reversed hazard rate function is defined by the expression

$$r(t;\theta,\lambda_1,\lambda_2) = \frac{f(t;\theta,\lambda_1,\lambda_2)}{F(t;\theta,\lambda_1,\lambda_2)}. \tag{18}$$

By substituting equations (7) and (8) into equation (18), we can obtain the reversed hazard rate function of the CRTL distribution.

3.6. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from the CRTL distribution with PDF (8), CDF (7), and parameter vector $\Theta = (\theta, \lambda_1, \lambda_2)^T$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics from this sample. The PDF $f_{(i:n)}(x)$ of i^{th} order statistics is given by

$$
\begin{aligned}
f_{(i:n)}(x) &= \frac{1}{B(i, n-i+1)} [F(x; \theta, \lambda_1, \lambda_2)]^{i-1} [1 - F(x; \theta, \lambda_1, \lambda_2)]^{n-i} f(x; \theta, \lambda_1, \lambda_2) \\
&= \frac{1}{B(i, n-i+1)} \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} f(x; \theta, \lambda_1, \lambda_2) F(x; \theta, \lambda_1, \lambda_2)^{i+s-1} \\
&= \frac{1}{B(i, n-i+1)} \sum_{s=0}^{n-i} (-1)^s \binom{n-i}{s} \left(\frac{\theta^2}{\theta+1} (1+x) e^{-\theta x} \left[\lambda_1 + 2(\lambda_2 - \lambda_1) \right.\right. \\
&\times \left.\left. \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) + 3(1-\lambda_2) \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right)^2 \right] \right) \\
&\times \left(\left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) \left[\lambda_1 + (\lambda_2 - \lambda_1) \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) \right.\right. \\
&+ \left.\left. (1-\lambda_2) \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right)^2 \right] \right)^{i+s-1}.
\end{aligned} \tag{19}
$$

The PDF of the minimum order statistics for the CRTL distribution is given by

$$
\begin{aligned}
f_{(1:n)}(x) &= \frac{1}{B(1, n)} (1 - F(x))^{n-1} f(x) \\
&= \frac{1}{B(1, n)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} F(x)^k f(x) \\
&= \frac{1}{B(1, n)} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \left(\left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) \left[\lambda_1 + (\lambda_2 - \lambda_1) \right.\right. \\
&\times \left.\left. \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) + (1-\lambda_2) \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right)^2 \right] \right)^k \\
&\times \left(\frac{\theta^2}{\theta+1} (1+x) e^{-\theta x} \left[\lambda_1 + 2(\lambda_2 - \lambda_1) \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) + 3(1-\lambda_2) \right.\right. \\
&\times \left.\left. \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right)^2 \right] \right).
\end{aligned} \tag{20}
$$

The PDF of the maximum order statistics for the CRTL distribution is given by

$$\begin{aligned}
f_{(n:n)}(x) &= \frac{1}{B(1,n)} F(x)^{n-1} f(x) \\
&= \frac{1}{B(1,n)} \Bigg(\left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) \Bigg[\lambda_1 + (\lambda_2 - \lambda_1) \\
&\times \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) + (1-\lambda_2) \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right)^2 \Bigg] \Bigg)^{n-1} \\
&\times \Bigg(\frac{\theta^2}{\theta+1} (1+x) e^{-\theta x} \Bigg[\lambda_1 + 2(\lambda_2 - \lambda_1) \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right) + 3(1-\lambda_2) \\
&\times \left(1 - e^{-\theta x} - \frac{\theta x}{\theta+1} e^{-\theta x}\right)^2 \Bigg] \Bigg). \qquad (21)
\end{aligned}$$

4. Parameter Estimation

In this section, we discuss the MLE method to estimate the parameters of the CRTL distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from the CRTL distribution with observed values $x_1, x_2, \ldots, x_n$ and $\Theta = (\theta, \lambda_1, \lambda_2)^T$ be the parameter vector. The likelihood function for Θ can be written as

$$\begin{aligned}
L(\Theta|x) = \frac{\theta^{2n}}{(\theta+1)^n} \prod_{i=1}^{n} (1+x_i) e^{-\theta x_i} \Bigg[\lambda_1 + 2(\lambda_2 - \lambda_1) \\
\times \left(1 - e^{-\theta x_i} - \frac{\theta x_i}{\theta+1} e^{-\theta x_i}\right) + 3(1-\lambda_2) \\
\times \left(1 - e^{-\theta x_i} - \frac{\theta x_i}{\theta+1} e^{-\theta x_i}\right)^2 \Bigg]. \qquad (22)
\end{aligned}$$

The log-likelihood function may be written as

$$\begin{aligned}
l(\Theta|x) = 2n \ln(\theta) - n \ln(\theta+1) + \sum_{i=1}^{n} \ln(1+x_i) - \theta \sum_{i=1}^{n} x_i \\
+ \sum_{i=1}^{n} \ln \Bigg[\lambda_1 + 2(\lambda_2 - \lambda_1) \left(1 - e^{-\theta x_i} - \frac{\theta x_i}{\theta+1} e^{-\theta x_i}\right) \\
+ 3(1-\lambda_2) \left(1 - e^{-\theta x_i} - \frac{\theta x_i}{\theta+1} e^{-\theta x_i}\right)^2 \Bigg]. \qquad (23)
\end{aligned}$$

To discuss the MLEs for θ, λ_1 and λ_2, we differentiate equation (23) with respect to θ, λ_1 and λ_2 respectively to obtain the score vector $\left(\frac{\partial l}{\partial \theta}, \frac{\partial l}{\partial \lambda_1}, \frac{\partial l}{\partial \lambda_2}\right)^T$ as shown below:

$$
\begin{aligned}
\frac{\partial l}{\partial \theta} &= \frac{2n}{\theta} - \frac{n}{\theta+1} - \sum_{i=1}^{n} x_i \\
&+ \sum_{i=1}^{n} \frac{2\theta x_i(\lambda_2-\lambda_1)e^{-\theta x_i}\left(\theta x_i + x_i + \theta + 2\right)}{(\theta+1)^2\Big[\lambda_1 + 2(\lambda_2-\lambda_1)G(x_i,\theta) + 3(1-\lambda_2)G^2(x_i,\theta)\Big]} \\
&+ \sum_{i=1}^{n} \frac{6(1-\lambda_2)\theta x_i e^{-2\theta x_i}\left(\theta x_i + \theta + x_i + 2\right)\left(\theta e^{\theta x_i} - \theta x_i + e^{\theta x_i} - \theta - 1\right)}{(\theta+1)^3\Big[\lambda_1 + 2(\lambda_2-\lambda_1)G(x_i,\theta) + 3(1-\lambda_2)G^2(x_i,\theta)\Big]},
\end{aligned}
\tag{24}
$$

$$
\frac{\partial l}{\partial \lambda_1} = \sum_{i=1}^{n} \frac{1 - 2G(x_i,\theta)}{\lambda_1 + 2(\lambda_2-\lambda_1)G(x_i,\theta) + 3(1-\lambda_2)G^2(x_i,\theta)}, \tag{25}
$$

$$
\frac{\partial l}{\partial \lambda_2} = \sum_{i=1}^{n} \frac{2G(x_i,\theta) - 3G^2(x_i,\theta)}{\lambda_1 + 2(\lambda_2-\lambda_1)G(x_i,\theta) + 3(1-\lambda_2)G^2(x_i,\theta)}, \tag{26}
$$

where $G(x_i;\theta) = 1 - e^{-\theta x_i} - \frac{\theta x_i}{\theta+1}e^{-\theta x_i}$.

The MLEs $\hat{\theta}, \hat{\lambda}_1, \hat{\lambda}_2$ of the unknown parameters $\theta, \lambda_1, \lambda_2$ respectively, can be obtained by solving non-linear equations $\frac{\partial l}{\partial \theta} = 0$, $\frac{\partial l}{\partial \lambda_1} = 0$, and $\frac{\partial l}{\partial \lambda_2} = 0$.

We can use numerical methods such as the quasi-Newton algorithm to numerically optimize the log-likelihood function given in equation (22), to get the MLEs of the parameters θ, λ_1, and λ_2. To compute the standard error and the asymptotic confidence interval, we use the usual large sample approximation in which the maximum likelihood estimators for Θ can be treated as being approximately normal. For the three-parameter CRTL distribution all second order derivatives exist. Hence we have

$$
\begin{pmatrix} \hat{\theta} \\ \hat{\lambda}_1 \\ \hat{\lambda}_2 \end{pmatrix} \sim \quad Normal\left[\begin{pmatrix} \theta \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \begin{pmatrix} \hat{V}_{\theta\theta} & \hat{V}_{\theta\lambda_1} & \hat{V}_{\theta\lambda_2} \\ \hat{V}_{\lambda_1\theta} & \hat{V}_{\lambda_1\lambda_1} & \hat{V}_{\lambda_1\lambda_2} \\ \hat{V}_{\lambda_2\theta} & \hat{V}_{\lambda_2\lambda_1} & \hat{V}_{\lambda_2\lambda_2} \end{pmatrix}\right] \tag{27}
$$

(28)

with $\hat{V}_{u,v} = V_{(u,v))}\big|_{(u,v)=(\hat{u},\hat{v})}$, and the asymptotic variance-covariance matrix of the MLEs is,

$$
\begin{pmatrix} \hat{V}_{\theta\theta} & \hat{V}_{\theta\lambda_1} & \hat{V}_{\theta\lambda_2} \\ \hat{V}_{\lambda_1\theta} & \hat{V}_{\lambda_1\lambda_1} & \hat{V}_{\lambda_1\lambda_2} \\ \hat{V}_{\lambda_2\theta} & \hat{V}_{\lambda_2\lambda_1} & \hat{V}_{\lambda_2\lambda_2} \end{pmatrix} = -E\begin{pmatrix} \hat{V}_{\theta\theta} & \hat{V}_{\theta\lambda_1} & \hat{V}_{\theta\lambda_2} \\ \hat{V}_{\lambda_1\theta} & \hat{V}_{\lambda_1\lambda_1} & \hat{V}_{\lambda_1\lambda_2} \\ \hat{V}_{\lambda_2\theta} & \hat{V}_{\lambda_2\lambda_1} & \hat{V}_{\lambda_2\lambda_2} \end{pmatrix}^{-1}
$$

where entries are obtained from

$$
V_{\theta\theta} = \frac{\partial^2}{\partial\theta^2}, \qquad V_{\theta\lambda_1} = \frac{\partial^2}{\partial\theta\partial\lambda_1}, \qquad V_{\theta\lambda_2} = \frac{\partial^2}{\partial\theta\partial\lambda_2}
$$

$$V_{\lambda_1\theta} = \frac{\partial^2}{\partial\lambda_1\partial\theta}, \qquad V_{\theta\lambda_1} = \frac{\partial^2}{\partial\theta\partial\lambda_1}, \qquad V_{\theta\lambda_2} = \frac{\partial^2}{\partial\theta\partial\lambda_2}$$

$$V_{\theta\theta} = \frac{\partial^2}{\partial\theta^2}, \qquad V_{\theta\lambda_1} = \frac{\partial^2}{\partial\theta\partial\lambda_1}, \qquad V_{\theta\lambda_2} = \frac{\partial^2}{\partial\theta\partial\lambda_2}$$

and $l(.)$ is the log-likelihood function given in (23). Approximate $100(1-\phi)\%$ two sided confidence intervals for θ, λ_1, and λ_2 are, respectively, given by

$$\hat{\theta} \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\theta\theta}} \quad , \qquad \hat{\lambda}_1 \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\lambda_1\lambda_1}} \quad , \qquad \text{and} \qquad \hat{\lambda}_2 \pm z_{\frac{\phi}{2}}\sqrt{\hat{V}_{\lambda_2\lambda_2}}$$

where z_ϕ is the upper ϕ^{th} percentile of the standard normal distribution.

5. Simulation Study

In this section, the MLEs of the CRTL distribution parameters θ, λ_1, and λ_2 are evaluated via a simulation study wherein two sets of parameters and sample sizes are tested. The simulation is repeated 1000 times for each combination of parameters and sample size. The estimates of θ, λ_1, and λ_2 and their standard deviations are presented in Table 1. From Table 1, we see that the expected numerical convergence of the parameters and their standard deviations converge to zero. Note that we used R software, R Core Team (2020) to carry out the simulation study and to produce all graphs on this manuscript.

Table 1: MLEs and Standard Deviations.

Sample Size	Actual Values			MLEs			Standard Deviations		
n	θ	λ_1	λ_2	$\hat{\theta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$	$\hat{\theta}$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
50	2.035	0.512	0.702	2.110	0.532	0.521	0.294	0.285	0.421
100	2.035	0.512	0.702	2.077	0.545	0.568	0.225	0.224	0.386
500	2.035	0.512	0.702	2.039	0.517	0.693	0.118	0.104	0.257
50	3.834	0.263	-0.300	3.756	0.260	-0.290	0.370	0.208	0.581
100	3.834	0.263	-0.300	3.823	0.266	-0.272	0.288	0.168	0.464
500	3.834	0.263	-0.300	3.835	0.263	-0.293	0.139	0.075	0.217

6. Applications

In this section, the CRTL distribution is compared with the Lindley (L), Transmuted Lindley (TL), Quasi Lindley (QL), Exponentiated Quasi Lindley (EQL), and Transmuted Generalized Quasi Lindley (TGQL) distributions using two application examples. For the first example, these distributions are used to model the survival times (in days) of guinea pigs infected with virulent tubercle bacilli. The second example considers failure times for the USS Halfbeak number 3 main propulsion diesel engine.

6.1. Guinea Pigs Survival Data

This data set consists of 72 survival times (in days) of guinea pigs infected with virulent tubercle bacilli. The data was observed and reported by Bjerkedal (1960). Other authors who studied this data include Shanker et al. (2016), and Ieren & Abdullahi (2020). Preliminary analysis showed that this data is skewed right with summary statistics minimum , first quartile (Q_1), median, mean, third quartile (Q_3), and maximum given in Table 2.

Table 2: Descriptive statistics for guinea pigs survival data.

Descriptive Statistics						
n	Minimum	Q_1	Median	Mean	Q_3	Maximum
72	0.080	1.080	1.560	1.837	2.303	7.000

The model selection is based on the measures of Akaike information criterion (AIC), the Bayesian information criterion (BIC), the consistent Akaike information criteria (CAIC) and the Hannan-Quinn information criterion (HQIC):

$$AIC = -2l(\hat{\Theta}) + 2q, \qquad BIC = -2l(\hat{\Theta}) + q\ln(n),$$
$$HQIC = -2l(\hat{\Theta}) + 2q\ln(\ln(n)), \quad \text{and} \quad CAIC = -2l(\hat{\Theta}) + \frac{2qn}{n-q-1},$$

where n is the sample size, q is the number of parameters in the model and $l(\hat{\Theta})$ denotes the log-likelihood function evaluated at the MLEs.

Here we note that the smaller the values of goodness-of-fit measures the better the fit of the data. The CRTL distribution is fitted to the data set. The results are compared to the L, QL, EQL, TL, and TGQL distributions based on the test statistics as shown in Table 3. Model parameters are estimated using the MLE method. Results from Table 3 show that the CRTL distribution is the best distribution for fitting this data since it has the lowest negative log-likelihood, AIC, BIC, HQIC, and CAIC test statistic values compared to the other distributions considered in this study.

Table 3: The AIC, BIC, HQIC, CAIC of the guinea pigs survival data.

Model	**Statistics**				
	$-\ell$	AIC	BIC	HQIC	CAIC
L	110.443	226.886	233.716	229.605	222.943
TL	102.677	209.353	213.907	211.166	209.527
CRTL	100.737	207.474	214.304	210.193	207.827
QL	103.720	211.439	215.993	213.252	211.613
EQL	102.277	210.553	217.383	213.272	210.906
TGQL	101.652	211.303	220.410	214.928	211.900

In addition, Figure 3 shows the quantile-quantile (Q-Q) plots for the CRTL, L, QL, EQL, TL, and TGQL distributions, each plotted against the ordered observations. The p^{th} quantile $\hat{Q}(p)$ was estimated from the p^{th} quantile of the fitted distribution and $p = (r - 0.5)/n$, $r = 1, \ldots, n$. The Q-Q plots show that the CRTL distribution provides a better fit for the data, compared to the other distributions in this study. Results from this graphical analysis agree with the results obtained in Table 3. Thus, we conclude that the CRTL distribution is the most appropriate of the distributions to model the guinea pigs survival data compared to the distributions considered in this study.

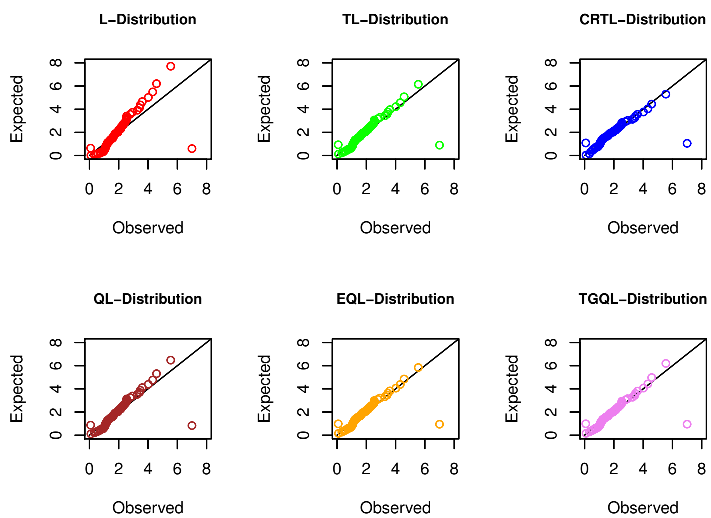

FIGURE 3: Q-Q plots for guinea pigs data.

6.2. USS Halfbeak Data

The second example uses a data set from the from USS Halfbeak diesel engine which was previously studied by Ascher & Feingold (1984), Meeker & Escobar (1998), Pekalp et al. (2014), and Deshpande et al. (2000). This data set consists of 71 cumulative failure times in operating hours (in thousands) to unscheduled maintenance actions for the USS Halfbeak number 3 main propulsion diesel engine. It is assumed that the system was observed until the 71st failure at 25518 hours, Pekalp et al. (2014). Preliminary analysis show that this data is skewed left with summary statistics given in Table 4.

TABLE 4: Descriptive statistics for USS halfbeak data.

Descriptive Statistics						
n	Minimum	Q_1	Median	Mean	Q_3	Maximum
71	1.382	19.119	21.461	19.400	22.768	25.518

The CRTL, L, QL, EQL, TL, and TGQL distributions are fitted to this data. Comparisons are based on the negative log-likelihood, AIC, BIC, HQIC, and CAIC test statistics. Results are shown in Table 5. Model parameters are estimated using the MLE method. Results from Table 5 show that the CRTL distribution is the best distribution for fitting this data since it has the lowest negative log-likelihood, AIC, BIC, HQIC, and CAIC test statistic values compared to the other distributions considered in this study.

TABLE 5: The AIC, BIC, HQIC, CAIC of the USS halfbeak data.

Model	Statistics				
	$-\ell$	AIC	BIC	HQIC	CAIC
L	262.513	527.026	529.288	527.925	527.083
TL	249.991	503.982	508.507	505.781	504.158
CRTL	241.1343	488.269	495.057	490.968	488.627
QL	260.365	524.730	529.256	526.529	524.906
EQL	246.767	499.534	506.322	502.233	499.892
TGQL	246.829	501.657	510.708	505.257	502.264

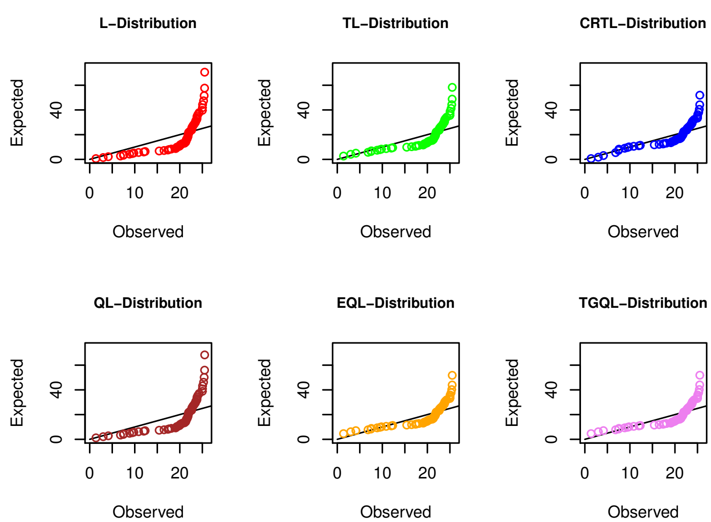

FIGURE 4: Q-Q plots for USS halfbeak data

Figure 4 shows the Q-Q plots for the CRTL, L, QL, EQL, TL, and TGQL distributions, each plotted against the ordered observations. The p^{th} quantile $\hat{Q}(p)$ was estimated from the p^{th} quantile of the fitted distribution as done in example 1. Results from the Q-Q plots show that the CRTL distribution provides a better fit for the data, compared to the other distributions in this study. Overall, we observe a close performance with the TGQL distribution in this case. This graphical analysis also agree with the results obtained in Table 5. Therefore, we conclude that the CRTL distribution is the most appropriate of the distributions to model the USS halfbeak number 3 main propulsion diesel engine data compared to the distributions considered in this study.

7. Conclusion

In this article, we have used the Lindley distribution as the baseline distribution and the cubic rank transmuted map approach to construct the generalized Lindley model. The expressions for several mathematical properties including moments and moment generating function, mean deviation, probability weighted moments, quantile function, reliability analysis and order statistics are derived. A simulation study is used to assess the performance of the MLEs of the parameters. We used two real data sets to show the goodness of fit of the proposed model.

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