In order to commemorate the Nobel Prize in Physics 2022, it is a great honor to publish the invited paper " IMPORTANCE OF QUANTUM ENTANGLEMENT” as a letter to the editor, written by professors Jorge Mahecha and Herbert Vinck of the Universidad de Antioquia and Universidad Nacional de Colombia respectively.

On behalf of the Editorial Committee of MOMENTO - Revista de Física, I want to thank professors Mahecha and Vinck for their kindness and valuable contribution.

Alvaro Mariño
Editor

IMPORTANCE OF QUANTUM ENTANGLEMENT

IMPORTANCIA DEL ENTRELAZAMIENTO CUÁNTICO

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Abstract

An electron has a binary intrinsic property (spin “up” and “down”) and can have binary shifts (move to the right or left). Its spin can be combination of these two, and can have moves that can be combinations of those two. It can also be formed four pairs (intrinsic property, motion): up-left, up-right, down-left, down right. Even combinations of these four possibilities, which is an example of quantum entanglement. Similarly, photons have a binary intrinsic property, polarization.
John Clauser and Alain Aspect experimentally observed that photon pairs can be in quantum entangled states formed with their polarization states. The third 2022 Nobel laureate in physics, along with the above researchers, Anton Zeilinger, experimentally demonstrated that quantum entanglement is applicable in information and quantum communication. Such applications, along with the quantum computing, quantum metrology, quantum microscopy, and others technologies, are called the “second quantum revolution”.

Keywords: quantum mechanics, quantum entanglement, teleportation, Bell tests, Nobel Prize.

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Resumen

Un electrón tiene una propiedad intrínseca binaria (espín "hacia arriba" y "hacia abajo") y puede tener desplazamientos binarios (moverse hacia la derecha o hacia la izquierda). Su espín puede ser combinación de los dos mencionados, y puede tener movimientos que pueden ser combinaciones de dichos dos. También se pueden formar cuatro pares (propiedad intrínseca, desplazamiento): arriba hacia la izquierda, arriba hacia la derecha, abajo hacia la izquierda, abajo hacia la derecha. Incluso combinaciones de estas cuatro posibilidades, que es un ejemplo del entrelazamiento cuántico. De manera similar, los fotones tienen una propiedad intrínseca binaria, la polarización.
John Clauser y Alain Aspect observaron experimentalmente que un par de fotones puede estar en estados cuánticos entrelazados formados con sus estados de polarización. El tercer premio Nobel de física 2022, junto con los investigadores mencionados anteriormente, Anton Zeilinger, demostró experimentalmente que el entrelazamiento cuántico es aplicable en información y comunicación cuántica. Tales aplicaciones, junto con la computación cuántica, metrología cuántica, microscopía cuántica, y otras tecnologías, se denominan "segunda revolución cuántica".

Palabras clave: mecánica cuántica, entrelazamiento cuántico, teleportación, pruebas de Bell, Premio Nobel.

Introduction

A theory of the periodic table of the chemical elements is not possible without assuming the **quantization** of the **energy** of the electrons in atoms and their **orbital angular momentum**. In addition, it is required to attribute to the electrons a new type of angular momentum, the **spin**. With these ideas, a model was achieved that describes the main attributes of said table. It assigns to each electron of a given atom a unique set of quantum numbers (n, l, m_l, m_s). This is the **shell model of the atom** from which the Mendeleev chemical periodicity can be obtained.

The idea of **probability** is not foreign to classical mechanics. In 1838 Joseph Liouville presented a probabilistic version of it. In *Les Méthodes Nouvelles de la Mécanique Céleste* by Henri Poincaré, published 1892 - 1899, it is argued that most mechanical models are intrinsically random. Therefore it is possible to replace the description in terms of the trajectories of the individual particles by one based on probability distributions.

What is new in quantum mechanics at this point consists in the introduction of the **complex probability amplitude distribution** whose modulus squared is the **probability density distribution**. This a union of complex numbers, superposition and probability, a concept strange in conventional **mathematical statistics**.

Before the invention of quantum mechanics was known that probability amplitudes provide models suitable for describing interference and diffraction phenomena (colloquially called **"wave"**).

A new definition of **quantum mechanics** is possible: "The set of mathematical models that are deduced from the idea of **complex amplitude of probability** and that are used to interpret phenomena of microscopic scales".

This represents a large separation from the models of the **classical mechanics**.

The angular momentum in quantum mechanics

In **three-dimensional** motion, consider the quantization of the **orbital angular momentum** and its projection on Z direction. The result is,

$$\begin{aligned} \hat{\boldsymbol{L}}^2 &\rightarrow \hbar^2 L(L+1), \quad L = 0, 1, 2, ... \\ \hat{L}_z &\rightarrow M_L \hbar, \qquad\quad M_L = 0, \pm 1, \pm 2, ... \pm L \end{aligned}$$

The **spin** is an **intrinsic angular momentum**, not associated with spatial motions, whose "L" takes value 1/2. Its quantization is described by,

$$\begin{aligned} \hat{\boldsymbol{S}}^2 &\rightarrow \hbar^2 S(S+1), \quad S = 1/2 \\ \hat{S}_z &\rightarrow M_S\hbar, \qquad\quad M_S = \pm 1/2. \end{aligned}$$

The total angular momentum $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$, the sum of one orbital angular momentum and one of spin, according to quantum mechanics, it is described with similar expressions,

$$\begin{aligned} \hat{\boldsymbol{J}}^2 &\rightarrow \hbar^2 J(J+1) \\ \hat{J}_z &\rightarrow M_J\hbar, \qquad\quad M_J = -J, -J+1, ...J. \end{aligned}$$

The **quantum number** J of the **total angular momentum** can take values $J = 1/2$ if $L = 0$ and $J = L \pm 1/2$ if $L \neq 0$.

Angular momentum quantum states

Two components of an angular momentum vector "do not commute," that means that they cannot be measured simultaneously with high precision, they are subject to the "uncertainty principle" ($\boldsymbol{\Delta L_x \Delta L_y \geq (\hbar^2/2)|M_L|}$). In this way, the **reality**, so appreciated by Einstein, Podolski and Rosen [1], in the case of this vector, is questioned, because the constancy of angular momentum vector, expressed by $[\hat{H}, \hat{\boldsymbol{L}}] = 0$, does not translate into the simultaneous existence of its components. Using the arguments of the EPR paper, we could say: "when the Z projection and the magnitude of the angular momentum vector are known, its X and Y projections have no physical reality."

In fact, the square, $\hat{\boldsymbol{L}}^2$, and an one of the components, for example $\hat{L}_z$, commute, so they have simultaneous **eigenstates**. Such states are **vectors of a $\boldsymbol{2L+1}$ dimensional Hilbert space** which can be labeled with the eigenvalues of those commuting operators, $\boldsymbol{|L, M_L\rangle}$.

For the case of orbital angular momentum:

$$\hat{\boldsymbol{L}}^2|L, M_L\rangle = \hbar^2 L(L+1)|L, M_L\rangle, \quad \hat{L}_z|L, M_L\rangle = M_L\hbar|L, M_L\rangle.$$

For a spin 1/2, $|\mathbf{1/2}, \boldsymbol{M_S}\rangle$, with $\boldsymbol{M_S} = \pm\mathbf{1/2}$, the Hilbert space is two dimensional:

$$\hat{\boldsymbol{S}}^2|1/2, M_S\rangle = \frac{3}{4}\hbar^2|1/2, M_S\rangle, \quad \hat{S}_z|1/2, M_S\rangle = M_S\hbar|1/2, M_S\rangle.$$

The Nobel prize 2022

The Nobel Prize in Physics 2022 was awarded jointly to Alain Aspect (Institut d'Optique Graduate School - Université Paris-Saclay and École Polytechnique, Palaiseau, France), John F. Clauser (J.F. Clauser & Assoc., Walnut Creek, California, USA) and Anton Zeilinger (University of Vienna, Austria) "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science."

In 1935, A. Einstein, B. Podolsky and N. Rosen argued that quantum mechanics considers physical quantities which have not reality, and therefore is an incomplete theory [1]. In 1964, J. S. Bell proposed one experiment to decide if the EPR's objection is valid or not [2]. The purpose of the experiment was to prove if some inequalities found by Bell, in order to test the existence of hidden-variables, are violated, indicating that the EPR's objection is not valid.

In 1969, Clauser presented ideas about a possible experiment to test local hidden-variable theories [3], which was performed, jointly with S. J. Freedman, in 1972 [4]. Experiment "provides strong evidence against local hidden-variable theories," they say.

In 1975, Aspect proposed an experimental scheme to test hidden-variable theories that satisfy the principle of separability of Einstein [5]. Experiment was performed jointly with J. Dalibard and G. Roger in 1982 [6]. They claimed that their results "are in good agreement with quantum mechanical predictions and violate Bell's inequalities by 5 standard deviations."

In 1992 C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters presented a theoretical proposal about "teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels" [7]. With their proposal, "Bob can convert the state of his EPR particle into an exact replica of the unknown state $|\phi\rangle$ which Alice destroyed," they concluded.

Zeilinger, jointly with D. Bouwmeester, J.W. Pan, K. Mattle, M. Eibl, H. Weinfurter, reported in 1997 one experimental realization of quantum teleportation [8]. They concluded that "During teleportation, an initial photon which carries the polarization that is to be transferred and one of a pair of entangled photons are subjected to a measurement such that the second photon of the entangled pair acquires the polarization of the initial photon. This latter photon can be arbitrarily far away from the initial one. Quantum teleportation will be a critical ingredient for quantum computation networks."

Qubits

According to quantum mechanics, the state of a quantum system is given by a vector of certain linear space, in general complex. The simplest case is a system whose state space is a two-dimensional Hilbert space, denoted by $\mathcal{H}$. Two-dimensional means that any vector, in Dirac notation, can be expanded into a pair of basis vectors conventionally denoted $|0\rangle$ and $|1\rangle$. Complex means that any vector $|\psi\rangle$ of $\mathcal{H}$ can be expanded as a linear combination of basis vectors with complex coefficients α and β,

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$$

The norm of any vector $|\psi\rangle$ in $\mathcal{H}$ is defined by

$$||\psi|| = \langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1.$$

where

$$\langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1|.$$

In the jargon of quantum mechanics $\langle\psi|$ is called "bra" and $|\psi\rangle$ "ket". The scalar product of $|\psi\rangle$ with other vector $|\phi\rangle = \gamma|0\rangle + \delta|1\rangle$ is

$$\langle\psi|\phi\rangle = \alpha^*\gamma + \beta^*\delta = \langle\phi|\psi\rangle^*.$$

The scalar products of pairs of basis vectors are $\langle 0|0\rangle = \langle 1|1\rangle = 1$, $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

Linear operators, denoted like $\widehat{A}$, transform the vectors of $\mathcal{H}$,

$$|\psi\rangle' = \widehat{A}|\psi\rangle = \alpha\widehat{A}|0\rangle + \beta\widehat{A}|1\rangle.$$

We can write $\widehat{A}|0\rangle = a_{00}|0\rangle + a_{10}|1\rangle$ and $\widehat{A}|1\rangle = a_{01}|0\rangle + a_{11}|1\rangle$.

Then,

$$|\psi\rangle' = (\alpha a_{00} + \beta a_{01})|0\rangle + (\alpha a_{10} + \beta a_{11})|1\rangle = \alpha'|0\rangle + \beta'|1\rangle.$$

This suggest associate to vector $|\psi\rangle$ a 2-dimensional vector, and to operator $\widehat{A}$ a 2×2 matrix, with complex components,

$$|\psi\rangle \to \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \widehat{A} \to \begin{pmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \end{pmatrix}, \quad \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Bloch sphere

The expansion coefficients α, β of an arbitrary one-qubit state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

satisfy $|\alpha|^2 + |\beta|^2 = 1$. Then, there exist θ and ϕ angles such that $\alpha = \cos\frac{\theta}{2}$, $\beta = e^{i\phi}\sin\frac{\theta}{2}$, and a general state can be expresed by

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle.$$

Due to the action of Pauli matrices on the basis states is,

$$\begin{array}{rclrcl} \sigma_x|0\rangle &=& |1\rangle, & \sigma_x|1\rangle &=& |0\rangle, \\ \sigma_y|0\rangle &=& i|1\rangle, & \sigma_y|1\rangle &=& -i|0\rangle, \\ \sigma_z|0\rangle &=& |0\rangle, & \sigma_z|1\rangle &=& -|1\rangle, \end{array}$$

then the average of Pauli matrix σ_x in this state is

$$\overline{\sigma_x} = \langle\psi|\sigma_x|\psi\rangle = (e^{i\phi} + e^{-i\phi})\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \sin\theta\cos\phi,$$

and $\overline{\sigma_y} = \sin\theta\sin\phi$, $\overline{\sigma_z} = \cos\theta$. This means that $\{\overline{\sigma_x}, \overline{\sigma_y}, \overline{\sigma_z}\}$ are the cartesian coordinates of a point on the surface of one sphere of radius one. See Figure 1.

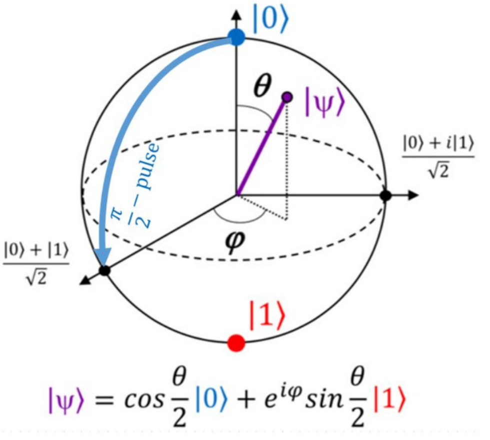

FIGURE 1. *Bloch sphere. Points on the surface represent one-qubit states* $|\psi\rangle$*. Blue arc represents a one-qubit gate. Google Image.*

Physical implementation of qubit gates

Let be a single-particle system formed by one **1/2 spin**. In quantum mechanics, it can be described by the spin commuting operators $\widehat{\boldsymbol{s}}^2$, $\widehat{s}_z$ and the correspondig eigenstates $|1/2, m_s\rangle$, for $m_s = \pm 1/2$.

Components of vector spin operator $\widehat{\boldsymbol{s}}$ are related to Pauli matrices

$$\widehat{s}_x = \frac{\hbar}{2}\sigma_x, \ \widehat{s}_y = \frac{\hbar}{2}\sigma_y, \ \widehat{s}_z = \frac{\hbar}{2}\sigma_z.$$

Action of Pauli σ_x matrix on the spin states $|\boldsymbol{s}, \boldsymbol{m_s}\rangle$,

$$|0\rangle \equiv |\mathbf{1/2, 1/2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ |1\rangle \equiv |\mathbf{1/2, -1/2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

is

$$\sigma_x|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle.$$

$$\sigma_x|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle.$$

$\boldsymbol{\sigma_x}$ produces the logical operation NOT.

Similarly,

$$\sigma_y|0\rangle = i|1\rangle, \ \sigma_y|1\rangle = -i|0\rangle, \ \sigma_z|0\rangle = |0\rangle, \ \sigma_z|1\rangle = -|1\rangle.$$

If are defined $\widehat{s}_{\pm} = \widehat{s}_x \pm i\widehat{s}_y$, it is noticed that

$$\widehat{s}_{+} = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\widehat{s}_{-} = \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\widehat{s}_{+}|0\rangle = 0,\ \widehat{s}_{+}|1\rangle = \hbar|0\rangle,\ \widehat{s}_{-}|0\rangle = \hbar|1\rangle,\ \widehat{s}_{-}|1\rangle = 0.$$

In terms of Pauli matrices, defining $\sigma_{\pm} = \sigma_x \pm i\sigma_y$,

$$\sigma_{+}|0\rangle = 0,\ \sigma_{+}|1\rangle = 2|0\rangle,\ \sigma_{-}|0\rangle = 2|1\rangle,\ \sigma_{-}|1\rangle = 0.$$

Any transformation of points of the Bloch sphere can be used as a one-qubit quantum gate. See Figure 1. Phase and Hadamard gates are examples.

Several Qubits Systems: Quantum Entanglement

Two-qubit states

Let be two single-qubit systems, A and B. Each of them has an associated Bloch sphere whose points represent states $|\psi\rangle_A = \alpha|0\rangle_A + \beta|1\rangle_A$ and $|\phi\rangle_B = \lambda|0\rangle_B + \mu|1\rangle_B$, respectively. States of the composite quantum system "A+B" can be defined, they are linear combinations of the orthogonal basis formed by direct product of the basis vectors of the individual single-qubit systems, $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$

$$|\Theta\rangle_{AB} = c_{00}|0\rangle_A \otimes |0\rangle_B + c_{01}|0\rangle_A \otimes |1\rangle_B + c_{10}|1\rangle_A \otimes |0\rangle_B + c_{11}|1\rangle_A \otimes |1\rangle_B.$$

Direct product of $|\psi\rangle_A$ and $|\phi\rangle_B$ gives,

$$|\psi\rangle_A \otimes |\phi\rangle_B = \alpha\lambda|0\rangle_A \otimes |0\rangle_B + \alpha\mu|0\rangle_A \otimes |1\rangle_B + \beta\lambda|1\rangle_A \otimes |0\rangle_B + \beta\mu|1\rangle_A \otimes |1\rangle_B.$$

Then, $|\Theta\rangle_{AB}$ can be written as a direct product if

$$c_{00}c_{11} = c_{01}c_{10},$$

or $(\alpha\lambda)(\beta\mu) = (\alpha\mu)(\beta\lambda)$.

The following two-qubit states, called Bell states, can not be "factorized," are called "entangled" states,

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

In $|\Psi^+\rangle$, for example, $c_{00} = c_{11} = 0$ and $c_{01} = c_{10} = 1/\sqrt{2}$, gives $c_{00}c_{11} \neq c_{01}c_{10}$.

Entanglement

Examples of two-qubit entangled states are the Bell states, which can be written with a simplified notation as, $\sqrt{2}|\phi^{\pm}\rangle = |00\rangle \pm |11\rangle$, $\sqrt{2}|\psi^{\pm}\rangle = |01\rangle \pm |10\rangle$. The four **binary pairs of symbols** (ψ,+), (ψ,-), (ϕ,+), (ϕ,-), can be expressed by using **two bits (*x*, *y*)** as follows, (ψ,+) = (1,0), (ψ,-) = (1,1), (ϕ,+) = (0,0), (ϕ,-) = (0,1). These four quantum states can be produced by using the 1-qubit Hadamard gate and 2-qubit CNOT gate [9],

$$|x^y\rangle = \widehat{CNOT}\widehat{H}_1|x, y\rangle.$$

A general 2-qubit entangled state is

$$|E\rangle = \cos\theta|0x\rangle + \sin\theta|1\overline{x}\rangle,$$

which give Bell states when $\theta = \pm\pi/4$, and separable states if $\theta = 0$ or $\theta = \pi/2$.

Many-qubit pure quantum states can be clasified according to their degree of entanglement. In the two-qubit case, an index whose value is 0 for unentangled states and 1 for the maximally entangled states can be defined. The entropy of entanglement, the concurrence, the quantum discord, the entanglement of formation, the entanglement cost, the distillable entanglement, the squashed entanglement, the log-negativity, the robustness monotones, the greatest crossnorm, are some of those entanglement measures [9]. For example, the von Neumann entropy of state $|E\rangle$ is 1 for $\theta = \pm\pi/4$, and 0 for $\theta = 0$ or $\theta = \pi/2$, and between 0 and 1 for some other values of θ.

Bell inequalities

For brevity we write the Bell state $|\Psi^-\rangle$ as $\sqrt{2}|\Psi\rangle = |01\rangle - |10\rangle$.

Originally, Bohm and other who were analyzing the EPR problem, thinked in a source emitting a pair of spin 1/2 particles in this entangled state. Then, we could write the state in this form,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),$$

in order to represent the spin states with projection parallel or antiparallel to Z axis.

It is evident that there is nothing that intrinsically favours the Z axis. If we had chosen another direction, $\hat{\boldsymbol{n}}$, instead of the south-north line of Bloch sphere to project the spins, we could write the pair of entangled spins in this form,

$$\frac{1}{\sqrt{2}}(|\uparrow_{\hat{\boldsymbol{n}}}\downarrow_{\hat{\boldsymbol{n}}}\rangle - |\downarrow_{\hat{\boldsymbol{n}}}\uparrow_{\hat{\boldsymbol{n}}}\rangle).$$

The polar angles of vector $\hat{\boldsymbol{n}}$ in the Bloch sphere can be used to expand the new pairs in terms of spins aligned or antialigned along Z axis,

$$|\uparrow_{\hat{\boldsymbol{n}}}\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow\rangle, \quad |\downarrow_{\hat{\boldsymbol{n}}}\rangle = -\sin\frac{\theta}{2}|\uparrow\rangle + e^{i\phi}\cos\frac{\theta}{2}|\downarrow\rangle.$$

Except for a global phase factor $e^{i\phi}$, it is found that

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle), = \frac{1}{\sqrt{2}}(|\uparrow_{\hat{\boldsymbol{n}}}\downarrow_{\hat{\boldsymbol{n}}}\rangle - |\downarrow_{\hat{\boldsymbol{n}}}\uparrow_{\hat{\boldsymbol{n}}}\rangle).$$

Measurements of the spin projection of A and B can be performed by using sets of measurement operators of the form $(|\uparrow\rangle\langle\uparrow|, |\downarrow\rangle\langle\downarrow|)$ or $(|\uparrow_{\hat{\boldsymbol{n}}}\rangle\langle\uparrow_{\hat{\boldsymbol{n}}}|, |\downarrow_{\hat{\boldsymbol{n}}}\rangle\langle\downarrow_{\hat{\boldsymbol{n}}}|)$. For example,

$$|\downarrow_A\rangle\langle\downarrow_A|\Psi\rangle = |\downarrow_A\rangle\langle\downarrow_A|\frac{1}{\sqrt{2}}(|\uparrow_A\rangle|\downarrow_B\rangle - |\downarrow_A\rangle|\uparrow_B\rangle) = -\frac{1}{\sqrt{2}}|\downarrow_A\rangle|\uparrow_B\rangle.$$

If now is performed a measurement of spin B, it is found that it is in state $|\uparrow_B\rangle$ with probability 1. If spin A is measured with result down, that measurement projects $|\Psi\rangle$ on a state where spin B is up. Analogous results are obtained if measurements of spin projection, along any axis, are performed: in all cases, the results of measurements of spin projections along any axis are anticorrelated.

It is easily proven that measurements $|\downarrow\rangle\langle\downarrow|$ and $|\downarrow_{\hat{\boldsymbol{n}}}\rangle\langle\downarrow_{\hat{\boldsymbol{n}}}|$, for A or B, do not commute, for any direction $\hat{\boldsymbol{n}}$ different from Z. Then, according to the objection raised by EPR, individual spins cannot be assigned a definite direction, that is, such a direction has no physical meaning for any of them. In order to recover that meaning, they consider that quantum mechanics needs some "hidden-variables" in order to deal with physical realities independent of the choise of measurement operators.

The ambiguity of the entangled state regarding the choice that A can make of a measurement operator within infinite possibilities and thereby affect the alignment of spin B, is what leads EPR to question the reality of said alignment.

In 1964 Bell found an inequality that must hold if the spin alignments are physical realities independent of the measurements. That is, if there are hidden-variables.

He considered three directions for measuring such alignments, corresponding to spin A and spin B. If measured along the $\hat{\boldsymbol{n}}$-axis and obtained upward, the result is said to be $\hat{\boldsymbol{n}}+$, and similarly for the other measurements. As there are 3 axes and two possible results for each one, there are $2^3 = 8$ possible observations, which are shown in the table. The axes are called $\hat{\boldsymbol{m}}$, $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{o}}$.

Bell proposed an experiment which consists in measuring all the 8 possible alignements and results.

Measurements	A	B
N_1	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}+)$	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}-)$
N_2	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}-)$	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}+)$
N_3	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}+)$	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}-)$
N_4	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}-)$	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}+)$
N_5	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}+)$	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}-)$
N_6	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}-)$	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}+)$
N_7	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}+)$	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}-)$
N_8	$(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}-)$	$(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}+)$

Due to measurements of spin projections of A and B are anticorrelated, if A particle belongs to the group $(\hat{\boldsymbol{m}}-, \hat{\boldsymbol{n}}+, \hat{\boldsymbol{o}}+)$, then necessarily measurements of B must to be into the group $(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}-, \hat{\boldsymbol{o}}-)$.

If $p(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+)$ is the probability of measuring projection of spin A along $\hat{\boldsymbol{m}}$ axis obtaining up and measuring projection of spin B along

$\hat{\boldsymbol{n}}$ obtaining up, and if the total number of measurements is $N_T = N_1 + ...N_8$, then

$$p(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+) = \frac{N_3 + N_4}{N_T}.$$

Similarly,

$$p(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{o}}+) = \frac{N_2 + N_4}{N_T}, \quad p(\hat{\boldsymbol{o}}+, \hat{\boldsymbol{n}}+) = \frac{N_3 + N_7}{N_T}.$$

Then, we note that $N_3 + N_4 \leq (N_3 + N_7) + (N_4 + N_2)$. Dividing by N_T, one gets the Bell inequality,

$$p(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+) \leq p(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{o}}+) + p(\hat{\boldsymbol{o}}+, \hat{\boldsymbol{n}}+).$$

The corresponding quantum calculations require considering probability amplitudes.

Quantum evaluation of probabilities of Bell inequality

If measurement of projection of spin A along $\hat{\boldsymbol{m}}$ axis gives +, then spin state of B along that axis will be −. Now, the probability amplitude of a measurement of projection of spin B along $\hat{\boldsymbol{n}}$ with result + will be

$$\langle \uparrow_{\hat{\boldsymbol{n}}} \mid \downarrow_{\hat{\boldsymbol{m}}} \rangle.$$

The involved states are

$$\mid \uparrow_{\hat{\boldsymbol{n}}} \rangle = \cos\frac{\theta}{2} \mid \uparrow\rangle + e^{i\phi} \sin\frac{\theta}{2} \mid \downarrow\rangle, \quad \mid \downarrow_{\hat{\boldsymbol{m}}} \rangle = -\sin\frac{\theta'}{2} \mid \uparrow\rangle + e^{i\phi'} \cos\frac{\theta'}{2} \mid \downarrow\rangle,$$

where the angles of vector $\hat{\boldsymbol{m}}$ are θ, ϕ, and those of vector $\hat{\boldsymbol{n}}$ are θ', ϕ'. An straightforward calculation gives

$$|\langle \uparrow_{\hat{\boldsymbol{n}}} \mid \downarrow_{\hat{\boldsymbol{m}}} \rangle|^2 = \sin^2\frac{\theta_{\hat{\boldsymbol{m}}\hat{\boldsymbol{n}}}}{2},$$

where $\theta_{\hat{\boldsymbol{m}}\hat{\boldsymbol{n}}}$ is the angle between axes $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{n}}$.
Measurement of alignment along any direction has probability 1/2 of resulting + and probability 1/2 of resulting −. This means that

$$p(\hat{\boldsymbol{m}}+, \hat{\boldsymbol{n}}+) = \frac{1}{2}\sin^2\frac{\theta_{\hat{\boldsymbol{m}}\hat{\boldsymbol{n}}}}{2}.$$

Similarly can be evaluated the other probabilities involved in the Bell's inequality, which in the quantum case takes the form

$$\sin^2\frac{\theta_{\hat{\boldsymbol{m}}\hat{\boldsymbol{n}}}}{2} \leq \sin^2\frac{\theta_{\hat{\boldsymbol{m}}\hat{\boldsymbol{o}}}}{2} + \sin^2\frac{\theta_{\hat{\boldsymbol{o}}\hat{\boldsymbol{n}}}}{2}.$$

We can choose the axes $\hat{\boldsymbol{m}}$, $\hat{\boldsymbol{n}}$, $\hat{\boldsymbol{o}}$ in such a way that that they are in the same plane and

$$\theta_{\hat{\boldsymbol{m}}\hat{\boldsymbol{n}}} = 2\theta, \quad \theta_{\hat{\boldsymbol{m}}\hat{\boldsymbol{o}}} = \theta_{\hat{\boldsymbol{o}}\hat{\boldsymbol{n}}} = \theta.$$

Then

$$\sin^2\theta \leq \sin^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}, \text{ or } \sin^2\theta \leq 2\sin^2\frac{\theta}{2}, \text{ or } 2\cos^2\frac{\theta}{2} \leq 1.$$

Clearly the inequality is violated for $0 < \theta < \pi/2$.

Quantum Teleportation

Let's suppose that Alice and Bob **share a pair of entangled states**, which we suppose is the $|\boldsymbol{\psi}^+\rangle$. Each store its qubit, by using a device to protect it from decoherence, and without measuring it to know if it is in the state $|0\rangle$ or $|1\rangle$.

Alice owns another qubit, $|\boldsymbol{\eta}\rangle_{\boldsymbol{a}} = \boldsymbol{\alpha}|\mathbf{0}\rangle + \boldsymbol{\beta}|\mathbf{1}\rangle$ and needs to send it, but not through a quantum channel but by sending a few ordinary bits through conventional communication networks. Note that she cannot send α and β, because she does not know them, and also because they are, in general, irrational numbers, their encoding would require an enormous number of bits.

Quantum teleportation is a means to do this task which uses the entangled state shared with Bob.

There are 3 qubits, $|\eta\rangle_a$ and the Bell pair $|\psi^+\rangle_{AB}$. We use "a" and "A" to designate the qubits that Alice has and "B" that of Bob. Then, the 3 qubit system is

$$|\eta\rangle_a \otimes |\psi^+\rangle_{AB} = (\alpha|0\rangle_a + \beta|1\rangle_a) \otimes \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B).$$

It can also be written as follows,

$$\frac{1}{\sqrt{2}}\alpha(|0\rangle_a|0\rangle_A|1\rangle_B + |0\rangle_a|1\rangle_A|0\rangle_B) + \frac{1}{\sqrt{2}}\beta(|1\rangle_a|0\rangle_A|1\rangle_B + |1\rangle_a|1\rangle_A|0\rangle_B).$$

Alice owns a device to create a Bell state with her two qubits [9],

$$|x^y\rangle_{aA} = \widehat{CNOT}\widehat{H}_a|x\rangle_a|y\rangle_A.$$

Of course, she can obtain 4 possible Bell, because she don't know if her qubits are $|0\rangle_a|0\rangle_A$, $|0\rangle_a|1\rangle_A$, $|1\rangle_a|0\rangle_A$, or $|1\rangle_a|1\rangle_A$.

Due to
$$|0\rangle_a|0\rangle_A = \frac{1}{\sqrt{2}}(|\phi^+|_{aA} + |\phi^-|_{aA}),$$
$$|1\rangle_a|1\rangle_A = \frac{1}{\sqrt{2}}(|\phi^+|_{aA} - |\phi^-|_{aA}),$$
$$|0\rangle_a|1\rangle_A = \frac{1}{\sqrt{2}}(|\psi^+|_{aA} + |\psi^-|_{aA}),$$
$$|1\rangle_a|0\rangle_A = \frac{1}{\sqrt{2}}(|\psi^+|_{aA} - |\psi^-|_{aA}),$$
the 3-qubit state can be written in the form
$$\frac{1}{2}|\psi^+\rangle_{aA}(\alpha|0\rangle_B + \beta|1\rangle_B)+\frac{1}{2}|\psi^-\rangle_{aA}(\alpha|0\rangle_B - \beta|1\rangle_B)+$$
$$\frac{1}{2}|\phi^+\rangle_{aA}(\alpha|1\rangle_B + \beta|0\rangle_B)+\frac{1}{2}|\phi^-\rangle_{aA}(\alpha|1\rangle_B - \beta|0\rangle_B).$$
Now, Alice can determine the Bell state of her two qubits. By applying
$$\widehat{H}_a\widehat{CNOT}|x^y\rangle_{aA} = |x\rangle_a|y\rangle_A,$$
she obtains the two qubits $|x\rangle$, $|y\rangle$ which specify the Bell state. From each pair are obtained the bits $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, with equal probabiliy, $1/4$.

Those measurements performed by Alice "collapse" the 3-qubit state in one of the four components. If she measures $|\psi^+\rangle_{aA}$, $|\psi^-\rangle_{aA}$, $|\phi^+\rangle_{aA}$, or $|\phi^-\rangle_{aA}$, the qubit of Bob will be prepared into the state $\alpha|0\rangle_B + \beta|1\rangle_B$, $\alpha|0\rangle_B - \beta|1\rangle_B$, $\alpha|1\rangle_B + \beta|0\rangle_B$, or $\alpha|1\rangle_B - \beta|0\rangle_B$, respectively, but he ignores the precise qubit.

If Alice send the two bits (x, y) through a conventional communication network, Bob can determine the precise qubit. If he receives $(0,0)$, his qubit is $\alpha|1\rangle_B + \beta|0\rangle_B$, which is converted in a qubit identical to the $|\eta\rangle_a$, that is $\alpha|0\rangle_B + \beta|1\rangle_B$, by appliyng σ_x to his qubit. Analogously, if he receives $(0,1)$, he must to do nothing, if $(1,0)$, he must to apply $i\sigma_y$, and if $(1,1)$, he must to apply σ_z. See Figure 2.

Grover's Search Algorithm

The theory of quantum information physics has reached a great level of maturity [10]. Here we will review a quantum algorithm which shows some of these ideas.

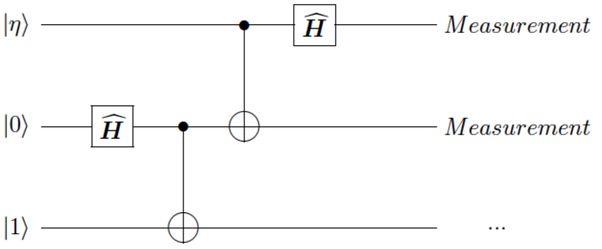

FIGURE 2. *Quantum circuit to perform the quantum teleportation scheme.*

If you want to find the position occupied by a number y in a list made up of N numbers, a conventional algorithm requires making the order of $N/2$ queries to the list of data. Grover's algorithm allows to find y with only order of $\sqrt{N}$ operations.

The database consists of a list of n-bit numbers, with a maximum size of $N = 2^n$ numbers $x \in \{0, 1, 2, ...2^n - 1\}$. One of those numbers is the searched y. Then, one can define a binary function which evaluates $f(x)$ as 0 if $x \neq y$ and 1 if $x = y$,

$$f(x) = \delta_{xy}.$$

The quantum algorithm requires to store one x, given in binary basis by one string of n zeros and ones $x_0 x_1 ... x_{n-1}$, into a quantum state of n qubits,

$$|x\rangle = |x_0\rangle \otimes |x_1\rangle \otimes ...|x_{n-1}\rangle \equiv |x_0 x_1 ... x_{n-1}\rangle. \quad x_i = 0 \text{ or } 1.$$

We define the oracle operator $\widehat{O}$ as follows,

$$\widehat{O}|x\rangle = (-1)^{f(x)}|x\rangle,$$

and one operator with eigenvalue 1 when acts on the state $|0\rangle$ and -1 when acts on the n-qubit state $|x\rangle$ for $x \neq 0$,

$$\widehat{F}_c|x\rangle = -(-1)^{\delta_{x0}}|x\rangle = (2|0\rangle\langle 0| - \widehat{I})|x\rangle.$$

The Grover's operator is defined by

$$\widehat{G} = \widehat{H}^{\otimes n}\widehat{F}_c\widehat{H}^{\otimes n}\widehat{O}.$$

Let be the initial state $|\Psi\rangle$ an equal weigth superposition of all the states $|x\rangle$,

$$|\Psi\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle = \frac{1}{\sqrt{N}}\sum_{x_0=0}^{1}\sum_{x_1=0}^{1}...\sum_{x_{n-1}=0}^{1}|x_0\rangle|x_1\rangle...|x_{n-1}\rangle = \widehat{H}^{\otimes n}|0\rangle^{\otimes n},$$

where it was used that $N = 2^n$ and $\widehat{H}|0\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$. That superposition is the core of the called "quantum parallelism".

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where it was used that $N = 2^n$ and $\widehat{H}|0\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$.
Taking into account that $\widehat{H}^2 = \widehat{I}$, the Grover's operator can be written as

$$\widehat{G} = \widehat{H}^{\otimes n}(2|0\rangle\langle 0|-\widehat{I})\widehat{H}^{\otimes n}\widehat{O} = (2\widehat{H}^{\otimes n}|0\rangle\langle 0|\widehat{H}^{\otimes n}-\widehat{I})\widehat{O} = (2|\Psi\rangle\langle\Psi|-\widehat{I})\widehat{O}.$$

The reason of defining operator $\widehat{G}$ is that it allows to perform a series of rotations which can project the state $|\Psi\rangle$ on the desired state $|y\rangle$.

We can decompose $|\Psi\rangle$ into two orthogonal components, one the state $|y\rangle$, and other formed by the set of states $|x\rangle$ which do not includes $|y\rangle$. Let be the normalised state

$$|\alpha\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq y}|x\rangle.$$

Then,

$$|\Psi\rangle = \sqrt{1-\frac{1}{N}}|\alpha\rangle + \frac{1}{\sqrt{N}}|y\rangle.$$

Next, we define angle θ such as

$$\cos\theta = \sqrt{1-\frac{1}{N}}, \quad \sin\theta = \frac{1}{\sqrt{N}}, \quad \tan\theta = \frac{1}{\sqrt{N-1}}.$$

Then,

$$|\Psi\rangle = \cos\theta|\alpha\rangle + \sin\theta|y\rangle.$$

If we apply operator $\widehat{O}$, we obtain

$$\widehat{O}|\Psi\rangle = \cos\theta|\alpha\rangle - \sin\theta|y\rangle.$$

This can be interpreted as a **reflection** in the plane of vectors $|\alpha\rangle$-$|y\rangle$ with respect to $|\alpha\rangle$. A vector orthogonal to $|\Psi\rangle$ is

$$|\Psi_\perp\rangle = \sin\theta|\alpha\rangle - \cos\theta|y\rangle.$$

We notice that

$$\widehat{O}|\Psi_\perp\rangle = \sin\theta|\alpha\rangle + \cos\theta|y\rangle.$$

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$$\cos\theta = \sqrt{1-\frac{1}{N}}, \quad \sin\theta = \frac{1}{\sqrt{N}}, \quad \tan\theta = \frac{1}{\sqrt{N-1}}.$$

Then,

$$|\Psi\rangle = \cos\theta|\alpha\rangle + \sin\theta|y\rangle.$$

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We notice that

$$\widehat{O}|\Psi_\perp\rangle = \sin\theta|\alpha\rangle + \cos\theta|y\rangle.$$

Now, let's consider the plane of vectors $|\Psi\rangle$-$|\Psi_\perp\rangle$ and apply the operator $2|\Psi\rangle\langle\Psi| - \widehat{I}$ to a vector of this plane,

$$(2|\Psi\rangle\langle\Psi| - \widehat{I})(\lambda|\Psi\rangle + \mu|\Psi_\perp\rangle) = \lambda|\Psi\rangle - \mu|\Psi_\perp\rangle,$$

it gives a reflection with respect to vector $|\Psi\rangle$, it leaves $|\Psi\rangle$ unchanged. Then, the action of operator $\widehat{G} = (2|\Psi\rangle\langle\Psi| - \widehat{I})\widehat{O}$ consists on two succesive reflections: around the vector $|\alpha\rangle$ and around the vector $|\Psi\rangle$.

$$\widehat{G}|\Psi\rangle = (2|\Psi\rangle\langle\Psi| - \widehat{I})\widehat{O}|\Psi\rangle.$$

Let's write $\widehat{O}|\Psi\rangle = \cos\theta|\alpha\rangle - \sin\theta|y\rangle$ as a combination of $|\Psi\rangle$ and $|\Psi_\perp\rangle$,

$$\cos\theta|\alpha\rangle - \sin\theta|y\rangle = A(\cos\theta|\alpha\rangle + \sin\theta|y\rangle) + B(\sin\theta|\alpha\rangle - \cos\theta|y\rangle).$$

It has the solution $A = \cos(2\theta)$, $B = \sin(2\theta)$. Then,

$$\widehat{G}|\Psi\rangle = (2|\Psi\rangle\langle\Psi| - \widehat{I})[\cos(2\theta)|\Psi\rangle + \sin(2\theta)|\Psi_\perp\rangle] = \cos(2\theta)|\Psi\rangle - \sin(2\theta)|\Psi_\perp\rangle,$$

$$\widehat{G}|\Psi\rangle = \cos(3\theta)|\alpha\rangle + \sin(3\theta)|y\rangle.$$

After j succesive applications of $\widehat{G}$ the n-qubit state is transformed as follows,

$$\widehat{G}^k|\Psi\rangle = \cos[(2j+1)\theta]|\alpha\rangle + \sin[(2j+1)\theta]|\, y\rangle.$$

The goal is obtaining a result which is the closest to the marked state $|y\rangle$. This is achieved when j is such that $\cos[(2j+1)\theta]$ attains the minimum possible value and $\sin[(2j+1)\theta]$ is very near to 1. The smallest j which satisfies the condition is certain k such that

$$(2k+1)\theta \approx \frac{\pi}{2}, \qquad \text{or } k = \left[\frac{\pi}{4\theta} - \frac{1}{2}\right],$$

where [...] denotes the nearest integer.

Since $|\Psi\rangle$ is an initial state where all the $|x\rangle$ are in superposition with amplitude $1/\sqrt{N}$,

$$\sin\theta = \langle y|\Psi\rangle = \frac{1}{\sqrt{N}}.$$

Therefore, for N large, $\theta \approx 1/\sqrt{N}$, and the number of applications of the Grover operator is of the order $k = O(\sqrt{N})$.

In the two-qubit case, problem consists in finding one item out of $N = 2^2 = 4$ items. $[\pi\sqrt{N}/4 - 1/2] = 1$, just one iteration is sufficient. See Figure 3.

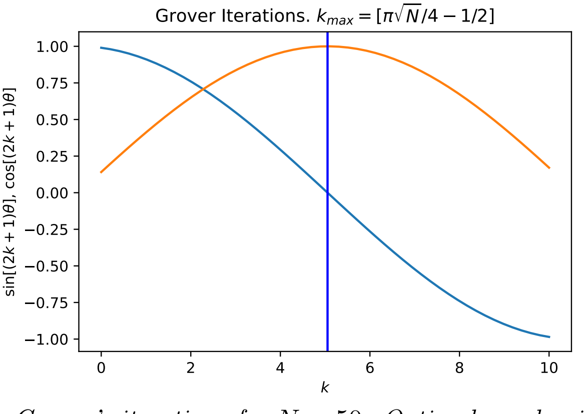

FIGURE 3. *Grover's iterations for $N = 50$. Optimal number is $[\pi\sqrt{N}/4 - 1/2]$. Vertical blue line represents the k_{max}.*

Concluding Remarks

EPR's objections to quantum mechanics can be explained considering two systems, A and B, that do not interact and whose vector of joint state, $|J, M_J\rangle_{AB}$, possessing quantum correlations, can be written in two different ways. To be specific, we can assume that it deals with two rotating molecules, where the states of molecule A are $|K, M_K\rangle_A$ and those for B are $|L, M_L\rangle_B$. By simplicity, which does not detract from the generality of the arguments, can assume that the system has zero total angular momentum, $|0, 0\rangle_{AB}$, which requires that $K = L$ and $M_K = -M_L$, the above assuming that M_K and M_L are projections of the angular momentum vector along the axis Z. But nothing prevents considering states with projection along the X axis, which we call $|L, N_L\rangle_A$ to the states of A and $|L, -N_L\rangle_B$ to those of B. Then

$$\sqrt{2L+1}|0,0\rangle_{AB} = \sum_{M=-L}^{L} (-1)^M |L, -M\rangle_B |L, M\rangle_A = \sum_{N=-L}^{L} (-1)^N |L, -N\rangle_B |L, N\rangle_A.$$

Applying the "collapse" of the wave function when making measurements, EPR would conclude that if the observable $\widehat{L}_Z$ is measured in the subsystem A and a certain value M is obtained, then B will be in the state $|L, -M\rangle_B$, but if $\widehat{L}_X$ is measured in the subsystem A and a certain N is obtained, then B will be in the state $|L, -N\rangle_B$. It is concluded that the measure in A of the incompatible observables $\widehat{L}_Z$ and $\widehat{L}_X$ have the ability to leave B in eigenstates of incompatible operators a despite the fact that at the time of measurement A and B do not interact.

This objection assumes that, even though $\widehat{L}_Z$ and $\widehat{L}_X$ do not commute, "intrinsically" the molecules, because they have constant angular momentum, must have definite values of all projections. The experiments performed by Clauser, Aspect and collaborators demonstrated that the objection is not valid. In 1931, María Göppert-Mayer considerated the simultaneous emission of two photons by one atom. The work was entitled "Elementary processes with two quantum transitions" [11]. It remains clear that the photons are in a polarization entangled state. This calculation gives the theoretical support of the experiments of Clauser, Aspect and collaborators, based on radiative cascade emitting pairs of photons correlated in polarization. Also Göppert calculated the second order susceptibility, $\chi^{(2)}$, relevant for the current way to produce entangled photons based on the spontaneous parametric down conversion using nonlinear crystals, used in the Zeilinger's experiments [12]. See Figure 4.

In July 2015, several institutions organized the International Conference Colombia in the International Year of Light [13]. In the list of speakers were the 2012 Nobel laureates in physics David Wineland and Serge Haroche, and Alain Aspect who won the 2022 Nobel prize in physics. See Figure 5.

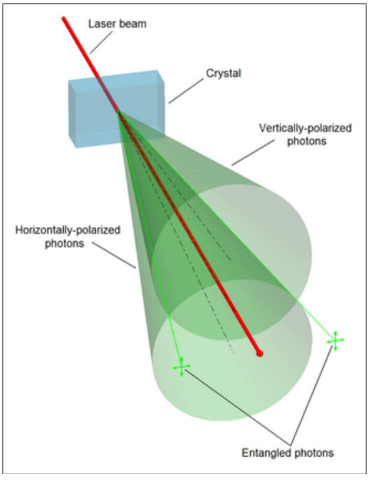

FIGURE 4. *Nonlinear crystal to produce polarization entangled photons by means of the SPDC mechanism. Google Image.*

FIGURE 5. *Some participants at the International Conference Colombia in the International Year of Light 2015.*

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