PROJECTILE MOTION IN A MEDIUM WITH QUADRATIC DRAG AT CONSTANT HORIZONTAL WIND

MOVIMIENTO DE PROYECTIL EN UN MEDIO CON RESISTENCIA CUADRÁTICA CON VIENTO HORIZONTAL CONSTANTE

Peter Chudinov[*], Vladimir Eltyshev and Yuri Barykin

Department of Engineering, Perm State Agro-Technological University, 614990, Perm, Russia.

Abstract

A classic problem of the motion of a projectile thrown at an angle to the horizon is studied. Air resistance force is taken into account with the use of the quadratic resistance law. The action of the wind is also taken into account, which is considered constant and horizontal (tailwind or headwind). The projectile velocity hodograph equation is used to take into account the effect of wind. Comparatively simple analytical approximations are proposed for the main variables of motion (cartesian projectile coordinates and time). All obtained formulas contain only elementary functions. The proposed formulas are universal, that is, they can be used for any initial conditions of throwing. In addition, they have acceptable accuracy over a wide range of the change of parameters. The motion of a golf ball, a tennis ball and shuttlecock of badminton are presented as examples. The calculation results show good agreement between the proposed analytical solutions and numerical solutions. The proposed analytical formulas can be useful for all researchers of this classical problem.

---

* chupet@mail.ru

Resumen

Se estudia un problema clásico del movimiento de un proyectil lanzado con un ángulo respecto al horizonte. Se tiene en cuenta la fuerza de resistencia del aire mediante la ley de resistencia cuadrática. También, se tiene en cuenta la acción del viento, que se considera constante y horizontal (viento de cola o viento en contra). Para tener en cuenta el efecto del viento se utiliza la ecuación hodográfica de la velocidad del proyectil. Se proponen aproximaciones analíticas comparativamente sencillas para las principales variables del movimiento (coordenadas cartesianas del proyectil y tiempo). Todas las fórmulas obtenidas contienen solo funciones elementales. Las fórmulas propuestas son universales, es decir, pueden utilizarse para cualquier condición inicial de lanzamiento. Además, tienen una precisión aceptable en un amplio rango de cambio de parámetros. Se presentan como ejemplos el movimiento de una pelota de golf, una pelota de tenis y un volante de bádminton. Los resultados de los cálculos muestran una buena concordancia entre las soluciones analíticas propuestas y las soluciones numéricas. Las fórmulas analíticas propuestas pueden ser útiles para todos los investigadores de este problema clásico.

Palabras clave: movimiento de proyectil, ley de resistencia cuadrática, viento horizontal.

1. Introduction

The study of projectile motion continues to be important in various fields of engineering, technology and sports. Currently, the study of parabolic motion, in the absence of any drag force, is a common example in introductory physics courses at all universities. The theory of parabolic motion allows you to analytically determine the trajectory and all-important characteristics of the movement of the projectile. Introduction of air resistance forces into the study of the motion, however, complicates the problem and makes it difficult to obtain analytical solutions. This especially applies to

the movement of the projectile, subjected to quadratic in velocity air drag force. The problem becomes even more complicated if the effect of wind is considered when the projectile moves in a medium with quadratic resistance. For a quadratic or more general non-linear drag force, results have been obtained only numerically. The detailed analysis of wind-influenced projectile motion in the case of linear and nonlinear (quadratic or nonquadratic) drag force is reviewed in [1].

Numerous studies have been dedicated to the numerical investigation of projectile motion in the presence of quadratic resistance of the medium and the action of the wind, for example [2, 3], [4–8] . This is because the differential equations of motion do not allow an analytical solution. Although, in the case of quadratic resistance, there is a closed relationship between the velocity and the corresponding angular parameter (the velocity hodograph equation). Only for the linear resistance force, closed-form solutions have been obtained [3], including expressions for the change in time of the velocity components and the shape of the trajectory. In this case, the special Lambert function W is used. Thus, the problem of the analytical description of the projectile motion with quadratic resistance and the action of the wind taken into account remains relevant. In this paper, an attempt is made to obtain analytical approximations for solving this problem, at least approximate ones. The main goal of this work is to give analytical approximations for the projectile trajectory. The proposed formulas contain only elementary functions, unlike study [3].

The research is based on two previously solved problems. The first task is to obtain a hodograph of the projectile velocity with a quadratic resistance of the medium, but without considering the wind. This problem has been solved and described long ago by Timoshenko et al [9]. Consideration of the action of a constant wind in the velocity hodograph equation was made by Lubarda et al [1]. The velocity hodograph equation taking into account the wind is used in this study. The second task is to successfully approximate the complex transcendental function that enters the hodograph

equation and the equations of motion and apply this approximation to solve the equations of motion of the projectile. This was done by Chudinov et al without considering the wind. Chudinov's idea is used in this study, taking into account the adopted wind model. The conditions of applicability of the quadratic resistance law are deemed to be fulfilled, i.e. Reynolds number Re lies within $1 \times 10^3 < Re < 2 \times 10^5$. Magnus forces are not included in this work.

2.   Equations   of   Projectile   Motion   and   Velocity Hodograph Equation

Here we state the formulation of the problem and the equations of the motion [10]. Let us consider the motion of a projectile with mass m launched at an angle θ_0 with an initial speed V_0 under the influence of the force of gravity and resistance force $R = mgkV^2$. Here, g is the acceleration of gravity, k is the drag constant and V is the speed of the object. Air resistance force R is proportional to the square of the speed of the projectile and is directed opposite the velocity vector (see Fig. 1).

The impact of the wind on the projectile will be modeled as a constant horizontal speed of the air $\bar{w}$. It is assumed that the projectile is initially located at the origin, and the point of impact of the projectile lies on the same horizontal line $y = 0$ (see Fig. 1). In ballistics in the absence of wind, the movement of a projectile is often studied in projections on natural axes. The equations of the projectile motion in this case have the form:

$$\frac{dV}{dt} = -g\sin\theta - gkV^2, \quad \frac{d\theta}{dt} = -\frac{g\cos\theta}{V}, \quad \frac{dx}{dt} = V\cos\theta, \quad \frac{dy}{dt} = V\sin\theta. \quad (1)$$

Here, θ is the angle between the tangent to the trajectory of the projectile and the horizontal, x and y are the Cartesian coordinates of the projectile. The well-known solution [9] of system (1) consists of an explicit analytical dependence of the velocity on the slope

angle of the trajectory (hodograph equation for projectile velocity) and three quadratures:

$$V(\theta) = \frac{V_0 \cos \theta_0}{\cos \theta \sqrt{1 + kV_0^2 \cos^2 \theta_0 \left(f(\theta_0) - f(\theta)\right)}}, f(\theta) = \frac{\sin \theta}{\cos^2 \theta} + \ln \tan \left(\frac{\theta}{2} + \frac{\pi}{4}\right), \quad (2)$$

$$x = x_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 d\theta, \ y = y_0 - \frac{1}{g}\int_{\theta_0}^{\theta} V^2 \tan \theta d\theta, \ t = t_0 - \frac{1}{g}\int_{\theta_0}^{\theta} \frac{V}{\cos \theta} d\theta. \ (3)$$

Here, t_0 is the initial value of the time, and x_0, y_0 are the initial values of the coordinates of the projectile.In the following formulas, we take $x_0 = 0$, $t_0 = 0$, $y_0 \geqslant 0$. Dependence $V(\theta)(2)$ is called the velocity hodograph equation. The drag coefficient k, used in formulas (1) – (2), can be calculated through the terminal speed: $k = 1/V_{term}^2$[11]. Terminal velocity is the maximum velocity attainable by an object as it falls through a fluid (air is the most common example).
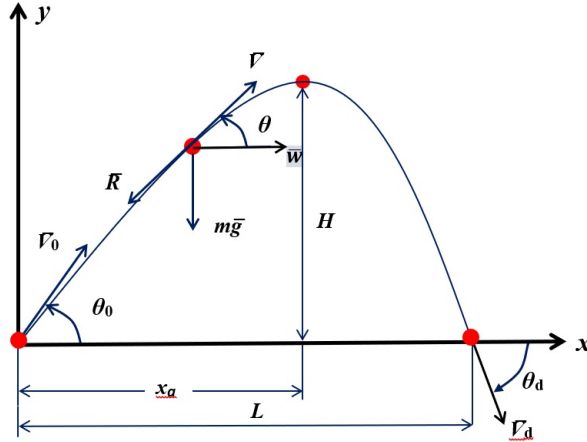

FIGURE 1. *Acting forces on the projectile and basic motion parameters.*

Let us write the hodograph equation (2) and integrals (3) under the action of a constant horizontal wind, following review [1]. In the presence of horizontal wind whose velocity vector $\bar{w}$ has

the components $(w, 0)$, the drag force in the model of quadratic resistance is:

$$\bar{R} = -c|\bar{V} - \bar{w}|(\bar{V} - \bar{w}) \tag{4}$$

where $c = gk$ is the drag coefficient. Then the differential equations of projectile motion in projections onto the Cartesian axes have the form:

$$\frac{dx}{dt} = V_x, \quad \frac{dV_x}{dt} = -c\sqrt{(V_x - w)^2 + V_y^2}\,(V_x - w),$$

$$\frac{dy}{dt} = V_y, \quad \frac{dV_y}{dt} = -c\sqrt{(V_x - w)^2 + V_y^2}\,V_y - g. \tag{5}$$

Initial conditions in this case $x_0 = y_0 = 0, V_x(0) = V_0 \cos\theta_0$, $V_y(0), = V_0 \sin\theta_0$. We introduce the relative velocity of the projectile by the equality $\bar{u} = \bar{V} - \bar{w}$ with components $u_x = V_x - w, u_y = V_y$. The magnitude of the relative velocity is $u = \sqrt{u_x^2 + u_y^2}$. Then, equations (5) take the form:

$$\frac{dx}{dt} = u_x + w, \quad \frac{du_x}{dt} = -c\sqrt{u_x^2 + u_y^2}\,u_x, \quad \frac{dy}{dt} = u_y, \quad \frac{du_y}{dt} - c\sqrt{u_x^2 + u_y^2}\,u_y - g. \tag{6}$$

Equations (6) have the same form as when the projectile moves without wind, i.e. the form of equations (5) at $w = 0$. Therefore, for equations (6), the relative velocity hodograph equation takes place, which is similar to equation (2):

$$u(\varphi) = \frac{u_0 \cos\varphi_0}{\cos\varphi\sqrt{1 + ku_0^2 cos^2\varphi_0\,(f(\varphi_0) - f(\varphi))}}, f(\varphi) = \frac{\sin\varphi}{\cos^2\varphi} + ln\tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right). \tag{7}$$

The auxiliary angular parameter φ is the angle between the relative velocity vector $\bar{u}$ and the positive axis x, and is determined by the formula [7]:

$$\varphi = \arctan\left(\frac{u_y}{u_x}\right) = \arctan\left(\frac{V\sin\theta}{V\cos\theta - w}\right), \quad \varphi_0 = \arctan\left(\frac{V_0\sin\theta_0}{V_0\cos\theta_0 - w}\right),$$

$$u_0 = \sqrt{u_x^2(0) + u_y^2(0)} = \sqrt{(V_0\cos\theta_0 - w)^2 + (V_0\sin\theta_0)^2} = \sqrt{V_0^2 - 2V_0 w\cos\theta_0 + w^2}.$$

In the absence of wind, the angle φ reduces to θ. Quadratures (3) in this case take the form:

$$x = x_0 - \frac{1}{g} \int_{\varphi_0}^{\varphi} u^2 d\varphi - \frac{w}{g} \int_{\varphi_0}^{\varphi} \frac{u}{\cos \varphi} d\varphi,$$

$$y = y_0 - \frac{1}{g} \int_{\varphi_0}^{\varphi} u^2 \tan \varphi d\varphi, \tag{8}$$

$$t = t_0 - \frac{1}{g} \int_{\varphi_0}^{\varphi} \frac{u}{\cos \varphi} d\varphi.$$

Since the dependence $u(\varphi)$ is known, the problem of projectile motion in a medium with quadratic resistance at a constant horizontal wind is reduced to the calculation of integrals (8).

3. Computational Formulas of the Problem

The main problem in calculating integrals (8) is that the complex form of the transcendental function $f(\varphi)$ in the velocity hodograph equation does not allow to find a solution in elementary functions. In this paper, we propose a convenient solution of the problem considering the wind, similar to the solution in [10]. According to the approach used by Chudinov et al [10], we divide the entire range of the trajectory angle $\varphi_0 \geqslant \varphi > -\pi/2$ into three intervals: $\varphi_0 \geqslant \varphi \geqslant 0, 0 \geqslant \varphi \geqslant \varphi_1, \varphi_1 \geqslant \varphi > -\pi/2$. Such a partition allows one to construct a solution over the entire interval of angle change φ. As it is known, the trajectory of a projectile has an asymptote when it moves in a medium with resistance. The value φ_1 is determined by the equality $\varphi_1 = -\frac{\varphi_0}{2} - \frac{\pi}{4}$. This choice of angle φ_1 allows for a good approximation of the function $f(\varphi)$, when the projectile moves along a trajectory approaching the asymptote. The first interval corresponds to the projectile lifting stage, the other two intervals correspond to the descent stage.

An approximation of the function $f(\varphi)$ of the following form was proposed by Chudinov et al [10]:

$$f_a(\varphi) = \alpha_1 \tan \varphi + \alpha_2 \tan^2 \varphi, \text{ on condition } \varphi \geqslant 0,$$
$$f_a(\varphi) = \alpha_1 \tan \varphi - \alpha_2 \tan^2 \varphi, \text{ on condition } \varphi \leq 0.$$

The coefficients α_1, α_2 are chosen in such a way that the functions $f(\varphi)$ and $f_a(\varphi)$ are closely related to each other. For this, the following conditions are used

$$f_a(\varphi_0) = f(\varphi_0), \quad f_a'(\varphi_0) = f'(\varphi_0). \tag{9}$$

From conditions (9) we have

$$\alpha_1 = 2\cot\varphi_0 \ln\tan\left(\frac{\varphi_0}{2} + \frac{\pi}{4}\right), \alpha_2 = \frac{1}{\sin\varphi_0} - \frac{\alpha_1}{2}\cot\varphi_0.$$

Next, we substitute the function $f_a(\varphi)$ instead of the function $f(\varphi)$ into the hodograph equation (7). Now, integrals (8) can be calculated in elementary functions. Omitting extensive standard intermediate calculations, we write down the final formulas.

On the first interval, the functions $x(\varphi), y(\varphi), t(\varphi)$ have the following form:

$$x_1(\varphi) = x_0 + \frac{2}{gk\alpha_1\Delta_1}\arctan\left(\frac{1 + 2b_2\tan\varphi}{\Delta_1}\right)\Big|_{\varphi_0}^{\varphi} - \frac{w}{g\sqrt{k\alpha_2}}\arcsin\left(\frac{1 + 2b_2\tan\varphi}{\Delta_3}\right)\Big|_{\varphi_0}^{\varphi},$$

$$y_1(\varphi) = y_0 - \frac{1}{gk\alpha_1\Delta_1 b_2}\arctan\left(\frac{1 + 2b_2\tan\varphi}{\Delta_1}\right)\Big|_{\varphi_0}^{\varphi} + \frac{1}{2gk\alpha_1 b_2}\ln\left(b_2\tan^2\varphi + \tan\varphi - b_1\right)\Big|_{\varphi_0}^{\varphi},$$

$$t_1(\varphi) = t_0 - \frac{1}{g\sqrt{k\alpha_2}}\arcsin\left(\frac{2b_2\tan\varphi + 1}{\Delta_3}\right)\Big|_{\varphi_0}^{\varphi}.$$

On the second interval we have

$$x_2(\varphi) = x_1(0) - x_1(\varphi_0) + \frac{2}{gk\alpha_1\Delta_2}\arctan\left(\frac{1 - 2b_2\tan\varphi}{\Delta_2}\right)\Big|_0^{\varphi}$$
$$- \frac{w}{g\sqrt{-k\alpha_2}}\arcsin\left(\frac{1 - 2b_2\tan\varphi}{\Delta_4}\right)\Big|_0^{\varphi},$$

$$y_2(\varphi) = y_1(0) - y_1(\varphi_0) + \frac{1}{gk\alpha_1\Delta_2 b_2}\arctan\left(\frac{1 - 2b_2\tan\varphi}{\Delta_2}\right)\Big|_0^{\varphi}$$
$$- \frac{1}{2gk\alpha_1 b_2}\ln\left(b_2\tan^2\varphi - \tan\varphi + b_1\right)\Big|_0^{\varphi},$$

$$t_2(\varphi) = t_1(0) - t_1(\varphi_0) + \frac{1}{g\sqrt{-k\alpha_2}}\arcsin\left(\frac{2b_2\tan\varphi - 1}{\Delta_4}\right)\Big|_0^{\varphi}.$$

On the third interval we have

$$x_3(\varphi) = x_1(0) - x_1(\varphi_0) + x_2(\varphi_1) - x_2(0) - \frac{2}{gk\beta_1 d_2} \arctan\left(\frac{2d_1 \tan\varphi - 1}{d_2}\right)\Big|_{\varphi_1}^{\varphi}$$

$$- \frac{w}{g\sqrt{k\beta_2}} \ln\left(1 - 2d_1 \tan\varphi - 2\sqrt{d_1}\sqrt{d_0 - \tan\varphi + d_1 \tan^2\varphi}\right)\Big|_{\varphi_1}^{\varphi},$$

$$y_3(\varphi) = y_1(0) - y_1(\varphi_0) + y_2(\varphi_1) - y_2(0) - \frac{2}{gk\beta_2 d_2} \arctan\left(\frac{2d_1 \tan\varphi - 1}{d_2}\right)\Big|_{\varphi_1}^{\varphi}$$

$$- \frac{1}{2gk\beta_2} \ln\left(d_0 - \tan\varphi + d_1 \tan^2\varphi\right)\Big|_{\varphi_1}^{\varphi},$$

(10)

$$t_3(\varphi) = t_1(0) - t_1(\varphi_0) + t_2(\varphi_1) - t_2(0)$$

$$- \frac{1}{g\sqrt{k\beta_2}} \ln\left(1 - 2d_1 \tan\varphi - 2\sqrt{d_1}\sqrt{d_0 - \tan\varphi + d_1 \tan^2\varphi}\right)\Big|_{\varphi_1}^{\varphi}.$$

Index value i in functions $x_i(\varphi), y_i(\varphi), t_i(\varphi)$corresponds to the number of the movement interval $(i = 1, 2, 3)$. In relations (10), the following notation is introduced:

$$b_1 = \frac{1}{\alpha_1}\left(\frac{1}{ku_0^2 \cos^2\varphi_0} + f(\varphi_0)\right), \quad b_2 = \frac{\alpha_2}{\alpha_1}, \quad \Delta_1 = \sqrt{-1 - 4b_1 b_2},$$

$$\Delta_2 = \sqrt{-1 + 4b_1 b_2}, \quad \Delta_3 = \sqrt{1 + 4b_1 b_2}, \quad \Delta_4 = \sqrt{1 - 4b_1 b_2},$$

$$\beta_2 = \frac{(f(\varphi_1) + f(89°)) \cos\varphi_1 - 2(\tan\varphi_1 + \tan 89°)}{(\tan\varphi_1 + \tan 89°)^2 \cos\varphi_1},$$

$$\beta_1 = \frac{2(1 + \beta_2 \sin\varphi_1)}{\cos\varphi_1}, \quad \beta_0 = f(\varphi_1) - \beta_1 \tan\varphi_1 + \beta_2 \tan^2\varphi_1,$$

$$d_0 = \frac{1}{\beta_1}\left(\frac{1}{ku_0^2 \cos^2\varphi_0} + f(\varphi_0) - \beta_0\right), \quad d_1 = \frac{\beta_2}{\beta_1}, d_2 = \sqrt{4d_0 d_1 - 1}.$$

Thus, on each of the intervals, the movement of the projectile is described by the equations:

$$\text{at} \quad i = 1 \quad x = x_1(\varphi), \quad y = y_1(\varphi), \quad t = t_1(\varphi);$$
$$\text{at} \quad i = 2 \quad x = x_2(\varphi), \quad y = y_2(\varphi), \quad t = t_2(\varphi);$$
$$\text{at} \quad i = 3 \quad x = x_3(\varphi), \quad y = y_3(\varphi), \quad t = t_3(\varphi).$$

Collectively, these equations analytically describe the motion of the projectile over the entire interval of change in the trajectory angle $\varphi_0 \geq \varphi \geq -\pi/2$. If during the motion of the projectile the trajectory angle φ is within the limits $\varphi_0 \geq \theta \geq \varphi_1$, then the

functions $x_3(\varphi), y_3(\varphi), t_3(\varphi)$ are not used to describe the movement. The projectile trajectory is given parametrically by the functions $x(\varphi), \ y(\varphi)$.

Formulas (10) are an improved version of the previously obtained formulas [10], taking into account the influence of wind on the movement of the projectile. At very small values of the drag coefficient $(k = 10^{-12})$ and in the absence of wind, formulas (10) are transformed into formulas of the theory of parabolic projectile motion. The value $k = 0$ cannot be used in formulas (10), since division by zero occurs. Using formulas for $x_1(\varphi), y_1(\varphi), t_1(\varphi)$, we can find the values of the x and y coordinates, and time t, corresponding to the top of the projectile trajectory. Assuming $\varphi = 0$ in these formulas, we have:

$$X_a = \ x_1(0) - x_1(\varphi_0), \ H = y_1(0) - y_1(\varphi_0), \ t_a = t_1(0) - t_1(\varphi_0).$$

Here, x_a is the abscissa of the vertex of the trajectory H is the maximum height of the projectile, t_a is the projectile rise time (see Fig. 1).

4.   Results of the Calculations

To check the applicability of analytical formulas (10), we consider the movement of various sports equipment – a golf ball, a tennis ball, a badminton shuttlecock. We will compare the results of numerical integration of the projectile motion equations (5) with the results obtained using formulas (10). The drag coefficient k for these projectiles is usually determined through the terminal velocity V_{term}[11]:

$$k = \frac{1}{V_{term}^2} = const.$$

The values of the drag coefficient k used in the calculations differ by a factor of 22. All of the figures below in the text show the

sport	V_{term}, m/s	$Re \times 10^5$	k, s^2/m^2	V_0, m/s
badminton	6.7	0.27	0.022	60
tennis	22	0.95	0.002	50
golf	32.09	0.90	0.000971	40; 60

TABLE 1. *Air Resistance Coefficients and Initial Speed in Calculations.*

trajectories of the projectile. The thick solid black lines are obtained by numerical integration of system (5) with the aid of the 4-th order Runge-Kutta method (RK4). The red dots are obtained using analytical formulas (10).
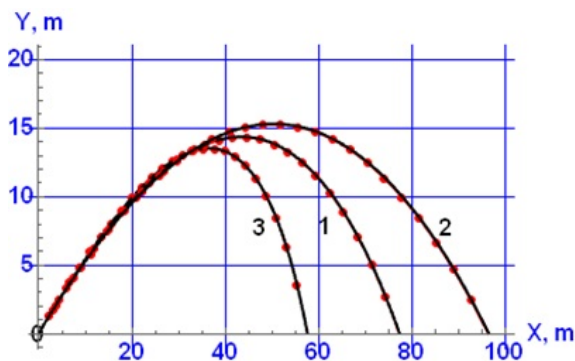

FIGURE 2. *Golf balls movement with parameters $V_0 = 40$ m/s, $g = 9.81$ m/s^2, $k = 0.000971$ s^2/m^2, $\theta_0 = 30°$.*

Figure 2 is constructed using the initial data and the drag coefficient taken from Figure 4b by Lubarda et al [[1]]. Curve 1 corresponds to the movement of a golf ball without wind, curve 2 to the movement with a tailwind $w = 10$ m/s, curve 3 to the movement of the ball with a headwind $w = -10$ m/s. Figure 2 shows the effect of wind on the parameters H, x_a and range of flight L of the projectile. A tailwind increases the values of the parameters H, L, a headwind reduces these values.

Generally speaking, the presence of a tailwind does not fundamentally change the shape of the well-known trajectories of projectiles (stone, ball, cannonball, etc.). The presence of a headwind can lead to the appearance of quite exotic trajectories.

For example, the projectile landing point can be either to the right or to the left of the throwing point; the projectile trajectory may have self-intersection; the trajectory may be closed (see Fig.3, Fig.6).
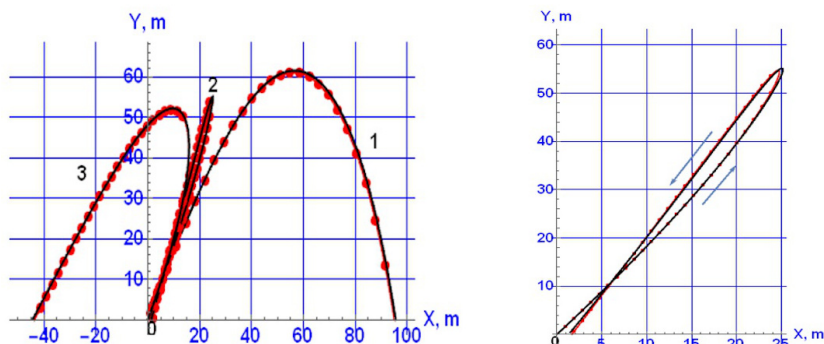

FIGURE 3. *Parameters for throwing a golf ball:* $V_0 = 60 \ m/s$; $\theta_0 = 60°$; $k = 0.000971 \ s^2/m^2$. *Trajectory 1 was plotted with* $w = 0$ *(no wind); trajectory 2 – with* $w = -20 \ m/s$; *trajectory 3 – with* $w = -30 \ m/s$.

Figure 3 shows the influence of the headwind on the shape of the trajectory and the position of the end point of the trajectory – to the right or left of the throwing point. Trajectory 2 in the left picture of Figure 3 has a complex form with self-intersection and is shown in the right picture of Figure 3. Figure 3 shows that the presence of a strong headwind significantly changes the shape of the projectile trajectory. Note also the good match in the right picture of Figure 3 of numerical (solid black line) and analytical (red dotted line) solutions even for such a complex trajectory. The arrows in the right picture of Figure 3 indicate the direction of the projectile along the trajectory.

Of all the trajectories of sport projectiles, the trajectory of the shuttlecock has the greatest asymmetry. This is explained by the relatively large value of the drag coefficient k. In addition, the trajectory of the shuttle approachesto the vertical asymptote very quickly. Figure 4 shows the trajectory of the shuttlecock under the influence of both tail and head wind. In Figure 4, the flight range L of the shuttlecock in the absence of wind (curve 1) is $L = 11.5$ m. In the absence of air resistance ($k = 0$) and in the

absence of wind, this value would be equal to $L = 367$ m. Figure 4 shows good agreement between the numerical and analytical solutions (10) for large values of the drag coefficient.
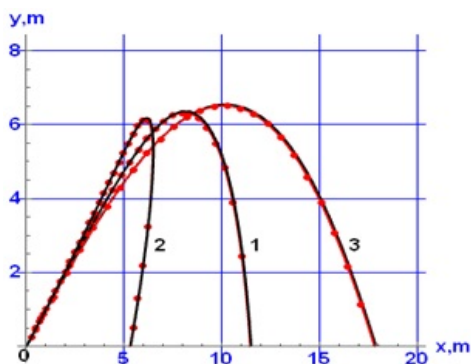

FIGURE 4. *Badminton shuttlecock throwing parameters:* $V_0 = 60$ m/s; $\theta_0 = 45°$, $k = 0.022$ s^2/m^2. *In the picture curve 1 was plotted at the value* $w = 0$ *(not wind), curve 3 – at* $w = 3$ m/s, ; *curve 2 – at* $w = -3m/s$.
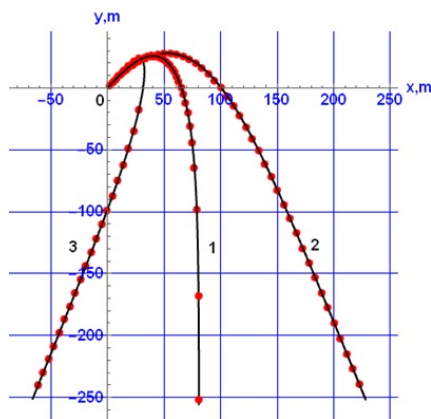

FIGURE 5. *Tennis ball throwing parameters:* $V_0 = 50$ m/s; $\theta_0 = 45°$, $k = 0.002$ s^2/m^2, $w = 0$; ± 10 m/s.

Analytical solutions (10) are constructed not only for the working values of the angle of inclination of the projectile trajectory θ (see Fig. 1), but also for critical values close to $\theta = -90°$. Such values of angle θ correspond to the movement of the projectile along a trajectory close to the asymptote. Figure 5 shows the

tennis ball trajectories approaching the asymptotes. Trajectory 1 corresponds to the movement of the ball without wind, trajectory 2 corresponds to movement with a tailwind at $w = 10 \, m/s$, trajectory 3 corresponds to the movement with a headwind at $w = -10 \, m/s$. The presence of wind turns the vertical asymptote into an oblique one. In this case, the slope of the asymptote occurs in the direction of the wind action.
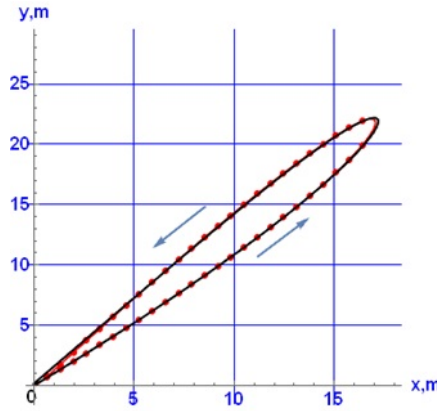

FIGURE 6. *Tennis ball throwing parameters:* $V_0 = 50 \, m/s$; $\theta_0 = 45°$, $k = 0.002 \, s^2/m^2$, $w = -19.75 \, m/s$.

Figure 6 shows the closed trajectory of a tennis ball at the corresponding value of the headwind speed. The ball returns to the point of throw at the selected value of the headwind speed. The arrows in Figure 6 show the direction of the ball's motion along the trajectory. All presented figures demonstrate excellent agreement between numerical solutions RK4 (solid black curves) and analytical solutions (red dotted curves), which are described by the formulas (10).

5.   Conclusions

In the proposed study, sufficiently accurate approximations were obtained for the main variables describing the motion of a projectile in a medium with quadratic drag in the presence of a constant horizontal wind (both tailwind and headwind). These analytical

approximations contain only elementary functions. Of course, the proposed analytical approximations do not replace the numerical solutions of the problem, but only supplement them. As examples of the use of formulas (10), the movement of various objects was considered: golf ball, shuttle of badminton, tennis ball. The calculation results testify to the universality of the proposed analytical solutions (10), which are an improved version of formulas [[10]]. They are operable over a wide range of initial throw conditions and drag coefficients. The relative maximum deviation of the analytical value (10) from the numerical value (RK4) at any point of the trajectory does not exceed 1 %. At very small values of the drag coefficient, formulas (10) are transformed into formulas of the theory of parabolic projectile motion. Along with some research value, it should be noted the educational and methodological benefits of solutions (10) for introductory physics courses studied by students at colleges and universities. Now, together with the parabolic motion of the projectile, it is possible to qualitatively study, without the use of numerical methods, a more realistic motion of the projectile in the presence of a quadratic resistance of the medium and the influence of the wind.

References

[1] M. Lubarda and V. Lubarda, Archive of App. Mech. **92**, 1997 (2022).

[2] P. W. Andersen, Eur. J. Phys. **36**, 068003 (2015).

[3] R. Bernardo, J. Esguerra, J. Vallejos, and J. Canda, Eur. J. Phys. **36**, 025016 (2015).

[4] J. McPhee and G. Andrews, Am. J. Phys. **56**, 933 (1988).

[5] N. de Mestre, The Anziam **33**, 65 (1991).

[6] R. Ozarslan, E. Bas, D. Baleanu, and B. Acay, AIMS Mathematics **5**, 467 (2020).

[7] S. Ray and J. Fröhlich, Arch. Appl. Mech **85**, 395 (2015).

[8] P. Veeresha, E. Ilhan, and H. Mehmet, Phys. Scripta **96**, 075209 (2021).

[9] S. Timoshenko and D. Young, *Advanced dynamics* (McGrow-Hill, 1948).

[10] P. Chudinov, V. Eltyshev, and Y. Barykin, Momento **62**, 79 (2021).

[11] C. Cohen, B. Darbois, G. Dupeux, E. Brunel, D. Quéré, and C. Clanet, Proc. Math. Phys. Eng. Sci. **470**, 20130497 (2014).