A BRIEF REVIEW OF FIRST-ORDER COSMOLOGICAL PERTURBATIONS INCLUDING BARYONIC MATTER FROM AN EULERIAN PERSPECTIVE

UNA CORTA REVISIÓN ACERCA DE PERTURBACIONES COSMOLÓGICAS A PRIMER ORDEN CON INCLUSIÓN DE MATERIA BARIÓNICA DESDE UNA PERSPECTIVA EULERIANA

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Abstract

In modern cosmology, the problem of large-scale structure formation has been studied through various analytical and computational methods and has become a cornerstone of astrophysics. The complexity of the equations that describe the evolution of small fluctuations in the matter field, with respect to the Friedmann-Lemaître-Robertson-Walker (FLRW) universe, commonly known as the theory linearized gravitational perturbations, makes it a valuable framework for describing the problem. Specifically, the approximation of sub-horizon scale allows us to explore scenarios where semi-analytical tools play a significant role in gaining a better understanding of how structures in our universe have evolved and how the cosmic web structure is formed. In this sense, these types of techniques have allowed for comparisons with extensive simulations and have provied a basis for contrasting with high-precision observations in this context. Therefore, in this paper, we present a semi-analytical description of the evolution of contrast density in cold dark matter (CDM), including baryonic

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matter, in a linear regime in Fourier space. We achieve this by using the Jeans filtering function (JFF), considering only growing solutions, and then comparing them with the numerical solutions calculated for the JFF equations to zero and first order, with the aim of explaining why decaying solutions must be included if one wishes to enhance precision in higher-order perturbations while also considering small scales. Finally, we discuss and extend some of the results obtained by Shoji et al. for various initial conditions in redshift.

Keywords: cosmology, growth factor, cold dark matter, baryonic matter, Jeans filtering function.

Resumen

En la cosmología moderna, el problema de la formación de estructura a gran escala ha sido estudiado a través de diferentes métodos analíticos y computacionales, y se ha convertido en una piedra angular en la astrofísica. La complejidad de las ecuaciones que describen la evolución de pequeñas fluctuaciones del campo de materia respecto al universo de Friedmann - Lemaître - Robertson - Walker (FLRW), lo que comúnmente se conoce como teoría de perturbaciones gravitacionales linealizadas, es una buena idea para describir el problema. Específicamente, la aproximación de escala sub-horizonte ha permitido abrir escenarios donde las herramientas semi-analíticas juegan un papel importante para alcanzar una mejor comprensión de cómo las estructuras significativas en nuestro universo han evolucionado y cómo se forma la estructura de la red cósmica. En este sentido, este tipo de técnicas han permitido realizar comparaciones con extensas simulaciones y han proporcionado una base para contrastar con observaciones de alta precisión en este contexto. Por lo tanto, en este trabajo, presentamos una descripción semi-analítica de la evolución de la densidad de contraste en la materia oscura fría (CDM), incluyendo la materia bariónica, en un régimen lineal en el espacio de Fourier. Lo conseguimos utilizando la función de filtrado de Jeans (JFF), considerando sólo soluciones crecientes, y comparándolas después con las soluciones numéricas calculadas para las ecuaciones JFF a cero y primer orden, con el objetivo de explicar por qué deben incluirse soluciones decrecientes si se desea mejorar la precisión en perturbaciones de orden superior considerando también escalas pequeñas. Finalmente, discutimos y extendemos algunos de los resultados obtenidos por Shoji et al. para diferentes condiciones de valor inicial en redshift.

Palabras clave: cosmología, factor de crecimiento, materia oscura fría, materia bariónica, función de filtrado de Jeans.

Introduction

If we accept the standard model of cosmology, ΛCDM, or the concordance model, the gravitational inestability is identified as responsible of the structure formation over this background. We can consider primordial small perturbations that grow gently and are influenced by two effects: the expansion of the universe and pressure effects [1]. This latter effect can be explained by the presence of baryonic matter and photons in regions of overdensity. Therefore, the growth perturbations is described through a power law rather than an exponential law [2].

We can describe structure formation in a smooth FLRW universe without baryonic matter to any order, only by finding solutions for CDM [3–5]. However, pressure plays a significance role, as it defines the Jeans scale λ_J [6, 7], or baryon bias [8]. Therefore, it is essential for high-precision observations and extensive simulations to understand how the pressure effect affects the growth, influenced by baryonic matter, and the dynamics and evolution of density field at any order. Including baryonic matter makes solving the Einstein-Boltzmann equations much more complex [1, 8].

Hence, we aim to describe what happens when we use the Jeans Filtering Function (JFF) or the baryon bias to first order, using semi-analytical tools. The structure of this document is as follows: First, Eulerian perturbation theory is employed, without baryonic matter, and then including this species, in the next three sections.

Afterward, we provide a discussion about the importance of decaying modes in first-order structure formation usign the Jeans Filtering Function, and finally, we present the conclusions.

Eulerian Perturbation Theory (EPT)

Following [9, 10], the starting point is an equation that describes the evolution of matter and radiation in an expanding universe. This relation is called the Boltzmann equation (1) and represents changes in the distribution function, denoted as $f \equiv f(t, \mathbf{x}, \mathbf{p})$, in phase space. The rate of change shows us how many particles enter and leave an element of volume,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{dx^i}{dt} + \frac{\partial f}{\partial p^i}\frac{dp^i}{dt} = c[f], \tag{1}$$

where the term $c[f]$ represents interactions between species. Initially, we can establish that structure formation is governed by CDM, because it dominates the matter density, and secondly, CDM decouples from radiation before baryons do. Therefore, the most important equation for describing structure formation is the Vlasov equation (2)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt} \cdot \nabla_{\mathbf{x}} f + \frac{d\mathbf{p}}{dt} \cdot \nabla_{\mathbf{p}} f = 0, \tag{2}$$

where we have considered a system of particles with mass m interacting only through the force of gravity. Covering the entire manifold with the coordinates $(t, \mathbf{x})$, and using the Eulerian coordinates defined as $\mathbf{r} = R(t)\mathbf{x}$, the matter dynamics is described with respect to a system of coordinates not comoving with the matter. In an expanding universe, all physical separations scale in proportion to the scale factor $R(t)$ [5, 11, 12]. Therefore, the physical velocity in the comoving coordinates $\mathbf{x}$ is

$$\mathbf{v}(t, \mathbf{x}) = \dot{R}(t)\mathbf{x} + R(t)\frac{d\mathbf{x}}{dt} = \dot{R}(t)\mathbf{x} + R(t)\mathbf{u}(t, \mathbf{x}), \tag{3}$$

with $\mathbf{u}(t, \mathbf{x}) = \dot{\mathbf{x}}$ as the velocity with respect to the comoving grid. Now, the Lagrangian for a particle with mass m, of CDM is

$$
\begin{aligned}
L = T - V = \frac{1}{2}m\left[\dot{R}\mathbf{x} + R\mathbf{u}(t,\mathbf{x})\right]^2 - m\Phi(\mathbf{x}), \\
L' = \frac{1}{2}m\left[\dot{R}x + R\dot{x}\right]^2 - m\Phi - \frac{d}{dt}\left[\frac{1}{2}m\dot{R}Rx^2\right], \\
L' = \frac{1}{2}mR^2\dot{x}^2 - m\left[\Phi + \frac{1}{2}R\ddot{R}x^2\right] = \frac{1}{2}mR^2\dot{x}^2 - m\phi. \qquad (4)
\end{aligned}
$$

Where, in the last equation, we have defined the transformation $L' = L - (d\Psi/dt)$, with $\Psi \equiv (1/2)m\dot{R}Rx^2$. Finally, from the expression (4), the scalar perturbations are described by the generalized potential $\phi \equiv \Phi + (1/2)R\ddot{R}x^2$. Using the Euler-Lagrange equation, we find the momentum definition

$$\mathbf{p} = mR^2\dot{\mathbf{x}} \quad \text{and} \quad \frac{d\mathbf{p}}{dt} = -m\nabla_{\mathbf{x}}\phi(\mathbf{x}). \qquad (5)$$

Then, with the help of the momentum equation (5), the Vlasov equation is

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{mR^2(t)} \cdot \nabla_{\mathbf{x}} f - m\nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} f = 0. \qquad (6)$$

This equation does not have explicit solutions, but if we use the moments of the distribution function, we can find very interesting physical consequences for describing the dynamics of the CDM fluid, including the baryonic component. Therefore, by taking the moments of the distribution function [1, 9].

$$\frac{m}{R^3}\int d^3p f(t,\mathbf{x},\mathbf{p}) = \rho(t,\mathbf{x}) \equiv \overline{\rho}(t)\left[1 + \delta(t,\mathbf{x})\right], \qquad (7)$$

$$\frac{1}{R^4}\int d^3p f(t,\mathbf{x},\mathbf{p})\mathbf{p} \equiv \rho(t,\mathbf{x})\mathbf{u}(t,\mathbf{x}), \qquad (8)$$

$$\frac{1}{mR^5}\int d^3p f(t,\mathbf{x},\mathbf{p})p^i p^j \equiv \rho(t,\mathbf{x})u^i(t,\mathbf{x})u^j(t,\mathbf{x}). \qquad (9)$$

Remember that $\overline{\rho}(t) \propto R^{-3}$ in the background, and in the last moment, equation (9), we could add a term associated with the stress tensor that, in the subhorizon approximation, may be composed of pressure and viscosity coefficients, like in standard fluid dynamics [13], and it can be defined as the equation of state of the cosmological fluid, characterizing the deviation of particle motions from a single coherent flow [5, 13]. By integrating over the momentum space, the Vlasov equation will be written as

$$\int d^3p \frac{\partial f}{\partial t} + \int d^3p \frac{\mathbf{p}}{mR^2(t)} \cdot \nabla_{\mathbf{x}} f - \int d^3p \Big[m\nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} f \Big] = 0.$$

By using integration by parts on the second term and Gauss's theorem on the last integral (since there isn't flux on the momentum surface) in the last equation,

$$\frac{\partial}{\partial t}\int d^3p f + \nabla_{\mathbf{x}} \cdot \int \frac{d^3p}{mR^2} f\mathbf{p} = 0. \tag{10}$$

If we use the relations (7) and (8), in the cosmic and conformal time, respectively (remember that $d\tau = dt/R$), we have the continuity equation

$$\frac{\partial \delta(t,\mathbf{x})}{\partial t} + \frac{1}{R}\nabla_{\mathbf{x}} \cdot \Big[\Big(1+\delta(t,\mathbf{x})\Big)\mathbf{u}(t,\mathbf{x})\Big] = 0, \tag{11}$$

$$\frac{\partial \delta(\tau,\mathbf{x})}{\partial \tau} + \nabla_{\mathbf{x}} \cdot \Big[\Big(1+\delta(\tau,\mathbf{x})\Big)\mathbf{u}(\tau,\mathbf{x})\Big] = 0. \tag{12}$$

Now, we want to derive the Euler equation. Therefore, by integrating over momentum space once again and multiplying by p^j (in components)

$$\frac{\partial}{\partial t}\int d^3p f p^j + \frac{1}{mR^2}\frac{\partial}{\partial x^i}\int d^3p f p^i p^j - \frac{\partial \phi}{\partial x^i} m \int d^3p \frac{\partial f}{\partial p^i} p^j = 0.$$

Integrating the last term by parts

$$\frac{\partial}{\partial t}\int d^3p f p^j + \frac{1}{mR^2}\frac{\partial}{\partial x^i}\int d^3p f p^i p^j - \frac{\partial \phi}{\partial x^i} m \int d^3p \Big\{ \frac{\partial}{\partial p^i}(f p^j) - f \underbrace{\frac{\partial}{\partial p^i} p^j}_{\delta_{ij}} \Big\} = 0.$$

With δ_{ij} being the Kronecker delta. By applying Gauss's theorem once again in the last integral, and using the equation (7), we have

$$\frac{\partial}{\partial t}\int d^3p f p^j + \frac{1}{mR^2}\frac{\partial}{\partial x^i}\int d^3p f p^i p^j + R^3\rho(t,\mathbf{x})\frac{\partial \phi}{\partial x^i} = 0.$$

If we apply this result to the last term of the expression: (which can be constructed from equation (10))

$$\frac{\partial}{\partial t}\left[R^3\overline{\rho}(t)\right]\frac{\partial \delta}{\partial t} + R^3\overline{\rho}(t)\frac{\partial^2 \delta}{\partial t^2} - 2R^{-3}\dot{R}\frac{\partial}{\partial x^j}\int d^3p f p^j + R^{-2}\frac{\partial}{\partial x^j}\left[\frac{\partial}{\partial t}\int d^3p f p^j\right] = 0.$$

Rearranging some terms and using (7) again, we can write (remember that $\overline{\rho}(t) \propto R^{-3}$):

$$R^3\overline{\rho}(t)\frac{\partial^2 \delta}{\partial t^2} - 2\frac{\dot{R}}{R^3}\left[-R^3\overline{\rho}(t)\frac{\partial \delta}{\partial t}R^2\right] = \frac{1}{mR^4}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\int d^3p f p^i p^j + R\overline{\rho}(t)\nabla_{\mathbf{x}}\cdot\left[\Big(1+\delta(t,\mathbf{x})\Big)\nabla_{\mathbf{x}}\phi\right],$$

therefore,

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{R}}{R}\frac{\partial \delta}{\partial t} = \frac{1}{\overline{\rho}(t)mR^7}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\int d^3p f p^i p^j + \frac{1}{R^2}\nabla_{\mathbf{x}}\cdot\left[\Big(1+\delta(t,\mathbf{x})\Big)\nabla_{\mathbf{x}}\phi\right], \quad (13)$$

and using the mean value for the product of velocities from(9), yields [9].

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{R}}{R}\frac{\partial \delta}{\partial t} = \frac{1}{R^2}\nabla_{\mathbf{x}}\cdot\left[\Big(1+\delta(t,\mathbf{x})\Big)\nabla_{\mathbf{x}}\phi\right] + \frac{1}{R^2}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\Big(1+\delta(t,\mathbf{x})\Big)u^i(t,\mathbf{x})u^j(t,\mathbf{x}). \quad (14)$$

It is convenient to make a comparison between the Vlasov equation for the developments that we have been reproducing [9], and the standard equations for an ideal fluid. The goal is to understand the physics of the cosmological fluid composed of CDM and baryonic matter. Therefore, we consider the equations of continuity and Euler in physical coordinates $(t, \mathbf{r})$

$$\frac{\partial \rho}{\partial t} + \nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}) = 0, \tag{15}$$

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v} \cdot \nabla_{\mathbf{r}} \right) \mathbf{v} \right] = -\nabla_{\mathbf{r}} P - \rho \nabla_{\mathbf{r}} \Phi. \tag{16}$$

And using Eulerian coordinates, with $\nabla_{\mathbf{r}} = R^{-1} \nabla_{\mathbf{x}}$, we can rewrite each term in the expressions (15) and (16). Therefore

$$\left(\frac{\partial}{\partial t} \right)_{\mathbf{r}} = \left(\frac{\partial}{\partial t} \right)_{\mathbf{x}} + \mathbf{r} \frac{\partial}{\partial t} \left(\frac{1}{R} \right) \cdot \frac{\partial}{\partial \mathbf{x}} = \left(\frac{\partial}{\partial t} \right)_{\mathbf{x}} - \frac{\dot{R}}{R} \mathbf{x} \cdot \nabla_{\mathbf{x}},$$

$$\left(\frac{\partial \rho}{\partial t} \right)_{\mathbf{r}} = \left(\frac{\partial \rho}{\partial t} \right)_{\mathbf{x}} - \frac{\dot{R}}{R} \left(\mathbf{x} \cdot \nabla_{\mathbf{x}} \right) \rho,$$

$$\nabla_{\mathbf{r}} \cdot (\rho \mathbf{v}) = \frac{1}{R} \nabla_{\mathbf{x}} \cdot \left[\rho \dot{R} \mathbf{x} + \rho \mathbf{u} \right] = 3 \frac{\dot{R}}{R} \rho + \frac{\dot{R}}{R} \mathbf{x} \cdot \nabla_{\mathbf{x}} \rho + \frac{1}{R} \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}).$$

Thus, we get the continuity equation in Eulerian coordinates [9],

$$\frac{\partial \rho}{\partial t} + 3 \frac{\dot{R}}{R} \rho + \frac{1}{R} \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0. \tag{17}$$

Now, by substituting the density contrast (7), into the last expression, we can observe that is the same equation as represented in (11). Next, we will rewrite the Euler equation (16) in the same manner as we did for the mass conservation equation

$$\rho \left[\ddot{R} \mathbf{x} + \left(\frac{\partial \mathbf{u}}{\partial t} \right) - \frac{\dot{R}}{R} \left(\mathbf{x} \cdot \nabla_{\mathbf{x}} \right) \left(\dot{R} \mathbf{x} + \mathbf{u} \right) \right]$$
$$+ \rho \left[\left(\dot{R} \mathbf{x} + \mathbf{u} \right) \cdot \frac{\nabla_{\mathbf{x}}}{R} \right] \left(\dot{R} \mathbf{x} + \mathbf{u} \right) = -\frac{1}{R} \nabla_{\mathbf{x}} P - \frac{\rho}{R} \nabla_{\mathbf{x}} \left[\phi - \frac{1}{2} R \ddot{R} x^2 \right].$$

We can obtain the Euler equation in cosmic and conformal time, respectively. Remember that $\mathcal{H}(\tau) = \frac{d}{d\tau} \ln R = \frac{dR}{dt} = \dot{R}$,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\dot{R}}{R} \mathbf{u} + \frac{1}{R} \left(\mathbf{u} \cdot \nabla_{\mathbf{x}} \right) \mathbf{u} = -\frac{1}{\rho R} \nabla_{\mathbf{x}} P - \frac{1}{R} \nabla_{\mathbf{x}} \phi, \tag{18}$$

$$\frac{\partial \mathbf{u}}{\partial \tau} + \mathcal{H}(\tau) \mathbf{u} + \left(\mathbf{u} \cdot \nabla_{\mathbf{x}} \right) \mathbf{u} = -\frac{1}{\rho} \nabla_{\mathbf{x}} P - \nabla_{\mathbf{x}} \phi. \tag{19}$$

This last equation (19), is comparable to the Vlasov equation when: (*i*) integrating over momentum space and using the moments of f, (*ii*) using the equation of continuity 17, and (*iii*) performing several lines of algebra. Therefore, we can write

$$\frac{\dot{R}}{R}u^j(t,\mathbf{x}) + \frac{1}{R}u^i(t,\mathbf{x})\frac{\partial}{\partial x^i}u^j(t,\mathbf{x}) + \frac{\partial u^i}{\partial t} = -\frac{1}{R}\frac{\partial \phi}{\partial x^i}. \tag{20}$$

This expression represent the same equation as (18), with one difference: the pressure term might be involved through the stress tensor in equation (9) (adding this term on the right-hand side), in the way described in [5]

$$\sigma_{ij}(t,\mathbf{x}) = -P\delta_{ij} + \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{u}\right) + \xi\delta_{ij}\nabla\cdot\mathbf{u}. \tag{21}$$

Considering the firts and second viscocity terms like a zero, we can finally identify an equivalent equation to (14), using the fundamental equations of hydrodynamics. If we write the Euler equation in index notation, and multiply it by $\rho(t,\mathbf{x})$, and then rewrite the first term on the left side of (18).

$$\frac{\partial}{\partial t}(u^\alpha \rho) - u^\alpha\frac{\partial \rho}{\partial t} + \frac{\dot{R}}{R}u^\alpha\rho + \frac{1}{R}\left[u^\beta\frac{\partial}{\partial x^\alpha}u^\alpha\right]\rho = -\frac{1}{R}\frac{\partial P}{\partial x^\alpha} - \frac{\rho}{R}\frac{\partial \phi}{\partial x^\alpha}.$$

Using equation (17), on the second term of the last expression on the left side, then taking the divergence, and finally using the contrast density, we can obtain

$$\frac{1}{R}\frac{\partial}{\partial t}\left[\overline{\rho}(t)\frac{\partial}{\partial x^\alpha}u^\alpha(1+\delta)\right] + 4\overline{\rho}(t)\frac{\dot{R}}{R^2}\frac{\partial}{\partial x^\alpha}\left[u^\alpha(1+\delta)\right]$$
$$= -\frac{\nabla^2_{\mathbf{x}}P}{R^2} - \frac{\overline{\rho}(t)}{R^2}\nabla_{\mathbf{x}}\cdot\left[(1+\delta)\nabla_{\mathbf{x}}\phi\right] - \frac{\overline{\rho}(t)}{R^2}\frac{\partial^2}{\partial x^\alpha\partial x^\beta}\left[(1+\delta)u^\alpha u^\beta\right],$$

To obtain the same relation as (14).

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{R}}{R}\frac{\partial\delta}{\partial t} = \frac{1}{R^2}\nabla^2_{\mathbf{x}}P + \frac{1}{R^2}\nabla_{\mathbf{x}}\cdot\left[(1+\delta)\nabla_{\mathbf{x}}\phi\right] + \frac{1}{R^2}\frac{\partial^2}{\partial x^\alpha\partial x^\beta}\left[(1+\delta)u^\alpha u^\beta\right]. \tag{22}$$

These equations presented in this section are fundamental in cosmological perturbation theory [5]. In the next section, we will work considering equation (22) without the last term [13], and we will describe solutions for the density contrast, in the linear approximation, for a fluid composed of CDM and baryonic matter [6, 8].

EPT to First Order

In order to find analytical and computational solutions to set of hydrodynamical equations described in the preceding section can be a very challenging task due to nonlinearity in the fields of contrast density and peculiar velocities. Therefore, the objective of this paper is to find perturbative solutions to first order for contrast density, where the pressure gradient is neglected to study the evolution of CDM fluctuations. On the other hand, the pressure gradient is considered and accompanies the evolution of baryonic matter. The aim is to gain a deeper understanding of baryonic matter and its influence on the large scale structure [2, 6, 8]. To describe this cosmological fluid, we need two continuity equations ($_{\text{C}}$ for CDM and $_{\text{B}}$ for baryonic matter)

$$\frac{\partial \delta_{\text{C}}(\tau, \mathbf{x})}{\partial \tau} + \nabla_{\mathbf{x}} \cdot \left[\Big(1 + \delta_{\text{C}}(\tau, \mathbf{x})\Big)\mathbf{u}_{\text{C}}(\tau, \mathbf{x})\right] = 0, \tag{23}$$

$$\frac{\partial \delta_{\text{B}}(\tau, \mathbf{x})}{\partial \tau} + \nabla_{\mathbf{x}} \cdot \left[\Big(1 + \delta_{\text{B}}(\tau, \mathbf{x})\Big)\mathbf{u}_{\text{B}}(\tau, \mathbf{x})\right] = 0. \tag{24}$$

And two Euler's equation, for CDM

$$\frac{\partial \mathbf{u}_{\text{C}}(\tau, \mathbf{x})}{\partial \tau} + \mathcal{H}(\tau)\mathbf{u}_{\text{C}}(\tau, \mathbf{x}) + \Big(\mathbf{u}_{\text{C}}(\tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}}\Big)\mathbf{u}_{\text{C}}(\tau, \mathbf{x}) = -\nabla_{\mathbf{x}}\phi(\tau, \mathbf{x}). \tag{25}$$

And baryonic matter

$$\frac{\partial \mathbf{u}_{\text{B}}(\tau, \mathbf{x})}{\partial \tau} + \mathcal{H}(\tau)\mathbf{u}_{\text{B}}(\tau, \mathbf{x}) + \Big(\mathbf{u}_{\text{B}}(\tau, \mathbf{x}) \cdot \nabla_{\mathbf{x}}\Big)\mathbf{u}_{\text{B}}(\tau, \mathbf{x}) + \frac{1}{\rho_{\text{B}}(\tau, \mathbf{x})}\nabla_{\mathbf{x}} P_{\text{B}} = -\nabla_{\mathbf{x}}\phi(\tau, \mathbf{x}). \tag{26}$$

It is important not to forget that these particles are interacting via a non-relativistic gravitational potential ϕ, and, of course, we have Poisson's equation $\nabla^2_{\mathbf{r}}\Phi = 4\pi G\rho(\mathbf{r})$. Remember that Φ is a Newtonian potential induced by mass density and if we use the definition $\phi \equiv \Phi + \frac{1}{2}R\ddot{R}x^2$ [9], it yields

$$\nabla^2_{\mathbf{x}}\phi = 4\pi G R^2 \overline{\rho}(t)\delta(t,\mathbf{x}) = \frac{3}{2}\mathcal{H}^2(\tau)\Omega(\tau)\delta(\tau,\mathbf{x}), \tag{27}$$

with $\Omega(\tau) \equiv \Omega_{\text{matter}}(\tau) = 8\pi G R^2(\tau)\rho(\tau)/3\mathcal{H}^2(\tau)$ [8]. Now, we want to work exclusively with the CDM and baryonic components in the linear regime. Thus, we can write the continuity and Euler's equations without second order terms, high order terms, or products involving the contrast density and peculiar velocity. Initially, we focus only with the CDM component and take the divergence of the Euler equation for this species, recognizing that any fluid is characterized by its divergence and curl according to the Helmholtz theorem. Therefore, the CDM fluid is entirely described by its divergence. (Note: If we calculate the curl of the Euler equation, we could demostrate that any vorticity decays in an expanding universe proportional to $1/R$)

$$\frac{\partial^2\delta_{\text{C}}(\tau,\mathbf{x})}{\partial\tau^2} + \mathcal{H}(\tau)\frac{\partial\delta_{\text{C}}(\tau,\mathbf{x})}{\partial\tau} - \frac{3}{2}\mathcal{H}^2\Omega(\tau)\delta_{\text{C}}(\tau,\mathbf{x}) = 0. \tag{28}$$

This equation is a special case of equation (22) [5]. Introducing the growth factor as $\delta_{\text{C}}(\tau,\mathbf{x}) = D(\tau)\delta_{\text{C}}(\tau_i,\mathbf{x})$, with τ_i denoting some initial conformal time. Thus, we get the second-order ordinary differential equation for the growth factor

$$\frac{d^2D(\tau)}{d\tau^2} + \mathcal{H}(\tau)\frac{dD(\tau)}{d\tau} - \frac{3}{2}\mathcal{H}^2(\tau)\Omega(\tau)D(\tau) = 0. \tag{29}$$

We can obtain solutions to this equation in general for any arbitrary cosmology. However, we will follow on a special case when we consider $\Omega_{\text{matter}} = 1$ (Einstein-de Sitter universe) [5, 6]. Therefore, the general solution for the growth factor is $D(\tau) = C_1\tau^{-3} + C_2\tau^2$ with C_1 and C_2 as integration constants. For $n = 2$ or $n = -3$, we will have growing and decreasing modes, respectively, thus

$D(\tau) \propto \tau^n \propto R^{n/2}$. Then, the density fluctuations are described by

$$\delta_{\mathrm{C}}(\tau, \mathbf{x}) = \left[\frac{R(\tau)}{R(\tau_i)}\right]^{n/2} \delta_{\mathrm{C}}(\tau_i, \mathbf{x}). \tag{30}$$

By using the Friedmann equations, we can express the equation (29) in terms of cosmic time and find solutions for the growth factor for each mode [14]. Additionally, we can derive solutions in terms of redshift. After some algebraic steps, yields [15]

$$(z+1)P(z)\frac{d^2D}{dz^2} + Q(z)\frac{dD}{dz} - \frac{3}{2}\Omega_{\mathrm{m},0}(z+1)^3 D = 0. \tag{31}$$

Where $P(z) \equiv \Omega_{\mathrm{m},0} z^3 + \left[\frac{3}{2}\Omega_{\mathrm{m},0} + q_0 + 1\right] z^2 + 2[1+q_0]z + 1$, $Q(z) \equiv \frac{1}{2}\Omega_{\mathrm{m},0} z^3 + \frac{3}{2}\Omega_{\mathrm{m},0} z^2 + \frac{3}{2}\Omega_{\mathrm{m},0} z + q_0$, and q_0 is the deceleration parameter [15]. Now, we can find the general solution for the decaying mode $D^-(z) \propto P(z)^{1/2}$, and for the growing mode:

$$D^+(z) = CP^{1/2}(z)\int_z^\infty \frac{s+1}{P^{3/2}(s)} ds. \tag{32}$$

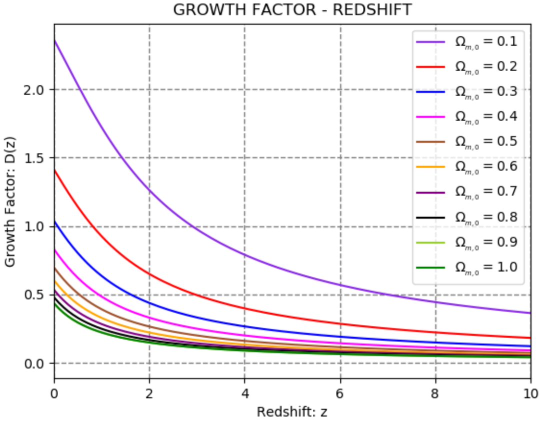

FIGURE 1. *Growth factor for different cosmologies [16]*

With C as a constant of proportionality and $P(z)$ definied by [15], we can find numeric solutions for an arbitrary cosmology, as shown in Figure 1. Therefore, if we consider $0 < \Omega_{\mathrm{m},0} \leqslant 1$, assuming $\Omega_{\mathrm{r},0} \approx 0$ and consequently, if $\Omega_{\mathrm{m},0} + \Omega_{\Lambda,0} = 1$, then the deceleration parameter is $q(t_0) = \frac{3}{2}\Omega_{\mathrm{m},0} - 1$. As we can see in Figure 1, regardless of the values of cosmological parameters, the growth factor always increases. However, in relation to observations and according to ΛCDM model, the best value for this factor is normalized to unit [1]. In our numerical approximation, this value is 1.038, where we have chosen 0.3175 for the dimensionless mass parameter [17]. For instance, if we take an Einstein-De Sitter model, the growth factor is 0.399.

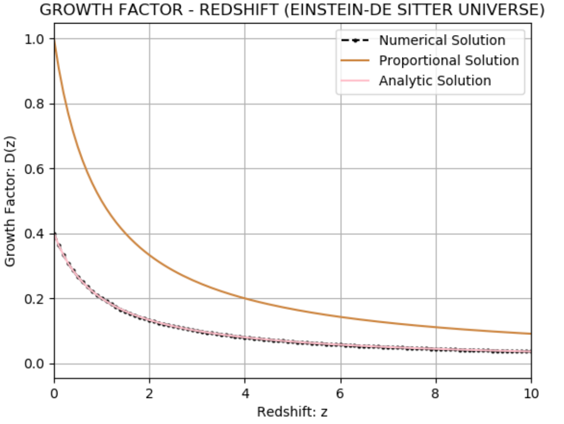

FIGURE 2. *Growth factor in the ΛCDM model. For the Einstein-De Sitter universe [16]*

Furthermore, if we are exclusively working with the Einstein-De Sitter model, we can get an analytic solution to equation (32), as shown in Figure 2, and compare it with numerical solution. For instance, if $\Omega_{\mathrm{m},0} = 1$, $q(t_0) = 0.5$ [18], we have for $P(z) = (z+1)^3$, and when we expand for the growth factor, we find $D^+(z) = \dfrac{2/5}{z+1}$.

We can observe in Figure 2, that the analytical solution is well-matched with the numerical solution. Additionally, we can appreciate the effect of the constant 2/5 on the proportional relationship of the solution, where $D^+(z) \propto \dfrac{1}{1+z}$, as shown by the orange line in this same figure. Following the work of Heat [15], we can assume that perturbations in the density matter field began to grow during the epoch when matter and radiation decoupled, which ocurred at redshift values between $1000 \leqslant z_b \leqslant 1500$, and continued until the redshift z_f when the growth of perturbations concluded. Additionally, based on the findings of Edwards and Heat [19], we can determine the redshift value at which the density contrast reaches unity, where the initial value will be strongly linked to the cosmological model employed (for this work, we have employed a redshift value near the radiation-matter decoupling epoch) [19]. This is

$$z_f \geq \frac{1}{\Omega_{\mathrm{m},0}} - 1. \tag{33}$$

Therefore, for the concordance model, the contrast density must reach unity by $z \geq 2.2$ (This value falls within the range proposed by Edwards and Sunyaver [19] $2 \leqslant z_f \leqslant 4$). Finally, we can calculate the amplification factor of a perturbation that begins to grow in the ΛCDM model after decoupling

$$A = \frac{D^+(z_f)}{D^+(z_b)} = \frac{D^+(2)}{D^+(1000)} = \frac{0.4235}{0.001293} = 327.5. \tag{34}$$

The values of the growth factor were obtained from the numerical solution to equation (32). This value allows us to calculate the decoupling fluctuation, given by $\delta = \frac{1}{A} = 0.003053$. This determines the fluctuation in the density field that must have originated at the decoupling epoch in order for the cosntrast density to reach unity [1, 15, 19]. In the next section, we are going to describe the evolution of contrast density, including the baryonic matter in a regime to first order.

EPT with CDM and Baryonic Matter

Now, we follow the work of Shoji and Komatsu [6] in this section. We aim to introduce baryonic matter into this theory to first order. Therefore, we consider the equations (23) to (26). The pressure effects are included through the stress tensor (21), considering the first and second viscosities to be zero; therefore, we can write this term as $\nabla_{\mathbf{x}}\sigma_{ij}(t, \mathbf{x}) = \nabla_i(-P\delta_{ij})$. Additionally, if we assume the barotropic relation $P(\rho)$, we obtain the pressure term in (26)

$$\begin{aligned}
\frac{\nabla_i (P_{_B}\delta_{ij})}{\rho_{\rm B}(\tau,\mathbf{x})} &= -\frac{1}{\rho_{\rm B}(\tau,\mathbf{x})}\frac{\partial}{\partial x^i}\Bigg[P(\rho_{\rm B})\delta_{ij}\Bigg],\\
&= -\frac{1}{\rho_{\rm B}(\tau,\mathbf{x})}\left[P(\rho_{\rm B})\frac{\partial}{\partial x^i}\delta_{ij}+\delta_{ij}\frac{\partial}{\partial x^i}P(\rho_{\rm B})\right],\\
&= -\frac{1}{\rho_{\rm B}(\tau,\mathbf{x})}\frac{\partial P}{\partial \rho_{\rm B}}\frac{\partial \rho_{\rm B}}{\partial x^i} = -\frac{1}{\rho_{\rm B}(\tau,\mathbf{x})}C_{_s}^2(\tau,\mathbf{x})\frac{\partial \rho_{\rm B}}{\partial x^i}.
\end{aligned}$$

Here, we have used the speed of sound, $C_{_s}$, and incorporated the baryonic contrast density $\rho_{\rm B}(\tau,\mathbf{x}) = \overline{\rho}_{\rm B}(\tau)\,[1+\delta_{\rm B}(\tau,\mathbf{x})]$,

$$\frac{\nabla_i (P_{_B}\delta_{ij})}{\rho_{\rm B}(\tau,\mathbf{x})} = -\frac{C_{_s}^2(\tau,\mathbf{x})}{[1+\delta_{\rm B}(\tau,\mathbf{x})]}\nabla_{\mathbf{x}}\delta_{\rm B}(\tau,\mathbf{x}). \tag{35}$$

And let's not forget Poisson's equation (27).

$$\nabla_{\mathbf{x}}\phi(\tau,\mathbf{x}) = 4\pi G R^2(\tau)\Bigg[\overline{\rho}_{\rm C}(\tau)\delta_{\rm C}(\tau,\mathbf{x}) + \overline{\rho}_{\rm B}(\tau)\delta_{\rm B}(\tau,\mathbf{x})\Bigg].$$

With $\delta(\tau,\mathbf{x}) = f_{\rm C}\delta_{\rm C}(\tau,\mathbf{x}) + f_{\rm B}\delta_{\rm B}(\tau,\mathbf{x})$ and [8]

$$f_{\rm C} \equiv \frac{\Omega_{\rm C}}{\Omega_{\rm m}} = \frac{\overline{\rho_{\rm C}}}{\overline{\rho_{\rm C}}+\overline{\rho_{\rm B}}} \qquad\qquad f_{\rm B} \equiv \frac{\Omega_{\rm B}}{\Omega_{\rm m}} = \frac{\overline{\rho_{\rm B}}}{\overline{\rho_{\rm C}}+\overline{\rho_{\rm B}}}. \tag{36}$$

It is a good idea to assume that we have an universe of the Einstein-de Sitter type, as this allows us to find analytic solutions. Thus, considering $\rho_{\rm m}(\tau) = \dfrac{3\mathcal{H}^2(\tau)\Omega_{\rm m}(\tau)}{8\pi G R^2(\tau)}$, into Poisson's equation, yields $\nabla_{\mathbf{x}}^2\phi(\tau,\mathbf{x}) = \dfrac{6}{\tau^2}\delta(\tau,\mathbf{x})$. Now, when calculating the divergence of Euler's equation (25) and (26) for each component, we have the following relations (with $\rho_{\rm m}$ as the total density):

$$\frac{\partial^2\delta_{\rm C}}{\partial\tau^2} + \frac{2}{\tau}\frac{\partial\delta_{\rm C}}{\partial\tau} - \frac{6}{\tau^2}\left[\frac{\overline{\rho}_{\rm C}}{\rho_{\rm m}}\delta_{\rm C} + \frac{\overline{\rho}_{\rm B}}{\rho_{\rm m}}\delta_{\rm B}\right] = 0, \tag{37}$$

$$\frac{\partial^2\delta_{\rm B}}{\partial\tau^2} + \frac{2}{\tau}\frac{\partial\delta_{\rm B}}{\partial\tau} - \frac{6}{\tau^2}\left[\frac{\overline{\rho}_{\rm C}}{\rho_{\rm m}}\delta_{\rm C} + \frac{\overline{\rho}_{\rm B}}{\rho_{\rm m}}\delta_{\rm B}\right] = -C_{_s}^2\nabla_{\mathbf{x}}^2\delta_{\rm B}. \tag{38}$$

These equations could describe the behavior and evolution of the contrast density for that mixed fluid. However, finding analytic solutions in real space may be a complex task. Therefore, it is necessary to transform this set of equations into Fourier space [1] $(\tau, \mathbf{k})$. Using

$$\mathcal{F}\left\{\tilde{\delta}(\tau,\mathbf{x})\right\} = \delta(\tau,\mathbf{x}) = \int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\delta}(\tau,\mathbf{k}), \tag{39}$$

$$\mathcal{F}\left\{\delta(\tau,\mathbf{x})\right\} = \tilde{\delta}(\tau,\mathbf{x}) = \int \frac{d^3\mathbf{x}}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}}\delta(\tau,\mathbf{k}). \tag{40}$$

Therefore, in Fourier space, these equations are

$$\frac{\partial^2\tilde{\delta}_{\rm C}(\tau,\mathbf{k})}{\partial\tau^2} + \frac{2}{\tau}\frac{\partial\tilde{\delta}_{\rm C}(\tau,\mathbf{k})}{\partial\tau} - \frac{6}{\tau^2}\tilde{\delta}(\tau,\mathbf{k}) = 0. \tag{41}$$

$$\frac{\partial^2\tilde{\delta}_{\rm B}(\tau,\mathbf{k})}{\partial\tau^2} + \frac{2}{\tau}\frac{\partial\tilde{\delta}_{\rm B}(\tau,\mathbf{k})}{\partial\tau} - \frac{6}{\tau^2}\tilde{\delta}(\tau,\mathbf{k}) = -C_s^2(\tau)k^2\tilde{\delta}_{\rm B}(\tau,\mathbf{k}). \tag{42}$$

Additionally, by using the definition of Jeans wave number [6] $k_{\rm J}(\tau) = \dfrac{\sqrt{6}}{C_s(\tau)}\tau$, we can rewrite the last equation

$$\frac{\partial^2\tilde{\delta}_{\rm B}(\tau,\mathbf{k})}{\partial\tau^2} + \frac{2}{\tau}\frac{\partial\tilde{\delta}_{\rm B}(\tau,\mathbf{k})}{\partial\tau} - \frac{6}{\tau^2}\left[\tilde{\delta}(\tau,\mathbf{k}) - \frac{k^2}{k_{\rm J}^2}\tilde{\delta}_{\rm B}(\tau,\mathbf{k})\right] = 0. \tag{43}$$

Here, we have considered that $k_{\rm J}$ is independent of conformal time. In general, the Jeans wave number depends on time, $k_{\rm J}(\tau)$, and the temperature of the species as well. However, we could simplify the problem and focus on the physical insights into the effects of pressure on the linear growth of structure under this assumption, knowing it is not completely realistic, especially if we aim to describe this effect at small scales [6]. In acording to Jeans wave number, the solutions for baryonic matter are divided into two categories: on one side, we have a growing solution for $k \ll k_{\rm J}$, and on the other hand, an oscillatory solution for $k \gg k_{\rm J}$, provided that $f_{\rm C} = 0$ and $f_{\rm B} = 1$. Initially, if we assume that CDM is the dominant source of gravity ($\delta \approx \delta_{\rm C}$), we can find solutions at each order of the set equations of (41) and (42). To find solutions of the contrast density of baryonic matter, we may use the definition of Jeans filtering function (JFF) [6, 8]

$$g(\tau, \mathbf{k}) \equiv \frac{\tilde{\delta}_{\mathrm{B}}(\tau, \mathbf{k})}{\tilde{\delta}_{\mathrm{C}}(\tau, \mathbf{k})}. \tag{44}$$

With this function, we will be able to describe how perturbations in the density field for the baryonic matter evolve as a function of the perturbation behavior of the contrast density for CDM, knowing the form of the Jeans filtering function at each order. At the zeroth order of iteration for the contrast density, CDM provides us with a solution from (41), if $\tilde{\delta}(\tau, \mathbf{k}) \to \tilde{\delta}_{\mathrm{C}}(\tau, \mathbf{k})$

$$\frac{\partial^2 \tilde{\delta}_{\mathrm{C}}(\tau, \mathbf{k})}{\partial \tau^2} + \frac{2}{\tau}\frac{\partial \tilde{\delta}_{\mathrm{C}}(\tau, \mathbf{k})}{\partial \tau} - \frac{6}{\tau^2}\tilde{\delta}_{\mathrm{C}}(\tau, \mathbf{k}) = 0. \tag{45}$$

This Cauchy-Euler equation has a solution for the growing mode as $\tilde{\delta}_{\mathrm{C}}(\tau, \mathbf{k}) \propto \tau^2$ at zeroth order. Then, the Jeans filtering function is $\tilde{\delta}^{(0)}_{\mathrm{B}}(\tau, \mathbf{k}) = g^{(0)}(\tau, \mathbf{k})\tilde{\delta}^{(0)}_{\mathrm{C}}(\tau, \mathbf{k}) = g^{(0)}(\tau, \mathbf{k})\tau^2$. Therefore, using this last result in (43), yields

$$\tau^2 \ddot{g}^{(0)}(\tau, \mathbf{k}) + 6\tau \dot{g}^{(0)}(\tau, \mathbf{k}) + 6\left[1 + \frac{k^2}{k_{\mathrm{J}}^2}\right] g^{(0)}(\tau, \mathbf{k}) = 6. \tag{46}$$

Remember the notation for derivatives, where $\cdot \equiv \partial/\partial\tau$ and $\cdot\cdot \equiv \partial^2/\partial\tau^2$. For the linearly independent solutions written as $g^{(0)}(\tau, \mathbf{k}) = \tau^n$ for the homogeneous equation, where n satisfies the quadratic expression $n^2 + 5n + \left[1 + \frac{k^2}{k_{\mathrm{J}}^2}\right] = 0$, and it has a solution

$$n^{(0)}_{\pm}(k) = -\frac{5}{2}\left[1 \pm \sqrt{1 - \frac{24}{25}\left[1 + \frac{k^2}{k_{\mathrm{J}}^2}\right]}\right]. \tag{47}$$

Therefore, $g^{(0)}(\tau, k) = C_1 \tau^{n^{(0)}_{+}(k)} + C_2 \tau^{n^{(0)}_{-}(k)}$, where C_1 and C_2 are integration constants. In general, we can express $g(\tau, k) \propto \tau^{n(k)}$. To find the particular solution of (46), we use the method of undetermined coefficients, obtaining for this solution $g^{(0)}_{\mathrm{par}}(\tau, k) = A$

with A as, $A = \left(1+\frac{k^2}{k_{\mathrm{J}}^2}\right)^{-1}$. Therefore, the general solution will be

$$g^{(0)}(\tau,k) = \underbrace{C_1\tau^{n_+^{(0)}(k)} + C_2\tau^{n_-^{(0)}(k)}}_{\mathcal{O}(\tau^{n(k)})} + \left[1+\frac{k^2}{k_{\mathrm{J}}^2}\right]^{-1}. \tag{48}$$

With $\mathcal{O}(\tau^{n(k)})$ representing decaying mode [6], if we neglect this mode, we have the filtering function $g^{(0)}(k) = \left(1+\frac{k^2}{k_{\mathrm{J}}^2}\right)^{-1}$. At zeroth order, the evolution of the contrast density of baryonic matter, approximately, is

$$\delta_{\mathrm{B}}^{(0)}(\tau,k) = \tau^2\left[1+\frac{k^2}{k_{\mathrm{J}}^2}\right]^{-1}. \tag{49}$$

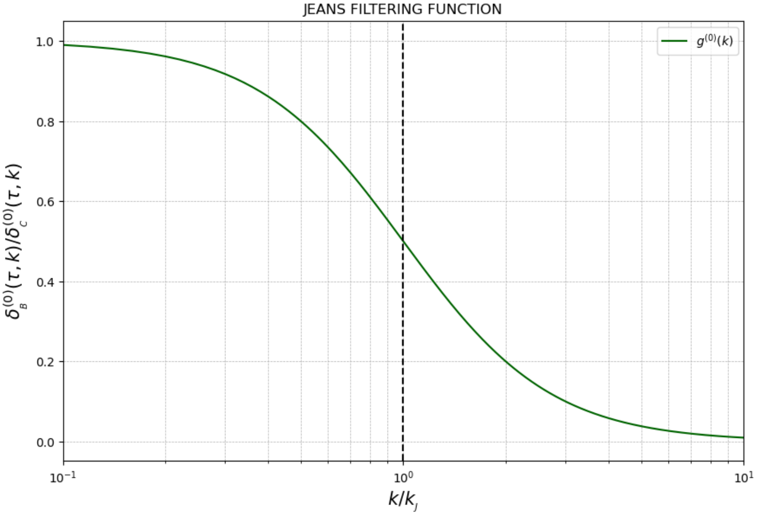

FIGURE 3. *Jeans filtering function to zeroth order*

Upon simple inspection of the equation (49), we can observe that if $k \gg k_{\mathrm{J}}$ then $\delta_{\mathrm{B}}^{(0)}(\tau,k) \to 0$ and if $k \ll k_{\mathrm{J}}$ then $\delta_{\mathrm{B}}^{(0)}(\tau,k) \to \tau^2$. In general, if we consider k is larger than k_{J}, then perturbations of baryonic matter are negligible and are dominated by CDM. Conversily, if k is much smaller than k_{J}, the fluctuations of baryonic matter follow the evolution of the CDM, proportional to the square of conformal time. We can observe in the Figure 3 that the baryonic perturbations in the case $k \ll k_{\text{J}}$ are entirely described by the behavior of CDM for all scales. Conversely, if we consider scales where $k \gg k_{\text{J}}$, we find that perturbations of baryonic matter are negligible and the CDM is the dominant source of gravity.

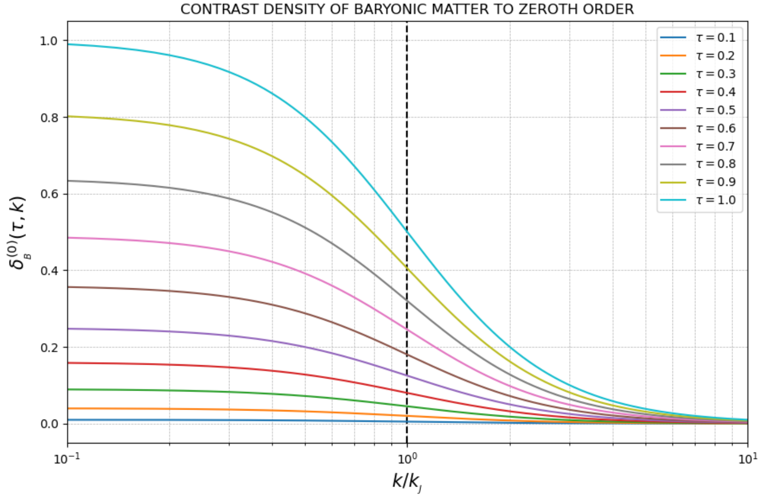

FIGURE 4. *Contrast density of baryonic matter to zeroth order for several values in* τ

It is important to point out that when considering scales near the Jeans wave number, baryonic perturbations could significantly contribute to structure formation. Additionally, in Figure 4, we can observe that for scales where $k \gg k_{\text{J}}$, the fluctuations in baryonic matter tend to zero, while for other sclaes $k \ll k_{\text{J}}$, they take on a constant value, which holds true when we fix several values of conformal time. Now, moving to first order of iteration, we can rewrite the contribution of CDM as $f_{\text{C}} = 1 - f_{\text{B}}$. Then we can express equation (41) as

$$\frac{\partial^2 \tilde{\delta}_{\text{C}}(\tau, \mathbf{k})}{\partial \tau^2} + \frac{2}{\tau}\frac{\partial \tilde{\delta}_{\text{C}}(\tau, \mathbf{k})}{\partial \tau} - \frac{6}{\tau^2}\left[1 - f_{\text{B}}\Big(1 - g(\tau, \mathbf{k})\Big)\right]\tilde{\delta}_{\text{C}}(\tau, \mathbf{k}) = 0.$$

Now, we need substitute the JFF to zeroth order, $g^{(0)}(k)$, into this last equation, just as we did with some equations before. This leads to a characteristic equation

$$\tau^n\left[n^2+n-6\Big(1-f_{\mathrm{B}}\left(1-g^{(0)}(k)\right)\Big)\right]=0,$$

if $\tau^n \neq 0$, then

$$n_{\pm}^{(1)}(k)=\frac{1}{2}\left[-1\pm 5\sqrt{1-\frac{24}{25}f_{\mathrm{B}}\left[1-g^{(0)}(k)\right]}\right]. \tag{50}$$

We can see that if the contribution of baryonic matter is zero, then we return to a solution proportional to τ^{-3} and τ^2 for CDM. Therefore, the general solution to first order for CDM is a combination: $\tilde{\delta}_{\mathrm{C}}(\tau,k)=C_1\tau^{n_+^{(1)}(k)}+C_2\tau^{n_-^{(1)}(k)}$, where C_1 and C_2 are integration constants once again. Approximating the function, square root in (50), through Taylor's expansion as,

$$\sqrt{1-\frac{24}{25}f_{\mathrm{B}}\left[1-g^{(0)}(k)\right]}\approx 1-\frac{12}{25}f_{\mathrm{B}}\left[1-g^{(0)}(k)\right],$$

we can describe the growing (+) and decaying (−) modes, respectively, as [6]

$$\begin{aligned} n_{+}^{(1)}(k)&=2-\frac{6}{5}f_{\mathrm{B}}\left[1-g^{(0)}(k)\right],\\ n_{-}^{(1)}(k)&=-3-\frac{6}{5}f_{\mathrm{B}}\left[1-g^{(0)}(k)\right]. \end{aligned} \tag{52}$$

In general $\tilde{\delta}_{\mathrm{C}}(\tau,k)\propto\tau^{n_{\pm}^{(1)}(k)}$. Therefore: (1) if $k\to 0(k\ll k_{\mathrm{J}})$ then $g^{(0)}(k)\to 1$ and consequently, $\tilde{\delta}_{\mathrm{C}}(\tau,k)\propto\tau^2$ so the structure formation is governed by CDM. (2) if $k\to\infty(k\gg k_{\mathrm{J}})$ then $g^{(0)}(k)\to 0$, and consequently $\tilde{\delta}_{\mathrm{C}}(\tau,k)\propto\tau^{2-(6/5)f_{\mathrm{B}}}$, and we have a contribution of baryonic matter to describe the evolution of the contrast density of CDM [6]. Now, calculating the JFF to first order, $g^{(1)}(k)$, only for growing mode:

$$g^{(1)}(\tau,\mathbf{k})=\frac{\tilde{\delta}_{\mathrm{B}}(\tau,\mathbf{k})}{\tilde{\delta}_{\mathrm{C}}(\tau,\mathbf{k})}\longrightarrow\tilde{\delta}_{\mathrm{B}}(\tau,\mathbf{k})=g^{(1)}(\tau,\mathbf{k})\tau^{n_+^{(1)}(k)}, \tag{53}$$

using this result in (42)

$$\ddot{g}^{(1)}(\tau,\mathbf{k})\tau^{n_+^{(1)}(k)} + 2\dot{g}^{(1)}(\tau,\mathbf{k})n_+^{(1)}(k)\tau^{n_+^{(1)}(k)-1} \\ + g^{(1)}(\tau,\mathbf{k})n_+^{(1)}(k)\left[n_+^{(1)}(k)-1\right]\tau^{n_+^{(1)}(k)-2} \\ + \frac{2}{\tau}\left[\dot{g}^{(1)}(\tau,\mathbf{k})\tau^{n_+^{(1)}(k)} + g^{(1)}(\tau,\mathbf{k})n_+^{(1)}(k)\tau^{n_+^{(1)}(k)-1}\right] \\ - \frac{6}{\tau^2}\left[(1-f_{\mathrm{B}})\tilde{\delta}_{\mathrm{C}}^{(1)}(\tau,\mathbf{k}) + f_{\mathrm{B}}\tilde{\delta}_{\mathrm{B}}^{(1)}(\tau,\mathbf{k}) - \frac{k^2}{k_{\mathrm{J}}^2}g^{(1)}(\tau,\mathbf{k})\tau^{n_+^{(1)}(k)}\right] = 0,$$

and after some algebric manipulations

$$\ddot{g}^{(1)}(\tau,\mathbf{k}) + \frac{2}{\tau}\left[n_+^{(1)}(k)+1\right]\dot{g}^{(1)}(\tau,\mathbf{k}) \\ + \frac{1}{\tau^2}\left[n_+^{(1)}(k)\left[n_+^{(1)}(k)-1\right] + 2n_+^{(1)}(k) - 6\left(f_{\mathrm{B}} - \frac{k^2}{k_{\mathrm{J}}^2}\right)\right]g^{(1)}(\tau,\mathbf{k}) \\ = \frac{6}{\tau^2}\left(1-f_{\mathrm{B}}\right).$$

By exploring the coefficients of $\dot{g}^{(1)}(\tau,\mathbf{k})$ and $g^{(1)}(\tau,\mathbf{k})$, respectively, we find: $2n_+^{(1)}(k) + 2 = 5\sqrt{1 - \frac{24}{25}f_{\mathrm{B}}\left[1-g^{(0)}(k)\right]} + 1$, and $n_+^{(1)}(k)\left[n_+^{(1)}(k)-1\right] + 2n_+^{(1)}(k) - 6\left(f_{\mathrm{B}} - \frac{k^2}{k_{\mathrm{J}}^2}\right)$ is equal to $6\left[1 + \frac{k^2}{k_{\mathrm{J}}^2} - f_{\mathrm{B}}\left(2 - g^{(0)}(k)\right)\right]$.

Therefore, the equation that describes the evolution of the JFF to first order is [6]

$$\ddot{g}^{(1)}(\tau,\mathbf{k}) + \frac{1}{\tau}\left[1 + 5\sqrt{1 - \frac{24}{25}f_{\mathrm{B}}\left(1-g^{(0)}(k)\right)}\right]\dot{g}^{(1)}(\tau,\mathbf{k}) \\ + \frac{6}{\tau^2}\left[1 + \frac{k^2}{k_{\mathrm{J}}^2} - f_{\mathrm{B}}\left(2-g^{(0)}(k)\right)\right]g^{(1)}(\tau,\mathbf{k}) = \frac{6}{\tau^2}\left(1-f_{\mathrm{B}}\right). \quad (54)$$

Solving the homogeneous equation (Cauchy-Euler equation), we obtain a general solution for the JFF

$$g^{(1)}(\tau,k) = C_1 \tau^{-\frac{5}{2}\sqrt{1-\frac{24}{25}f_{\rm B}\left[1-g^{(0)}(k)\right]}+\frac{1}{2}\sqrt{1-24\left(\frac{k^2}{k_{\rm J}^2}-f_{\rm B}\right)}} + C_2 \tau^{-\frac{5}{2}\sqrt{1-\frac{24}{25}f_{\rm B}\left[1-g^{(0)}(k)\right]}-\frac{1}{2}\sqrt{1-24\left(\frac{k^2}{k_{\rm J}^2}-f_{\rm B}\right)}}, \quad (55)$$

and with a particular solution: $g^{(1)}(k) = \dfrac{1-f_{\rm B}}{1+\dfrac{k^2}{k_{\rm J}^2}-f_{\rm B}\left[2-g^{(0)}(k)\right]}$.

That is to say, the solution for the growing mode is

$$g^{(1)}(k) = \frac{1-f_{\rm B}}{1-f_{\rm B}+\dfrac{k^2}{k_{\rm J}^2}\left[1-\dfrac{f_{\rm B}}{1+k^2/k_{\rm J}^2}\right]}. \quad (56)$$

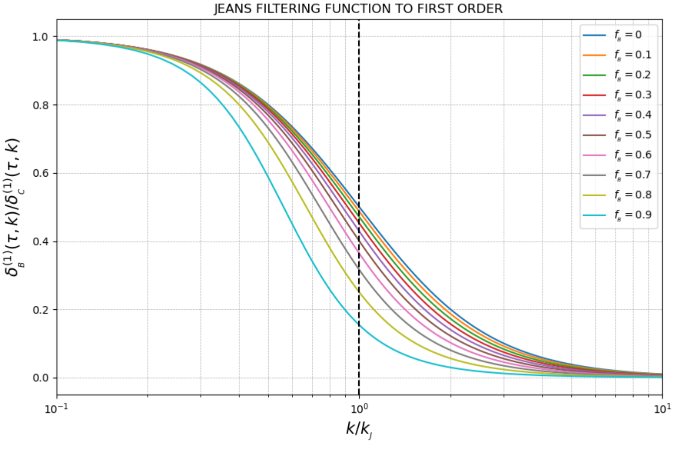

FIGURE 5. *Jeans filtering function to first order for several contributions of baryonic matter f_B*

Then, the behavior of fluctuations of baryonic matter $\tilde{\delta}_{\rm B}(\tau, k)$ are described with the function:

$$\frac{\tilde{\delta}^{(1)}_{\rm B}(\tau,k)}{\tilde{\delta}^{(1)}_{\rm C}(\tau,k)} = \frac{1-f_{\rm B}}{1-f_{\rm B}+\dfrac{k^2}{k_{\rm J}^2}\left[1-\dfrac{f_{\rm B}}{1+k^2/k_{\rm J}^2}\right]},$$
$$\tilde{\delta}^{(1)}_{\rm B}(\tau,k) = \frac{1-f_{\rm B}}{1-f_{\rm B}+\dfrac{k^2}{k_{\rm J}^2}\left[1-\dfrac{f_{\rm B}}{1+k^2/k_{\rm J}^2}\right]}\tau^{2-\frac{6}{5}f_{\rm B}\left[1-g^{(0)}(k)\right]}. \tag{57}$$

In this last expression, we can see that if the contribution of $f_{\rm B}$ is zero, then we can revert to equation (49) (zeroth order iteration). Furthermore, if we consider $f_{\rm B} = 1$, we lose the fluctuations of baryonic matter. As shown in Figure 5, for different values of f_B, when $k \gg k_{\rm J}$ or $k \ll k_{\rm J}$, the evolution of baryonic contrast density follows the same path as that of CDM contrast density. However, when we consider that $k \approx k_{\rm J}$, the evolution in these fields are very different.

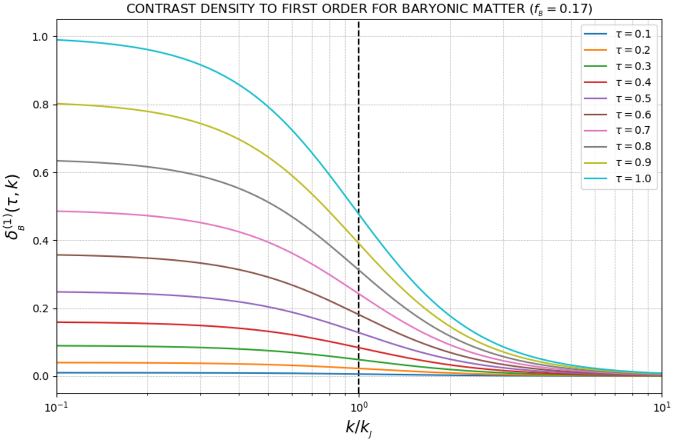

FIGURE 6. *Contrast density of baryonic matter to first order with a baryonic contribution of* $f_B = 0.17$ *[3]*

Additionally, in Figures 6 and 7, we can observe the behavior of baryonic contrast density for various conformal times or different values of $f_{\rm B}$. If we consider the ratio of the two Jeans Filtering Function at zeroth and first order of iteration, equations for $g^{(0)}(k)$

and $g^{(1)}(k)$, we can say that as $k/k_{\text{J}} \to \infty (k \gg k_{\text{J}})$ [6], there is a signifiicant difference between these functions, as depicted in Figure 8. Therefore, we could say for large scales $(k \to 0)$, it is sufficient to use the JFF to zeroth order to describe the evolution of a fluid composed of baryonic and CDM matter, as shown in Figure 8 once again.

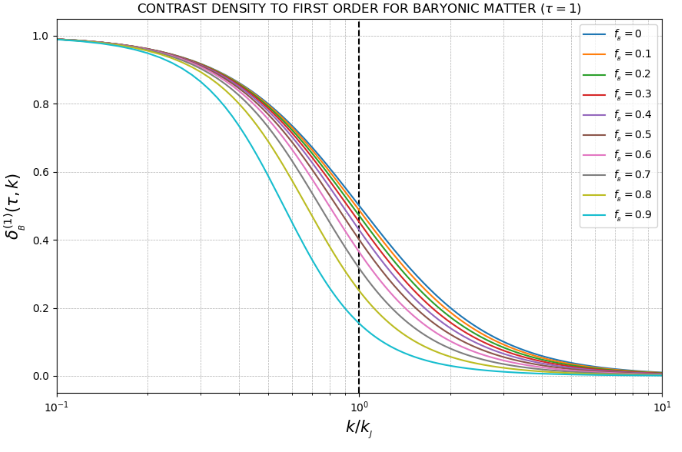

FIGURE 7. *Contrast density of baryonic matter to first order with the condition* $\tau = 1$

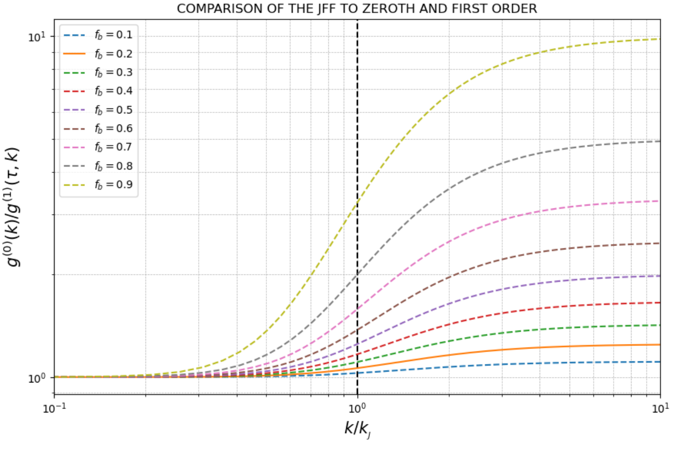

FIGURE 8. *Comparison of the Jeans Filtering Function to zeroth and first order for different values of* f_B. *The continuous line shows* $f_B = 0.2$

Furthermore, we can observe that, when aiming for greater precision in cosmology to explain structure formation at small scales (when $k \gg k_{\mathrm{J}}$), the difference between the JFF at zeroth and first orders must be considered. And, at the end of the day, for high orders (when $k \ll k_{\mathrm{J}}$) we can choose the JFF at zeroth order as shown in Figure 8. In this graph, the continuous line shows the behavior of the ratio between the zeroth and first order for the JFF of the baryonic component according to the current observations.

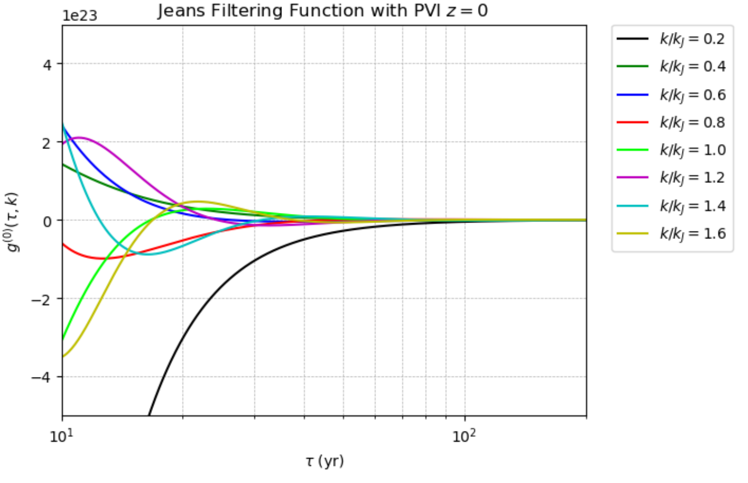

FIGURE 9. *General solution of the Jeans Filtering Function for initial value problem $z = 0$*

Finally, we can determine the general structure of the JFF using the initial conditions $g^{(0)}(\tau, k) = 1$ and $\dot{g}^{(0)}(\tau, k) = 0$ for $0, 4, 8$ and 10 in redshift [6]. In Figures 9 to 12, we can observe the general behaivor of the Jeans Filtering Functions at zeroth order as a function of conformal time. In each of these cases, we find that if you want to improve the precision of the baryonic fluctuations, the decaying modes cannot be neglected for small scales, and this modes introduce the oscillatory behavior in the solutions in early times. Furthermore, if we plot $g(\tau, k)$ as a function of k scale, for the condition $z = 0, 4$, and 8, we can find the behavior as shown in Figure 13.
In the left panel of Figure 13, we calculate the function of JFF,

$$\Delta g^{(0)}(\tau, k) = g^{(0)}(\tau, k) - \left[1 + \frac{k^2}{k_{\mathrm{J}}^2}\right]^{-1}, \tag{1}$$

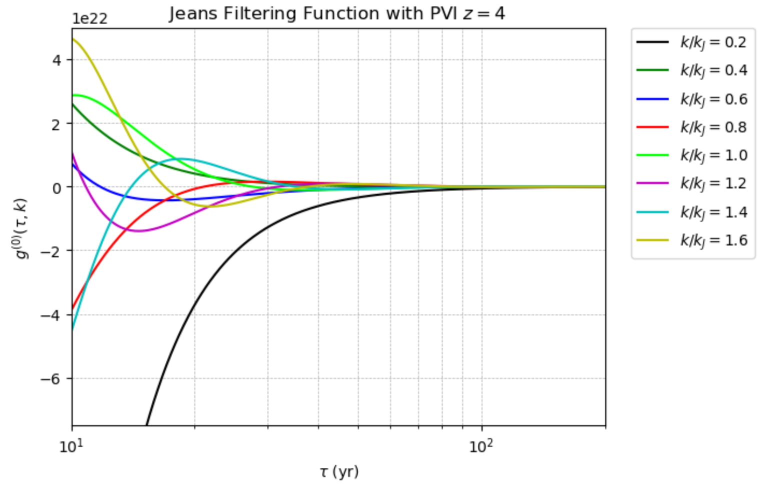

FIGURE 10. *General solution of the Jeans Filtering Function for initial value problem* $z = 4$

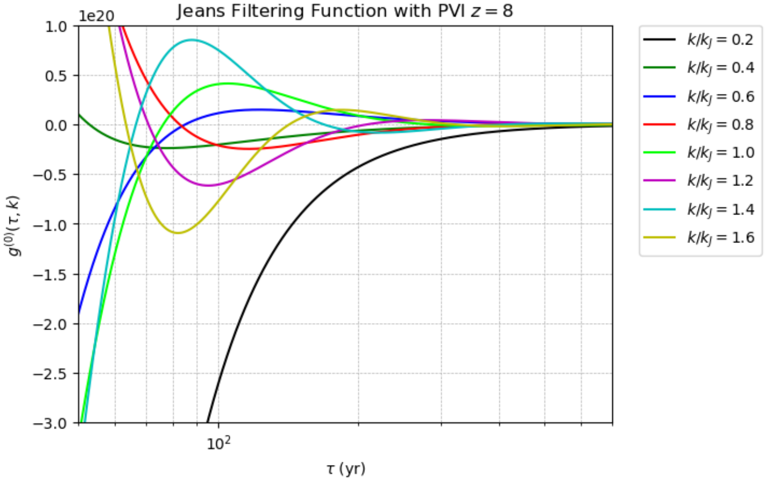

FIGURE 11. *General solution of the Jeans Filtering Function for initial value problem* $z = 8$

for different values of redshift [6]. This analysis aims to show that, when considering low scales, it is not sufficient to consider only growing modes because oscillations play a crucial role in describing the evolution of baryonic matter fluctuations. In the right panel, we can observe that the red line (an approximation to the growth mode

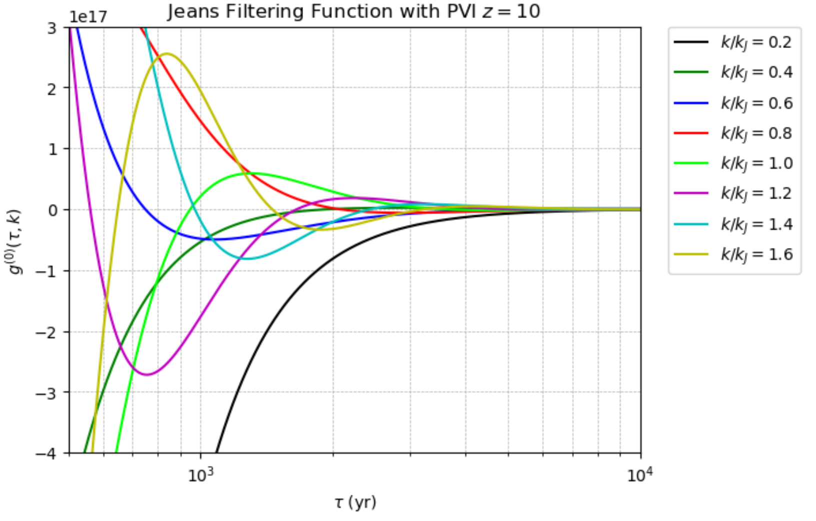

FIGURE 12. *General solution of the Jeans Filtering Function for initial value problem $z = 10$*

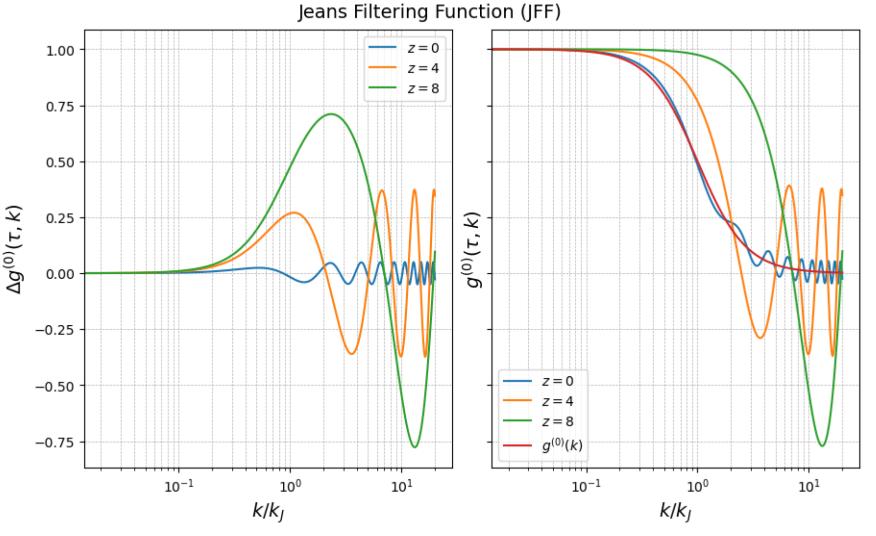

FIGURE 13. *General solution of the Jeans Filtering Function*

shown in Figure 3) for $k \gg k_J$ exhibits a high level of imprecision. However, as $k \to 0$, its approximation for only growing modes, provies a much better description of the evolution of fluctuations.

Discussion and Conclusions

In this paper, we have determined that for a universe under the ΛCDM model, the level of fluctuation with respect to the FLRW metric results in a particular order of magnitude growth for galaxies, approximately on the order of $\sim 10^{-3}$ [19]. That is to say, the structures in the universe, such as galaxies, evolve due to a fluctuation near $\sim 3 \times 10^{-3}$ with respect to unity [1, 15].

However, it is important to note that this situation strongly depends on the cosmological model. Additionally, we have obtained that the growth factor for the ΛCDM model, could be normalized to unity as shown in Figure 1. If we only consider growing modes in the linear regime at zeroth order, when we introduce baryonic matter, we can obtain the predictions of ΛCDM. In this scenario, the effects of fluctuations in baryonic matter follow the fluctuations of cold dark matter for large scales, and are suppressed at small scales. However, when the scale is comparable to the Jeans scale, the contribution of baryonic matter becomes significant, as shown in Figures 3 and 4.

Similar results are obtained, for first-order effects and different values of conformal time or the contribution of baryonic matter, as depicted in Figures 5, 6 and 7. When we use this approximation, we can employ the Jeans filtering function at zeroth order to describe high-order terms [6], as demonstrated in the last section, supported by Figure 8. It was very important to demostrate that, when we find numerical solutions to the Jeans filtering function at zeroth order, we can not neglect the decaying modes if we aim to increase the precision in this type of tools, as shown through the graphs 9 to 13. At this point, as shown in Figure 13, it is significant to observe that if we want to work on small scales, we cannot neglect the decaying modes. Therefore, we consider that including this type of modes could be important in the development of high-precision theories, including those involving baryonic matter.

Finally, we believe that this work expands upon the research conducted by Shoji and Komatsu [6] at first order in delta (δ), and for this order, we reproduce the zeroth and first iterations of the Jeans Filtering Functions, $g^{(0)}(\tau, k)$ and $g^{(1)}(\tau, k)$. We aim for this work to assits us in understanding the baryonic physics in the large scale structure and, why not, employ it in various simulations to comprehend different asthrophysical processes [20].

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