

q - DIFFERENTIAL OPERATORS AND DERIVATIONS ON THE QUADRATIC RELATIVISTIC INVARIANT ALGEBRAS

OPERADORES DE DIFERENCIALES q - Y DERIVACIONES DE LAS ALGEBRAS INVARIANTES RELATIVISTAS CUADRÁTICAS

Julio C. Jaramillo-Quiceno

Universidad Nacional de Colombia, Sede Bogotá, Departamento de Física, Colombia.

(Received: jan./2024. Accepted: may./2024)

Abstract

This study aims to develop a new algebra based on the Minkowskian product or relativistic Lorentz invariants. This leads to the notion of q - invariant algebras, defining q - deformed quadratic relativistic algebras and establishing some q - differential operators and derivations. Then, from these algebras, we define the q - relativistic invariant function on a free algebra $\mathbb{k}\langle x, y, z, u \rangle$ with the objective of formulating the q - differential quadratic operators. On the other hand, we define the q - quadratic differential operators on the Clifford algebra $Cl_{0,n}$. We consider the case of a polynomial function in the noncommutative quadratic variables x^2, y^2, z^2 and u^2 , obtaining the q - differential quadratic operators for these functions with their respective properties. Furthermore, we formulate the q - quadratic Dirac differential operators. On the other hand, on the algebra Ψ with the generators x, y , we have proposed the extended derivation with its corresponding properties, in order to apply it to the q - relativistic invariant algebras and make a relationship with the q - Dirac quadratic operators. On a function of non-commutative quadratic variables, we

define the *q*- quadratic differentiation operators D_{q^2} with their properties, and finally some applications for further work.

Keywords: Lorentz invariant, *q*- relativistic invariant algebras, quadratic relativistic algebras, *q*- difference operators and derivations.

Resumen

Este estudio tiene como objetivo desarrollar una nueva álgebra basada en el producto minkowskiano o invariantes relativistas de Lorentz. Esto lleva a la noción de *q*-álgebras invariantes, definiendo *q*-álgebras relativistas cuadráticas deformadas y estableciendo algunos *q*-operadores diferenciales y derivaciones. Luego, a partir de estas álgebras, definimos la *q*- función invariante relativista sobre un álgebra libre $\mathbb{k}\langle x, y, z, u \rangle$ con el objetivo de formular los *q*- operadores diferenciales cuadráticos. Por otro lado, definimos los *q*- operadores diferenciales cuadráticos sobre el álgebra de Clifford $Cl_{0,n}$. Consideramos el caso de una función polinómica en las variables cuadráticas no conmutativas x^2, y^2, z^2 y u^2 , obteniendo los *q*- operadores cuadráticos diferenciales para estas funciones con sus correspondientes propiedades. Además, formulamos los *q*- operadores diferenciales cuadráticos de Dirac. Por otro lado, sobre el álgebra Ψ con los generadores x, y , hemos propuesto la derivación extendida con sus respectivas propiedades, con el fin de aplicarla a las *q*-álgebras invariantes relativistas y realizar una relación con los *q*- operadores cuadráticos de Dirac. Sobre una función de variables cuadráticas no conmutativas, definimos los *q*- operadores de derivación cuadráticos D_{q^2} con sus propiedades, y finalmente algunas aplicaciones para trabajos posteriores.

Palabras clave: invariante de Lorentz, *q*- álgebras invariantes relativistas, álgebras cuadráticas relativistas, *q*- operadores diferenciales y derivaciones.

1. Introduction

The study on q - difference operators appeared already at the beginning of the last century by Jackson [1], and subsequently studied and developed in [2, 3]. The Quantum difference operators have an interesting role due to their applications in several mathematical areas such as orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations and the **theory of relativity** [2]. The quadratic algebra relation $yx = \alpha x^2 + \beta xy + \gamma y^2$ was introduced by Golovashkin and Maksimov [4] and Rosengren [5], and subsequently studied and developed in [6, 7]. Let \mathbb{k} be a unital commutative associative ring. We present a construction method for q - quadratic relativistic algebras starting from a scalar product or Minkowskian product [8]. The purpose of this article is to define the q - difference operators and derivations on the quadratic relativistic invariant algebras. The paper is organized as follows: in section 2, we present some preliminaries. In section 3, we present the q - invariant relativistic algebras. In the final section, we present the q - difference operators.

2. Preliminaries

The main objective of the principle of special relativity is to establish invariant laws of physics that hold true in different reference frames [9]. Let $e = \{e_1, e_2, e_3, e_4\}$ be an orthonormal basis of \mathbb{R}^4 equipped with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$, and let $x = x^a e_a$ and $y = y^a e_a$ be the coordinate expressions of $x, y \in \mathbb{R}^4$ in this basis. Then

$$xy = x_4 y_4 - x_1 y_1 - x_2 y_2 - x_3 y_3. \quad (1)$$

We call xy the scalar product or Minkowskian product [8]. We will consider the 4 - vectors $x = (x_1, x_2, x_3, x_4), y = (-x_1, -x_2, -x_3, x_4)$, and Λ the Lorentz transformation, which is defined as a linear transformation i. e $x^\mu = \Lambda x_\nu$. From the definition of a Lorentz transformation, [9] it is clear that the norm of a 4-vector is an invariant, i.e.

$$x'y' = (\Lambda x)(\Lambda y) = xy, \tag{2}$$

in the special case where the $x = y$:

$$(x')^2 = x_4^2 - x_1^2 - x_2^2 - x_3^2 = x^2, \tag{3}$$

so is an invariant. In this work, we will denote (1) as $x^\alpha y_\alpha$ for all $\alpha = 1, 2, 3$ and 4. On other hand, let Ψ be an algebra over field \mathbb{k} , and recall that a \mathbb{k} - linear map $\partial : \Psi \rightarrow \Psi$ is a derivation of Ψ if for all $x, y \in \Psi$, we have $\partial(xy) = \partial(x)y + x\partial(y)$ [7]. The space of all derivations of Ψ is a Lie algebra with respect to the operation of commutation, and we will denote this algebra by $Der(\Psi)$ [7]. Let R be a ring, σ a ring endomorphism of R , and δ an σ -derivation on R . We shall write $S = R[x; \sigma, \delta]$ provided: (i) S is a ring, containing R as subring, (ii) x is an element of S , (iii) S is a free left R -module with basis $\{1, x, x^2, \dots\}$, and (iv) $xy = \sigma(x)y + \delta(x)$ for all $y \in R$. Such a ring S is called a skew polynomial ring over R , or an Ore extension of R (see [10], p. 34).

Let $\{e_1, e_2, \dots, e_n\}$ be an orthonormal basis of \mathbb{R}^n . The Clifford algebra $Cl_{0,n}$ is the free algebra over \mathbb{R}^n generated modulo the relation

$$x^2 = -|x|^2 e_0, \tag{4}$$

where e_0 is the identity of $Cl_{0,n}$. For the algebra $Cl_{0,n}$ we have the anticommutation relationship

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \tag{5}$$

where δ_{ij} is the Kronecker symbol [11]. Let Ω be a domain of \mathbb{R}^{n+1} , $f = \sum_X f_X e_X : \Omega \rightarrow X$ is an X - valued C^1 function, $f_X(x)$ are real valued functions on f [12]. We give a short introduction to Dirac Operator, see e.g. [11–13]. The Dirac operator is defined by

$$D := \sum_{j=1}^n e_j \frac{\partial}{\partial x_j}.$$

q - Invariant algebras

DEFINITION 0.0.1. Let $x^\alpha y_\alpha$ be the lorentzian invariants. The q -invariant relativistic algebra over \mathbb{k} is the \mathbb{k} -algebra $\Psi^\alpha(\mathbb{k}, q)$ given by the generators $x^\alpha, y^\alpha, \alpha = 1, 2, 3, 4$, and defining the relations

$$[x^\alpha y_\alpha]_4 = q^2 x^4 y_4 - x^1 y_1 - x^2 y_2 - x^3 y_3, \quad (6)$$

$$[x^\alpha y_\alpha]_1 = x^4 y_4 - q^2 x^1 y_1 - x^2 y_2 - x^3 y_3, \quad (7)$$

$$[x^\alpha y_\alpha]_2 = x^4 y_4 - x^1 y_1 - q^2 x^2 y_2 - x^3 y_3, \quad (8)$$

$$[x^\alpha y_\alpha]_3 = x^4 y_4 - x^1 y_1 - x^2 y_2 - q^2 x^3 y_3, \quad (9)$$

i.e.,

$$\Psi_1(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle x^1, y_1 \rangle}{\langle [x^\alpha y_\alpha]_1 = x^4 y_4 - q^2 x^1 y_1 - x^2 y_2 - x^3 y_3 \rangle},$$

$$\Psi_2(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle x^2, y_2 \rangle}{\langle [x^\alpha y_\alpha]_2 = x^4 y_4 - x^1 y_1 - q^2 x^2 y_2 - x^3 y_3 \rangle},$$

$$\Psi_3(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle x^3, y_3 \rangle}{\langle [x^\alpha y_\alpha]_3 = x^4 y_4 - x^1 y_1 - x^2 y_2 - q^2 x^3 y_3 \rangle},$$

$$\Psi_4(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle x^4, y_4 \rangle}{\langle [x^\alpha y_\alpha]_4 = q^2 x^4 y_4 - x^1 y_1 - x^2 y_2 - x^3 y_3 \rangle}.$$

PROPOSITION 0.0.1.

We will make the following assumptions: $x^\alpha = y_\alpha$. Therefore the relations (6), (7), (8), and (9) can be written as

$$q^2 y_4^2 - y_1^2 - y_2^2 - y_3^2, \quad (10)$$

$$y_4^2 - q^2 y_1^2 - y_2^2 - y_3^2, \quad (11)$$

$$y_4^2 - y_1^2 - q^2 y_2^2 - y_3^2, \quad (12)$$

$$y_4^2 - y_1^2 - y_2^2 - q^2 y_3^2, \quad (13)$$

we will denote by u_q^2 the expression (10), x_q^2 (11), y_q^2 (12), and z_q^2 (13).

Notation

For simplicity of notation, we write (10), (11), (12) and (13) as

$$u_q^2 = q^2u^2 - x^2 - y^2 - z^2, \tag{14}$$

$$x_q^2 = u^2 - q^2x^2 - y^2 - z^2, \tag{15}$$

$$y_q^2 = u^2 - x^2 - q^2y^2 - z^2, \tag{16}$$

$$z_q^2 = u^2 - x^2 - y^2 - q^2z^2. \tag{17}$$

PROPOSITION 0.0.2. *The *q* - Quadratic relativistic invariant algebra $L(q^2)$ over \mathbb{k} is the \mathbb{k} - algebra $\Psi_\beta(\mathbb{k}, q^2)$, $1 \leq \beta \leq 6$, given by generators x, y, z, u , and is defined by the commutation relations*

$$yx = u^2 - xy - q^2y^2 - z^2, \tag{18}$$

$$zx = u^2 - xz - y^2 - q^2z^2, \tag{19}$$

$$yz = u^2 - x^2 - q^2y^2 - zy, \tag{20}$$

$$ux = q^2u^2 - xu - y^2 - z^2, \tag{21}$$

$$uy = q^2u^2 - x^2 - yu - z^2, \tag{22}$$

$$uz = q^2u^2 - x^2 - y^2 - zu. \tag{23}$$

The algebra $L(q^2)$ can be constructed as a quotient algebra

$$\Psi_1(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle y, x \rangle}{yx - u^2 + xy + q^2y^2 + z^2},$$

$$\Psi_2(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle x, z \rangle}{zx - u^2 + xz + y^2 + q^2z^2},$$

$$\Psi_3(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle y, z \rangle}{yz - u^2 + x^2 + q^2y^2 + zy},$$

$$\Psi_4(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle u, x \rangle}{ux - q^2u^2 + xu + y^2 + z^2},$$

$$\Psi_5(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle u, y \rangle}{uy - q^2u^2 + x^2 + yu + z^2},$$

$$\Psi_6(\mathbb{k}, q^2) = \frac{\mathbb{k}\langle u, z \rangle}{uz - q^2u^2 + x^2 + y^2 + zu}.$$

of the free associative algebra with four generators x, y, z and u by the ideal generated by $yx - u^2 + xy + q^2y^2 + z^2, zx - u^2 + xz + y^2 + q^2z^2, yz - u^2 + x^2 + q^2y^2 + zy, ux - q^2u^2 + xu + y^2 + z^2, uy - q^2u^2 + x^2 + yu + z^2, \text{ and } uz - q^2u^2 + x^2 + y^2 + zu, \text{ respectively.}$

REMARK 1. Consider the general Clifford number $a = \sum_X e_X a_X$, the relation (4) proposed in [13], and the Lorentzian invariants defined by (6), (7), (8) and (9). The general Clifford Lorentzian invariants are determined by the following expressions

$$[x^\alpha y_\alpha]_4 = -q e_0 x^4 y_4 + e_0 x^1 y_1 + e_0 x^2 y_2 + e_0 x^3 y_3, \quad (24)$$

$$[x^\alpha y_\alpha]_1 = -e_0 x^4 y_4 + q e_0 x^1 y_1 + e_0 x^2 y_2 + e_0 x^3 y_3, \quad (25)$$

$$[x^\alpha y_\alpha]_2 = -e_0 x^4 y_4 + e_0 x^1 y_1 + q e_0 x^2 y_2 + e_0 x^3 y_3, \quad (26)$$

$$[x^\alpha y_\alpha]_3 = -e_0 x^4 y_4 + e_0 x^1 y_1 + e_0 x^2 y_2 + q e_0 x^3 y_3. \quad (27)$$

3. q - Difference Operators and Derivations

DEFINITION 0.0.2. Let \mathbb{k} be a field and $\langle x, y, z, u \rangle$ any set; we shall denote the free \mathbb{k} - algebra on $\langle x, y, z, u \rangle$ by $\mathbb{k}\langle x, y, z, u \rangle$. An element $f \in \mathbb{k}\langle x, y, z, u \rangle$ may be thought of as a polynomial function in four noncommuting variables x, y, z, u .

DEFINITION 0.0.3. Let Ψ be an \mathbb{k} - algebra on $\langle x, y, z, u \rangle$. We call f the q - invariant relativistic function or q - invariant function $f : \Psi \longrightarrow \Psi$, and will be referred to Ψ as \mathbb{k} - invariant relativistic algebra.

PROPOSITION 0.0.3. We will consider f as a polynomial function that depends on the quadratic variables $x^2, y^2, z^2, u^2, ux, uy, uz, xy$ and xz , thus $f(ux, uy, uz, xy, xz, yz, x^2, y^2, z^2, u^2)$.

From the above proposition, we will introduce the q - difference operator over $f \in \mathbb{k}\langle x, y, z, u \rangle$ as follows.

PROPOSITION 0.0.4. Let $f \in \mathbb{k}\langle x, y, z, u \rangle$ be a q - invariant relativistic function. For $q \in \mathbb{R} - \{0\}$, the q - quadratic difference

operators over $f \in \mathbb{k}\langle x, y, z, u \rangle$ are

$$\frac{\partial_{q^2} f}{\partial_{q^2}(ux)} = \frac{f(q^2u^2 + ux) - f(ux)}{q^2u^2}, \tag{28}$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(uy)} = \frac{f(q^2u^2 + uy) - f(uy)}{q^2u^2}, \tag{29}$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(uz)} = \frac{f(q^2u^2 + uz) - f(uz)}{q^2u^2}, \tag{30}$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(yz)} = \frac{f(q^2y^2 + yz) - f(yz)}{q^2y^2}, \tag{31}$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(xy)} = \frac{f(q^2y^2 + xy) - f(xy)}{q^2y^2}, \tag{32}$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(xz)} = \frac{f(q^2z^2 + xz) - f(xz)}{q^2z^2}. \tag{33}$$

Proof Proposition 0.0.4

Consider the Jackson derivative [1]

$$\frac{\partial_{q^2} f}{\partial_{q^2} \lambda} = \frac{f(q^2\lambda) - f(\lambda)}{q^2\lambda - \lambda}. \tag{34}$$

To deduce (28) from (34), take

$$q^2\lambda = q^2u^2 + ux, \tag{35}$$

$$\lambda = ux. \tag{36}$$

Combining (35) and (36) with (34) we obtain (28). The same reasoning applies to the Eqs. (29), (30), (31), (32) and (33), and the proof is complete.

In the following, we present some examples of $q-$ difference operator which has one fixed function $f \in \mathbb{k}\langle x, y, z, u \rangle$:

DEFINITION 0.0.4. *The function $f = \sum_{\Psi} e_{\Psi} f_{\Psi} : \Omega \rightarrow \Psi$ is said to be a Ψ - valued C^1 function, if only if $f_{\Psi}(x)$ are real valued functions on x [12, 13].*

LEMMA 1. *The $q-$ invariant relativistic function is a Ψ - valued C^1 function.*

PROPOSITION 0.0.5. *Let f be a Ψ - valued C^1 function and (4). For $q \in \mathbb{R} - \{0\}$, the q - quadratic derivatives over the Clifford algebra $Cl_{0,n}$ are determined by the following expressions*

$$\frac{\partial_{q^2} f}{\partial_{q^2}(ux)} = \frac{f(-q^2|u^2|e_0 + ux) - f(ux)}{q^2 u^2}, \quad (37)$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(uy)} = \frac{f(-q^2|u^2|e_0 + uy) - f(uy)}{q^2 u^2}, \quad (38)$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(uz)} = \frac{f(-q^2|u^2|e_0 + uz) - f(uz)}{q^2 u^2}, \quad (39)$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(yz)} = \frac{f(-q^2|y^2|e_0 + yz) - f(yz)}{q^2 y^2}, \quad (40)$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(yx)} = \frac{f(-q^2|y^2|e_0 + yx) - f(yx)}{q^2 y^2}, \quad (41)$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(zx)} = \frac{f(-q^2|z^2|e_0 + zx) - f(zx)}{q^2 z^2}. \quad (42)$$

This construction is due ([12, 13]).

Proof of Proposition 0.0.5

This construction follows [13]. According to (4), consider $u^2 = -|u^2|e_0$, $y^2 = -|y^2|e_0$ and $z^2 = -|z^2|e_0$ and substituting into (28), (29), (30), (31), (32) and (33). Which completes the proof.

EXAMPLE 0.1. *Let $f(xy) = qxy + q^2$ for $q \in (0, 1)$. Applying (32) for this case, we obtain*

$$\begin{aligned} \frac{\partial_{q^2} f}{\partial_{q^2}(xy)} &= \frac{q(q^2 y^2 + xy) + q^2 - qxy - q^2}{q^2 y^2}, \\ &= \frac{q^3 y^2 + qxy + q^2 - qxy - q^2}{q^2 y^2}, \\ &= q^2. \end{aligned} \quad (43)$$

DEFINITION 0.0.5. *Let $f \in \mathbb{k}$ be a polynomial function in noncommuting quadratic variables u^2, x^2, y^2 and $z^2 \in L(q^2)$. We*

define the q - quadratic difference operators for f as:

$$\frac{d_{q^2} f}{d_{q^2} u^2} = \frac{f(q^2 u^2 + x^2) - f(x^2)}{q^2 u^2}, \tag{44}$$

$$\frac{d_{q^2} f}{d_{q^2} x^2} = \frac{f(u^2 + q^2 x^2) - f(q^2 x^2)}{u^2}, \tag{45}$$

$$\frac{d_{q^2} f}{d_{q^2} y^2} = \frac{f(u^2 + q^2 y^2) - f(q^2 y^2)}{u^2}, \tag{46}$$

$$\frac{d_{q^2} f}{d_{q^2} z^2} = \frac{f(u^2 + q^2 z^2) - f(q^2 z^2)}{u^2}. \tag{47}$$

provided that the derivatives (44) exists at $q \in \mathbb{R} - \{0\}$, and (45), (46) and (47) exists at $u^2 \in \mathbb{R} - \{0\}$.

REMARK 2. The expressions (45), (46), and (47) can be written of the following form

$$\frac{d_{q^2} f}{d_{q^2} x_i^2} = \frac{f(u^2 + q^2 x_i^2) - f(q^2 x_i^2)}{u^2}, \quad \forall i = 1, 2, 3. \tag{48}$$

EXAMPLE 0.2.

1. $f(x^2) = (x^2)^m$

$$\frac{d_{q^2} f}{d_{q^2} x^2} = \sum_{k=1}^m \begin{bmatrix} m \\ k \end{bmatrix}_q u^{2m-2k} (q^2 x^2)^k, \tag{49}$$

where $\begin{bmatrix} m \\ k \end{bmatrix}_q$ is the q - combinatorics (see [14]).

2. $f(u^2) = u^2 + b, \forall b \in \mathbb{R}$

$$\begin{aligned} \frac{d_{q^2} f}{d_{q^2} u^2} &= \frac{q^2 u^2 + x^2 + b - (x^2 + b)}{q^2 u^2}, \\ &= 1. \end{aligned} \tag{50}$$

LEMMA 2. If $u^2 = u_0^2$, then for this case, we can establish that (44) is the q - quadratic relativistic derivative of f at u_0^2 . The

same reasoning applies to the expressions (45), (46), and (47) for $u^2 \in \mathbb{R} - \{0\}$.

REMARK 3. We establish that f is q - differentiable on \mathbb{k} if $\frac{d_{q^2}f}{d_{q^2}u^2}(u^2 = u_0^2)$, $\frac{d_{q^2}f}{d_{q^2}x^2}(x^2 = a^2)$, $\frac{d_{q^2}f}{d_{q^2}y^2}(y^2 = b^2)$ and $\frac{d_{q^2}f}{d_{q^2}z^2}(z^2 = c^2)$ exists for $a^2, b^2, c^2, u_0^2 \in \mathbb{k}$.

THEOREM 3. Assume that $g \in \mathbb{k}$ and $h \in \mathbb{k}$ are q - differentiable functions at $x_0^2 \in \mathbb{k}$. Then:

i The product $gh \in \mathbb{k}$ is q - differentiable at x_0^2 and

$$\frac{d_{q^2}(hg)}{d_{q^2}x_i^2} = h(u^2 - q^2x_i^2) \frac{d_{q^2}g}{d_{q^2}x_i^2} + g(u^2 - q^2x_i^2) \frac{d_{q^2}h}{d_{q^2}x_i^2}, \quad \forall i = 1, 2, 3. \tag{51}$$

ii g/h is q - differentiable at x_0^2 and

$$\frac{d_{q^2}(h/g)}{d_{q^2}x_i^2} = \frac{\frac{d_{q^2}g}{d_{q^2}x_i^2}h(q^2x_i^2) - \frac{d_{q^2}h}{d_{q^2}x_i^2}g(q^2x_i^2)}{[g(u^2 - q^2x_i^2)g(q^2x_i^2)]} \quad \forall i = 1, 2, 3, \tag{52}$$

$$g(u^2 - q^2x_i^2)g(q^2x_i^2) \neq 0.$$

Now, we introduce the notion of q - quadratic Dirac operator, following Faustino - Kälher [13], Yafang and Jinyuan [12] and Coulembier and Sommen [11].

DEFINITION 0.0.6. Our aim is to define a q - quadratic Dirac operator. Let f be a map from $\mathbb{R}^n \longrightarrow \mathbb{R}$ and Ψ - valued C^1 function. The q - quadratic differential operators over f are determined by the following expressions

$$\frac{\partial_q f}{\partial_q x} = \frac{f((x + q^2e_0u)u) - f(xu)}{qu}, \tag{53}$$

$$\frac{\partial_q f}{\partial_q y} = \frac{f((y + q^2e_0u)u) - f(yu)}{qu}, \tag{54}$$

$$\frac{\partial_q f}{\partial_q z} = \frac{f(y(z + q^2 e_y y)) - f(yz)}{qy}, \tag{55}$$

$$\frac{\partial_q f}{\partial_q u} = \frac{f((u - q^2 e_y y)y) - f(uy)}{qy}, \tag{56}$$

$$\frac{\partial_q f}{\partial_q x} = \frac{f((x + q^2 e_z z)z) - f(xz)}{qz}, \tag{57}$$

$$\frac{\partial_q f}{\partial_q z} = \frac{f((z + q^2 e_0 u)u) - f(zu)}{qu}, \tag{58}$$

$$\frac{\partial_q f}{\partial_q y} = \frac{f((y + q^2 e_x x)x) - f(yx)}{qx}, \tag{59}$$

$$\frac{\partial_q f}{\partial_q z} = \frac{f((z + q^2 e_x x)x) - f(zx)}{qx}. \tag{60}$$

where e_x, e_y and e_z are the generators for the Clifford algebras. According to references [11–13] and the above expressions, we define the *q*- quadratic Dirac operator as

$$D^q f = e_x \frac{\partial_q f}{\partial_q x} + e_y \frac{\partial_q f}{\partial_q y} + e_z \frac{\partial_q f}{\partial_q z}. \tag{61}$$

0.0.1 Extended Derivation

In this section, we introduce the concept of *Extended Derivation* d , and their properties.

PROPOSITION 0.0.6. *Let x, y be a generators of the algebra Ψ . Given an element $\psi \in \Psi$ consider the **Extended Derivation** d of the algebra Ψ as a linear transformation $d : \Psi \rightarrow \Psi$ such that*

$$d(xy) = \partial(xy) + D\psi. \tag{62}$$

In the following, we state some clear properties of the extended derivation.

- (i) $d_x(xy) = (\partial x)y + D[\psi(y)];$
- (ii) $d_y(xy) = x(\partial y) + D[\psi(x)];$
- (iii) $(d_x + d_y)(xy) = \partial(xy) + D[\psi(y) + \psi(x)];$

$$(iv) \quad D\psi = 0;$$

$$(v) \quad d([y, x]) = \partial([y, x]).$$

Proof Proposition 0.0.6

For (iii) we first compute (i)+(ii)

$$d_x(xy) + d_y(xy) = (\partial x)y + D[\psi(y)] + x(\partial y) + D[\psi(x)],$$

Now $d_x(xy) + d_y(xy) = (d_x + d_y)(xy)$ and $D[\psi(x)] + D[\psi(y)] = D[\psi(x) + \psi(y)]$, thus

$$(d_x + d_y)(xy) = (\partial x)y + x(\partial y) + D[\psi(y) + \psi(x)],$$

and using the Leibniz rule $\partial(xy) = x(\partial y) + (\partial x)y$ yields (iii). To show the property (iv), we assume that ψ is an arbitrary function that depends of the other generators u, z , i.e, $\psi = F(u, z)$, therefore $D\psi = D(F(u, z)) = 0$. And finally for (v), we first show that it suffices to consider the relation (18). Applying (62)

$$d(yx) = -\partial(xy) + D(u^2 - q^2y^2 - z^2), \quad (63)$$

from (63) we have

$$d(xy) = -\partial(yx) + D(u^2 - q^2y^2 - z^2) \quad (64)$$

subtracting (63) from (64) we obtain

$$d(yx) - d(xy) = -\partial(xy) + \partial(yx) \quad (65)$$

and $d(yx) - d(xy) = \partial(yx) - \partial(xy) = \partial([y, x])$ as claimed. Thus completes the proof.

Now we will establish the relation between the q -quadratic Dirac operator introduced in definition 0.0.6 and extended derivation in the following proposition

PROPOSITION 0.0.7. *Let f be a Ψ - valued C^1 function, and the extended derivation mentioned in Proposition 0.0.4. The Ordinary*

derivative over f at points u_0, x_0, y_0 and z_0 are determined by the following expressions

$$\begin{aligned} \frac{df}{dx} &= e_x \left[\frac{f((x + q^2 e_0 u_0) u_0) - f(x u_0)}{q u_0} \right] + d(x u_0), \\ \frac{df}{dy} &= e_y \left[\frac{f((y + q^2 e_0 u_0) u_0) - f(y u_0)}{q u_0} \right] + d(y u_0), \\ \frac{df}{dy} &= e_z \left[\frac{f(y_0(z + q^2 e_y y_0)) - f(y_0 z)}{q y_0} \right] + d(y_0 z), \\ \frac{df}{du} &= e_0 \left[\frac{f((u - q^2 e_y y_0) y_0) - f(u y_0)}{q y_0} \right] + d(u y_0), \\ \frac{df}{dx} &= e_x \left[\frac{f((x + q^2 e_z z_0) z_0) - f(x z_0)}{q z_0} \right] + d(x z_0), \\ \frac{df}{dz} &= e_z \left[\frac{f(z + q^2 e_0 u_0) u_0 - f(z u_0)}{q u_0} \right] + d(z u_0), \\ \frac{df}{dy} &= e_y \left[\frac{f((y + q^2 e_x x_0) x_0) - f(y x_0)}{q x_0} \right] + d(y x_0), \\ \frac{df}{dz} &= e_z \left[\frac{f((z + q^2 e_x x_0) x_0) - f(z x_0)}{q x_0} \right] + d(z x_0). \end{aligned}$$

Taking into account the above, we can define the expressions for the extended derivations at points x_0, y_0, z_0 and u_0 in the following remark.

REMARK 4. *The following properties for the extended derivations holds:*

- (i) $d(yx_0) = -\partial(x_0y) + D(u^2 - z^2),$
- (ii) $d(xy_0) = -\partial(y_0x) + D(u^2 - z^2),$
- (iii) $d(y_0z) = -\partial(z_0y) + D(u^2 - x^2),$
- (iv) $d(z_0y) = -\partial(yz_0) + D(u^2 - x^2),$
- (v) $d(u_0x) = -\partial(xu_0) + D(-y^2 - z^2),$
- (vi) $d(x_0u) = -\partial(ux_0) + D(-y^2 - z^2),$

- (vii) $d(u_0y) = -\partial(yu_0) + D(-x^2 - z^2)$,
- (viii) $d(y_0u) = -\partial(uy_0) + D(-x^2 - z^2)$,
- (ix) $d(u_0z) = -\partial(zu_0) + D(-x^2 - y^2)$,
- (x) $d(z_0u) = -\partial(uz_0) + D(-x^2 - y^2)$.

PROOF 0.0.1. *The same reasoning applies to the case (i) using (63) at x_0 . Similary argument apply from the case (ii) to (x).* Finally, we introduce the notion of q - Quadratic Derivation Operator D_{q^2} in the following definition.

0.0.2 Quadratic Derivation Operator D_{q^2}

DEFINITION 0.0.7. *Let $f \in \mathbb{k}$ be a polynomial function in non-commuting quadratic variables u^2, x^2, y^2 and z^2 , and consider the q - quadratic difference operators defined in (44), (45), (46), and (47). We can say that q - quadratic derivation operators D_{q^2} are determinated by the following expressions:*

$$D_{q^2}^{u^2} f = \left[\frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} u^2} - \frac{d_{q^2}}{d_{q^2} x^2} - \frac{d_{q^2}}{d_{q^2} y^2} - \frac{d_{q^2}}{d_{q^2} z^2} \right] f, \quad (66)$$

$$D_{q^2}^{x^2} f = \left[\frac{d_{q^2}}{d_{q^2} u^2} - \frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} x^2} - \frac{d_{q^2}}{d_{q^2} y^2} - \frac{d_{q^2}}{d_{q^2} z^2} \right] f, \quad (67)$$

$$D_{q^2}^{y^2} f = \left[\frac{d_{q^2}}{d_{q^2} u^2} - \frac{d_{q^2}}{d_{q^2} x^2} - \frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} y^2} - \frac{d_{q^2}}{d_{q^2} z^2} \right] f, \quad (68)$$

$$D_{q^2}^{z^2} f = \left[\frac{d_{q^2}}{d_{q^2} u^2} - \frac{d_{q^2}}{d_{q^2} x^2} - \frac{d_{q^2}}{d_{q^2} y^2} - \frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} z^2} \right] f. \quad (69)$$

LEMMA 4. *The expressions (66), (67), (68) and (69) can be represented in matricial form*

$$\begin{bmatrix} D_{q^2}^{u^2} \\ D_{q^2}^{x^2} \\ D_{q^2}^{y^2} \\ D_{q^2}^{z^2} \end{bmatrix} f = \begin{bmatrix} \frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{d_{q^2}}{d_{q^2} x^2} & -\frac{d_{q^2}}{d_{q^2} y^2} & -\frac{d_{q^2}}{d_{q^2} z^2} \\ \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} x^2} & -\frac{d_{q^2}}{d_{q^2} y^2} & -\frac{d_{q^2}}{d_{q^2} z^2} \\ \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{d_{q^2}}{d_{q^2} x^2} & -\frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} y^2} & -\frac{d_{q^2}}{d_{q^2} z^2} \\ \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{d_{q^2}}{d_{q^2} x^2} & -\frac{d_{q^2}}{d_{q^2} y^2} & -\frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} z^2} \end{bmatrix} f, \quad (70)$$

and can be rewritten as

$$D_{q^2}^\mu f = \partial_{q^2}^{\alpha\beta,\mu} f, \quad \forall \mu = \{u^2, x^2, y^2, z^2\}, \tag{71}$$

$$\partial_{q^2}^{\alpha\beta,\mu} = \begin{bmatrix} \frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{d_{q^2}}{d_{q^2} x^2} & -\frac{d_{q^2}}{d_{q^2} y^2} & -\frac{d_{q^2}}{d_{q^2} z^2} \\ \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} x^2} & -\frac{d_{q^2}}{d_{q^2} y^2} & -\frac{d_{q^2}}{d_{q^2} z^2} \\ \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{d_{q^2}}{d_{q^2} x^2} & -\frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} y^2} & -\frac{d_{q^2}}{d_{q^2} z^2} \\ \frac{d_{q^2}}{d_{q^2} u^2} & -\frac{d_{q^2}}{d_{q^2} x^2} & -\frac{d_{q^2}}{d_{q^2} y^2} & -\frac{1}{q^2} \frac{d_{q^2}}{d_{q^2} z^2} \end{bmatrix}. \tag{72}$$

PROPOSITION 0.0.8. *Let $D_{q^2}^\mu$ be the q - quadratic diferential operator of $f \in \mathbb{k}$. In the following, we state some clear properties of the q - quadratic differential operator.*

i $D_{q^2}^{\mu,n} f = \Gamma(\frac{1}{q^m}, \frac{1}{q^n}) \left[\frac{d_{q^2} f}{d_{q^2} \mu} \right]^n, \quad \forall n, m \in \mathbb{N}.$

ii $[D_{q^2}^{\mu,n} f]^p = \left\{ \Gamma(\frac{1}{q^m}, \frac{1}{q^n}) \right\}^p \left[\frac{d_{q^2} f}{d_{q^2} \mu} \right]^{np}, \quad \forall n, m, p \in \mathbb{N}$

iii $D_{q^2}^{\mu,n} D_{q^2}^{\nu,n} = 0$ if only if $\mu \neq \nu$,

where $\Gamma(\frac{1}{q^m}, \frac{1}{q^n})$ are the coefficients that depends of $\frac{1}{q^m}, \frac{1}{q^n}$.

Proof Proposition 0.0.8

We first show that it suffices to consider the case $D_{q^2}^{\mu,2} = D_{q^2}^\mu(D_{q^2}^\mu f)$ obtaining the following matrix elements:

$$\partial_{q^2}^{11,\mu} f = \left[\frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} u^2} \right]^2 - \frac{d_{q^2} f}{d_{q^2} x^2} \frac{d_{q^2} f}{d_{q^2} u^2} - \frac{d_{q^2} f}{d_{q^2} y^2} \frac{d_{q^2} f}{d_{q^2} u^2} - \frac{d_{q^2} f}{d_{q^2} z^2} \frac{d_{q^2} f}{d_{q^2} u^2}, \tag{73}$$

$$\partial_{q^2}^{12,\mu} f = -\frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} u^2} \frac{d_{q^2} f}{d_{q^2} x^2} + \frac{1}{q^2} \left[\frac{d_{q^2} f}{d_{q^2} x^2} \right]^2 + \frac{d_{q^2} f}{d_{q^2} y^2} \frac{d_{q^2} f}{d_{q^2} x^2} + \frac{d_{q^2} f}{d_{q^2} z^2} \frac{d_{q^2} f}{d_{q^2} x^2}, \tag{74}$$

$$\partial_{q^2}^{13,\mu} f = -\frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} u^2} \frac{d_{q^2} f}{d_{q^2} y^2} + \frac{d_{q^2} f}{d_{q^2} x^2} \frac{d_{q^2} f}{d_{q^2} y^2} + \frac{1}{q^2} \left[\frac{d_{q^2} f}{d_{q^2} y^2} \right]^2 + \frac{d_{q^2} f}{d_{q^2} z^2} \frac{d_{q^2} f}{d_{q^2} y^2}, \tag{75}$$

$$\partial_q^{14,\mu} f = -\frac{1}{q^2} \frac{d_q^2 f}{d_q^2 u^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{d_q^2 f}{d_q^2 x^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{d_q^2 f}{d_q^2 y^2} \frac{d_q^2 f}{d_q^2 z^2} + \left[\frac{1}{q^2} \frac{d_q^2 f}{d_q^2 z^2} \right]^2, \quad (76)$$

$$\partial_q^{21,\mu} f = \frac{1}{q^2} \left[\frac{d_q^2 f}{d_q^2 u^2} \right]^2 - \frac{1}{q^2} \frac{d_q^2 f}{d_q^2 x^2} \frac{d_q^2 f}{d_q^2 u^2} - \frac{d_q^2 f}{d_q^2 y^2} \frac{d_q^2 f}{d_q^2 u^2} - \frac{d_q^2 f}{d_q^2 z^2} \frac{d_q^2 f}{d_q^2 u^2}, \quad (77)$$

$$\partial_q^{22,\mu} f = -\frac{d_q^2 f}{d_q^2 u^2} \frac{d_q^2 f}{d_q^2 x^2} + \left[\frac{1}{q^2} \frac{d_q^2 f}{d_q^2 x^2} \right]^2 + \frac{d_q^2 f}{d_q^2 y^2} \frac{d_q^2 f}{d_q^2 x^2} + \frac{d_q^2 f}{d_q^2 z^2} \frac{d_q^2 f}{d_q^2 x^2}, \quad (78)$$

$$\partial_q^{23,\mu} f = -\frac{d_q^2 f}{d_q^2 u^2} \frac{d_q^2 f}{d_q^2 y^2} + \frac{1}{q^2} \frac{d_q^2 f}{d_q^2 x^2} \frac{d_q^2 f}{d_q^2 y^2} + \frac{1}{q^2} \left[\frac{d_q^2 f}{d_q^2 y^2} \right]^2 + \frac{d_q^2 f}{d_q^2 z^2} \frac{d_q^2 f}{d_q^2 y^2}, \quad (79)$$

$$\partial_q^{24,\mu} f = -\frac{d_q^2 f}{d_q^2 u^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{1}{q^2} \frac{d_q^2 f}{d_q^2 x^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{d_q^2 f}{d_q^2 y^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{1}{q^2} \left[\frac{d_q^2 f}{d_q^2 z^2} \right]^2, \quad (80)$$

$$\partial_q^{31,\mu} f = \frac{1}{q^2} \left[\frac{d_q^2 f}{d_q^2 u^2} \right]^2 - \frac{d_q^2 f}{d_q^2 x^2} \frac{d_q^2 f}{d_q^2 u^2} - \frac{1}{q^2} \frac{d_q^2 f}{d_q^2 y^2} \frac{d_q^2 f}{d_q^2 u^2} - \frac{d_q^2 f}{d_q^2 z^2} \frac{d_q^2 f}{d_q^2 u^2}, \quad (81)$$

$$\partial_q^{32,\mu} f = -\frac{d_q^2 f}{d_q^2 u^2} \frac{d_q^2 f}{d_q^2 x^2} + \frac{1}{q^2} \left[\frac{d_q^2 f}{d_q^2 x^2} \right]^2 + \frac{1}{q^2} \frac{d_q^2 f}{d_q^2 y^2} \frac{d_q^2 f}{d_q^2 x^2} + \frac{d_q^2 f}{d_q^2 z^2} \frac{d_q^2 f}{d_q^2 x^2}, \quad (82)$$

$$\partial_q^{33,\mu} f = -\frac{d_q^2 f}{d_q^2 u^2} \frac{d_q^2 f}{d_q^2 y^2} + \frac{d_q^2 f}{d_q^2 x^2} \frac{d_q^2 f}{d_q^2 y^2} + \left[\frac{1}{q^2} \frac{d_q^2 f}{d_q^2 y^2} \right]^2 + \frac{d_q^2 f}{d_q^2 z^2} \frac{d_q^2 f}{d_q^2 y^2}, \quad (83)$$

$$\partial_q^{34,\mu} f = -\frac{d_q^2 f}{d_q^2 u^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{d_q^2 f}{d_q^2 x^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{1}{q^2} \frac{d_q^2 f}{d_q^2 y^2} \frac{d_q^2 f}{d_q^2 z^2} + \frac{1}{q^2} \left[\frac{d_q^2 f}{d_q^2 z^2} \right]^2, \quad (84)$$

$$\partial_{q^2}^{41,\mu} f = \frac{1}{q^2} \left[\frac{d_{q^2} f}{d_{q^2} u^2} \right]^2 - \frac{d_{q^2} f}{d_{q^2} x^2} \frac{d_{q^2} f}{d_{q^2} u^2} - \frac{d_{q^2} f}{d_{q^2} y^2} \frac{d_{q^2} f}{d_{q^2} u^2} - \frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} z^2} \frac{d_{q^2} f}{d_{q^2} u^2}, \tag{85}$$

$$\partial_{q^2}^{42,\mu} f = -\frac{d_{q^2} f}{d_{q^2} u^2} \frac{d_{q^2} f}{d_{q^2} x^2} + \frac{1}{q^2} \left[\frac{d_{q^2} f}{d_{q^2} x^2} \right]^2 + \frac{d_{q^2} f}{d_{q^2} y^2} \frac{d_{q^2} f}{d_{q^2} x^2} + \frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} z^2} \frac{d_{q^2} f}{d_{q^2} x^2}, \tag{86}$$

$$\partial_{q^2}^{43,\mu} f = -\frac{d_{q^2} f}{d_{q^2} u^2} \frac{d_{q^2} f}{d_{q^2} y^2} + \frac{d_{q^2} f}{d_{q^2} x^2} \frac{d_{q^2} f}{d_{q^2} y^2} + \frac{1}{q^2} \left[\frac{d_{q^2} f}{d_{q^2} y^2} \right]^2 + \frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} z^2} \frac{d_{q^2} f}{d_{q^2} y^2}, \tag{87}$$

$$\partial_{q^2}^{44,\mu} f = -\frac{d_{q^2} f}{d_{q^2} u^2} \frac{d_{q^2} f}{d_{q^2} z^2} + \frac{d_{q^2} f}{d_{q^2} x^2} \frac{d_{q^2} f}{d_{q^2} z^2} + \frac{d_{q^2} f}{d_{q^2} y^2} \frac{d_{q^2} f}{d_{q^2} z^2} + \left[\frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} z^2} \right]^2. \tag{88}$$

If $\frac{d_{q^2} f}{d_{q^2} \mu} \frac{d_{q^2} f}{d_{q^2} \nu} = 0$ for $\mu \neq \nu$, then $\left[\frac{1}{q^2} \frac{d_{q^2} f}{d_{q^2} \mu} \right]^2$ and $\frac{1}{q^2} \left[\frac{d_{q^2} f}{d_{q^2} \mu^2} \right]^2 \neq 0$. Thus we may assume that $D_{q^2}^{\mu,2} f = \Gamma(\frac{1}{q^2}, \frac{1}{q}) \left[\frac{d_{q^2} f}{d_{q^2} \mu^2} \right]^2$. For the general case, we conclude that statement (i) holds. For (ii), the base of induction $p = 1$ is evident, which implies (i). Now, we prove that it is true for $p = l + 1$. We have

$$\begin{aligned} [D_{q^2}^{\mu,n} f]^{l+1} &= \left\{ \Gamma\left(\frac{1}{q^m}, \frac{1}{q^n}\right) \right\}^{l+1} \left[\frac{d_{q^2} f}{d_{q^2} \mu} \right]^{n(l+1)}, \\ &= \left\{ \Gamma\left(\frac{1}{q^m}, \frac{1}{q^n}\right) \right\}^l \left\{ \Gamma\left(\frac{1}{q^m}, \frac{1}{q^n}\right) \right\} \left[\frac{d_{q^2} f}{d_{q^2} \mu} \right]^{nl} \left[\frac{d_{q^2} f}{d_{q^2} \mu} \right]^n, \\ &= \left\{ \Gamma\left(\frac{1}{q^m}, \frac{1}{q^n}\right) \right\} \left[\frac{d_{q^2} f}{d_{q^2} \mu} \right]^n \left\{ \Gamma\left(\frac{1}{q^m}, \frac{1}{q^n}\right) \right\}^l \left[\frac{d_{q^2} f}{d_{q^2} \mu} \right]^{nl}, \\ &= [D_{q^2}^{\mu,n} f][D_{q^2}^{\mu,n} f]^l. \end{aligned}$$

Hence (ii) holds for all $l \in \mathbb{N}$. For the statement (iii) we note that if $\mu = \nu$, then we have $[D_{q^2}^{\mu,n} f]^2 = \Gamma(\frac{1}{q^2}, \frac{1}{q}) \left[\frac{d_{q^2} f}{d_{q^2} z^2} \right]^2$ by (i).

4. Some Comments and Suggestions for Further Work

The main point of this paper has been to show the definition of the q -Differential and derivations for the quadratic relativistic algebras. There are five further topics arising from this paper which are worth investigating. First, one might consider the relativistic dynamics for subatomic particles with variable transfer momenta. Secondly there is the problem of describing the q -relativistic wave equation for q -bosons or fermions with mass m

$$i\partial_k^q f^k + mf = Ef_0, \quad k = 1, 2, 3. \quad (89)$$

for some Clifford-algebra-valued spinors $f_k = \begin{bmatrix} \psi^\alpha \\ \varphi_{\dot{\beta}} \end{bmatrix}_k e_k$, $f_0 = \begin{bmatrix} \psi^\alpha \\ \varphi_{\dot{\beta}} \end{bmatrix}_0 e_0$ and $f = \begin{bmatrix} \psi^\alpha \\ \varphi_{\dot{\beta}} \end{bmatrix} |e_0|^2$ which is monogenic with respect to the usual Dirac operator D subject to Dirichlet condition proposed in [15] and the relations of the q -Heisenberg algebra for the q -Dirac operators. Lastly, based in [16], one might consider the q -spinor differential and integral calculus for the q -Dirac operators (53), (54), (55), (56), (57), (58), (59) and (60) considering the fermionic case $q = -1$ and bosonic case $q = +1$. An other suggestion is the problem of describing the *Relativistic Maxwell Electrodynamical Algebra*, which is defined by the following commutation relations

$$e_i e^i = |e_0|^2 - e_X^2, \quad X = 1, 2, 3, \quad (90)$$

$$f_{ij} = \partial_i e_j - \partial_j e_i, \quad (91)$$

$$\partial_i e^i = \partial_j e^j = 0, \quad (92)$$

$$D_i = \partial_i - b e_i, \quad b \in \mathbb{R} \quad (93)$$

$$\partial^i f_{ij} = \Gamma_j, \quad (94)$$

$$\partial_j f_0^{ij} = 0, \quad (95)$$

where $f_0^{ij} = \varepsilon^{ij0} f_{ij}$, $f^{ij} = 0$ if $i = j$ and $f^{ij} \neq 0$ otherwise. And finally, from (91) and taking into account the above, one

can propose the *q*- Relativistic Dirac - Maxwell algebra, which is subject to relations

$$f_{ij}^q = D_i^q e_j - D_j^q e_i, \tag{96}$$

$$\mathbf{D}^q = \boldsymbol{\partial}^q - b\mathbf{e} \quad b \in \mathbb{R}, \tag{97}$$

being $\mathbf{D}^q = \mathbf{e}^i D_i^q$, $\boldsymbol{\partial}^q = \mathbf{e}^i \partial_i^q$ and $\mathbf{e} = \mathbf{e}^i e_i$, where $\mathbf{e}^i, i = 1 \cdots n$ are the generators for the Clifford algebras, and (97) is called the *Covariant Derivative*.

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