QUANTUM MECHANICS AND TOPOLOGY

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1. Introduction

Physics defined on multiply connected manifolds is an old topic in theoretical physics. In the context of the path integral formalism it was studied by the first time by Schulman [1] in 1968 and rigorously formulated by Laidlaw and de Witt [2] and Dowker [3] in 1971.

The central point is that in a multiply connected manifold the paths have different weights in the sum over histories and the problem –which does not exist in the "standard" quantum mechanics– is how to define a quantum theory taking this fact into account.

The problem of to how define a quantum theory on a topologically non–trivial manifold is not only an academic problem because it finds experimentally realizable systems such as the Aharonov–Bohm effect [4] and the anyons that could be an explanation to the quantum Hall effect and, maybe, to high temperature superconductivity.

The purpose of this lecture is to explain some aspects of the quantum theory defined on multiply connected manifolds in the context of the path integral formulation and the applications that these ideas find in anyon physics in one and two dimensions.

In section 2 we start by explaining some examples that involve non–trivial topological aspects; this section does not involve calculations and its unique purpose is to introduce several useful concepts. In section 3 we introduce the formal definition of a multiply connected manifold. In section 4 path integrals on arbitrary manifolds. In section 5 several applications are studied. In section 6 anyons in two dimensions and in section 7 anyons in one dimension. Section 8 is dedicated to the conclusions.

2. Systems Defined on Non–Trivial Topological Manifolds

Let us start by discussing the most popular example of a quantum theory defined on a multiply connected manifold, namely the Aharonov–Bohm effect.

The Aharonov–Bohm effect consists in the experimental arrangement shown in figure 1.

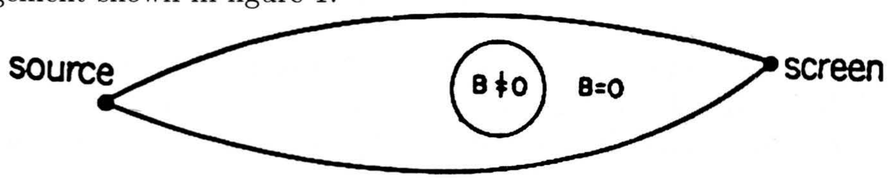

Figure 1. The solenoid has infinite lenght with an inner constant magnetic field; we assume also that the solenoid is impenetrable.

The electrons can follow infinite paths, as shown in figure 2.

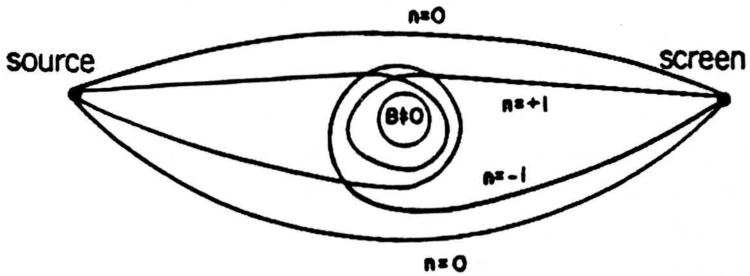

Figure 2. Some of the the infinites path of the electron.

The important question is that theoretically we expect that interference lines can be observed on the screen, such as in a diffraction experiment, and that the lines be dependent only on the magnetic field inside of the solenoid.

This example tell us that the electromagnetic potentials -that classically are unobservable- are quantum mechanically responsible for the observability of the interference pattern. The experimental question relative to the Aharonov–Bohm effect was only solved with a serie of experiments performed by Tonomura and collaborators in the beginning of the eighties [5], $\cdots$ twentyfive years after the Aharonov and Bohm prediction.

From a theoretical point of view, we can see the Aharonov–Bohm effect as a phenomenon that occurrs because the $\Re^2$ manifold (that is the plane where the paths live) has a point removed (the point where the solenoid is) and, as a consequence, the configuration space of this system is $\Re^2 - \{0\}$.

The Aharonov–Bohm effect is an example of a mechanism that appears in many examples of recent physics, one of them is the problem of two–anyons.

In order to explain this problem, let us consider the motion of two non–relativistic particles moving on a plane. The motion is regular everywhere except in the point where the particles collide.

The colision condition in the point $x_1 = x_2$ is equivalent to the replacement

$$\Re^2 \to \Re^2 - \{0\}\,, \tag{1}$$

and, in consequence, the manifold (configuration space) has also a point removed.

We can see formally this example as a similar phenomenon to the Aharonov–Bohm effect, each particle has a flux–tube attached to it and in the case of the two particles, we can exactly map it into the Aharonov–Bohm effect.

Of course, there are questions to which we should give an answer: What is the analogue of the magnetic field for the case of two particles?; How to implement technically this fact?; etc.

There is also another problem closely related with the previous ones, namely cosmic strings. The cosmic strings are solutions of the Einstein field equations when point–like matter is present. The solution is

$$ds^2 = dt^2 - r^2\, d\phi'^2 - dr^2 - dz^2\,, \tag{2}$$

where ϕ' is the defect angle.

The cosmic strings are generally assumed to be singularities that remained after the formation of our universe and could be experimentally detectable. The manifold when we project to a plane is again $\Re^2 - \{0\}$.

In the next sections we will try to formalize these examples by developing appropriate computational techniques.

3. Rudiments of Homotopy Theory

The configuration space where we compute the propagator of a free particle is an example of a simply connected manifold. When we give two points of the manifold we can draw infinite, topologically equivalent, paths between these two points.

The word 'deformable' has a technical connotation for a mathematical operation called homotopy transformation that we will define below [6].

The idea of a homotopy is the following; we will say that two curves are homotopically equivalent if it is possible to deform continuosly one into the other, in other words, two continuous applications f and g of the space X to the space Y, $f, g : X \to Y$ are homotopic (simbolically $f \sim g$) if there exists a continuous function $F : X \times I \to X$, where I is the closed interval $[0, 1]$, such that

$$F(x,t)|_{t=0} = f(x), \tag{3}$$

and

$$F(x,t)|_{t=1} = g(x), \tag{4}$$

with $(x, t) \in X$.

It is clear that the idea of homotopy defines a class of equivalence between applications, *i.e.*,

1. $f \sim f$,

2. $f \sim g \Rightarrow g \sim f$,

3. $f \sim g$ and $g \sim h \Rightarrow f \sim h$,

for all continuous functions f, g and h.

If G is the space of all continuous applications between X and Y, then a relation of equivalence has the property of decomposing the space G in classes of equivalence or disjoint sets of functions which are homotopically equivalent.

If the functions g and f are homotopic, then they belong to the same class of homotopy, otherwise they are non–homotopically equivalent. We will denote the homotopy class by $[\alpha]$ where the set $[\alpha]$ is the set of all paths that are homotopically equivalent. In the case of the Aharonov–Bohm effect in figure 2 the paths 1 and 2 belong to the same class of homotopy.

Now, we will restrict our considerations only to the applications that are closed curves or loops; we will say that the loops α and β with basis in x_0 (*i.e* the point where the extremes coincide) are equivalent if there exists a function $H : I \times I \to X$ such that $H(t,0) = \alpha$, $H(t',0) = \beta$ and $H(0,s) = H(1,s) = x_0\ \forall s \in I$.

The function $H(s,t)$ is a homotopy. Therefore, if α, β and γ, are loops with basis in $x_0 \in X$, then

1. $\alpha \sim \alpha$, *i.e.* any loop is equivalent itself.

2. If $\alpha \sim \beta$, then there exists a homotopy $H : I \times I \to X$ with $H(t,0) = \alpha$, $H(t,1) = \beta$ and $H(0,s) = H(1,s) = x_0$.

3. $\alpha \sim \gamma$ if $\alpha \sim \beta$ and $\beta \sim \gamma$.

It is possible to define a homotopy $L(t,s)$ between α and γ as follows

$$L(t,\, s) = \begin{cases} H(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ H(s, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{5}$$

and consequently, $\alpha \sim \gamma$.

We can think of a more tangible example by considering the Aharonov–Bohm effect "joining" the two extremes in figure 2. Observing the figure 2, we see that there are paths that can be classified by a "topological invariant": the winding number n.

In general, the loops can be summed up and the resulting sum is another loop that links $\sum n$ times the hole. The set of all loops is a group that is isomorphic to the integer group. However, in order to implement this fact it is necessary to define the product of loops. The definition is the following; let α and β be curves in X with $\alpha(1) = \beta(0)$, then the product $*$ of curves is

$$(\alpha * \beta)(t) = \begin{cases} H(2\,s,\,t)\,, & 0 \le s \le \frac{1}{2} \\ H(2\,s-1,\,t)\,, & \frac{1}{2} \le s \le 1 \end{cases}. \tag{6}$$

Once these definitions are given, we can show that the set of all the homotopy classes of loops $\{[\alpha]\}$ with basis in x_0 of X is a group and it is denoted by $\pi_1(X, x_0)$ and is formally equivalent to

$$\pi_1(X, x_0) = \{[\alpha]\}\,. \tag{7}$$

The set π_1 endowed of the operation $*$ defines the first homotopy group or fundamental group. The group π_1 is the first of an infinite set of $(n > 0)$ higher–order homotopy groups. π_1 eventually might be non–Abelian while the higher homotopy groups are all Abelian.

4. Path Integrals on Multiply Connected Manifolds

In this section we will introduce the concept of path integrals on multiply connected manifolds. Let us start by considering path integrals on simply connected manifolds and the most simple application, namely, the free non–relativistic particle; this example is a warm–up exercise and will be useful when we consider the path integral over a multiply connected manifold at the end of this section.

The idea of the path integral consists into summing over all the paths between the initial and final points A and B. The propagation amplitude between these two points is equivalent to the computation of the formal sum

$$G[B,\,A] \sim \sum_{paths} \text{something}\,. \tag{8}$$

The previous expression has two difficulties: firstly, it has the technical problem of how to define the sum between paths, and;

second, it has the physical problem of how to define *something*. This second problem is equivalent to postulate the Schrödinger equation in the conventional quantum mechanics and it is equivalent to make the replacement

$$\text{something} \to e^{\frac{i}{\hbar}S}, \tag{9}$$

where S is the action.

The first problem is technically more difficult and, in essence, its solution consists into the replacement (for details see *e.g.* [7])

$$\sum_{paths} e^{\frac{i}{\hbar}S} \to \int \prod_t dx(t) e^{\frac{i}{\hbar}\int dt L(x,\dot{x})}, \tag{10}$$

where $L(x, \dot{x})$ is the Lagrangian of the system.

In general although we can give a discretization prescription, the physical quantities are well defined only by giving correctly the boundary conditions. Thus, if we are interested in computing the propagator of a particle, we must give the boundary conditions

$$\begin{aligned} x(t_1) &= x_1\,, \\ x(t_2) &= x_2\,, \end{aligned} \tag{11}$$

and then the expression

$$G[x_2, x_1] = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}S}\,, \tag{12}$$

together with (11) defines the propagation amplitude or propagator of the system. Here $\mathcal{D}x(t) = \prod_t dx(t)$.

We can verify explicitly how to work out these ideas by considering explicitly the most simple example, namely the motion of a free non–relativistic particle in one dimension described by the Lagrangian

$$L = \frac{1}{2}\,\dot{x}^2\,. \tag{13}$$

We are interested in computing the propagation amplitude $G[x_2, x_1]$ with the boundary conditions (11). In order to compute

$$G[x_2,\, x_1] = \int \mathcal{D}x(t)\, e^{\frac{i}{\hbar}\int_1^2 dt \frac{1}{2}\dot{x}^2}\,, \tag{14}$$

we start by making the following change of variables

$$x(t) = x_1 + \frac{\Delta x}{\Delta t}(t - t_1) + y(t) = x_{cl} + y(t)\,, \tag{15}$$

where x_{cl} is the solution of the classical equation of motion $\ddot{x} = 0$ and $y(t)$ is a quantum fluctuation that, by consistency, satisfies the boundary condition

$$\begin{aligned} y(t_1) &= 0\,, \\ y(t_2) &= 0\,. \end{aligned} \tag{16}$$

When (15) is replaced in (14) we find that

$$G[x_2,\, x_1] = e^{i\frac{(\Delta x)^2}{2\Delta t}} \int \mathcal{D}y(t)\, e^{\frac{i}{2}\int_1^2 dt y(-\partial_t^2)y}\,. \tag{17}$$

The integral in y is Gaussian and the result of the integration is

$$\det(-\partial_t^2)^{-\frac{1}{2}}\,, \tag{18}$$

Now, we should compute the determinant, the procedure is the following: we start by solving the eingenvalue equation

$$\partial_t^2 \psi_n = \lambda_n \psi_n\,, \tag{19}$$

with Dirichlet boundary conditions, and afterwards we use the formula

$$\det(-\partial_t^2) = \prod_n \lambda_n\,. \tag{20}$$

Using $\psi_n(t_1) = 0$, $\psi_n(t_2) = 0$, we find that $\lambda_n = (n\pi/\Delta t)^2$ and (20) becomes

$$\det(-\partial_t^2) = \prod_{-\infty}^{+\infty} \left(\frac{n\pi}{\Delta t}\right)^2\,. \tag{21}$$

However, (21) is a divergent quantity.

In order to make sense of the divergent expression (21), we regularize appropiately this product; firstly we observe that (21) has the general form $\prod an^b$. Then we can write

$$\prod a\, n^b = e^{\sum_n \ln(an^b)} = e^{\sum_n \ln a + b \sum_n \ln n}\,, \tag{22}$$

and using the Riemann ζ–function

$$\zeta(s) = \sum_n \frac{1}{n^s}\,, \tag{23}$$

we see that

$$\begin{aligned} \prod a\, n^b &= e^{\ln a \lim_{s\to 0} \sum_{n=1}^{n=\infty} n^{-s} + b \lim_{s\to 0} \frac{d}{ds} \sum n^{-s}} \\ &= e^{\log a \zeta(0) + b\zeta'(0)}\,. \end{aligned} \tag{24}$$

By analytic continuation we see that $\zeta(0) = -1/2$ and $\zeta'(0) = -(1/2)\ln 2\pi$, and then

$$\prod a\, n^b = a^{-1/2}\,(2\,\pi)^{b/2}\,, \tag{25}$$

so that (22) is simply $(1/\Delta t)$ and the propagator becomes

$$G[x_2,\, x_1] = \frac{1}{\sqrt{\Delta t}}\, e^{i(\Delta x)^2/\Delta t}\,, \tag{26}$$

which is the standard result.

In the previous problem we have assumed that the manifold is defined on

$$-\infty < x < \infty\,. \tag{27}$$

The next question is: What happens if the manifold has another topological structure such as a circle or a torus, etc.?

If the manifold has the topology of a circle, the boundary conditions (11) do not define completely the problem and we must modify (11) in the following way

$$\begin{aligned} x(t_1) &= x_1 \,, \\ x(t_2) &= x_2 + 2\,n\,\pi \,, \end{aligned} \tag{28}$$

where n is an integer number (winding number).

A physical system described by the Lagrangian

$$L = \frac{1}{2}\,\dot{x}^2 \,, \tag{29}$$

with the boundary conditions (28), is called quantum rotator and it is the most simple example defined on a multiply connected manifold.

The strategy that we will follow below (which in essence is due to Schulman) is to solve this example in detail and afterwards to derive a general formula.

Let us start by making in (29) the identification $x \to \phi$. Then, (28) becomes

$$\begin{aligned} \phi(t_1) &= \phi_1 \,, \\ \phi(t_2) &= \phi_2 + 2\,n\,\pi \,. \end{aligned} \tag{30}$$

The propagation amplitude for this case becomes

$$G_n[\phi_2,\, \phi_1] = \int \mathcal{D}\phi(t)\, e^{\frac{i}{\hbar}\int_1^2 dt \frac{1}{2}\dot{\phi}^2} \,, \tag{31}$$

provided that the boundary condition (30) are assumed. When (31) is computed using (30), the propagation amplitude will depend on n, for this reason we have written $G_n[\phi_2, \phi_1]$.

We solve this problem in complete analogy with the free non–relativistic particle. In fact, by making the change of variables

$$\phi(t) = \phi_1 + \frac{\Delta\phi + 2n\pi}{\Delta t}\,(t - t_1) + \psi(t) = \phi_{cl} + \psi(t) \,, \tag{32}$$

with ϕ_{cl} the classical solution of the equation of motion and, by consistency, the quantum fluctuations satisfy $\psi(t_1) = 0$, $\psi(t_2) = 0$.

Replacing (32) in (31)

$$
\begin{aligned}
G_n[\phi_2, \phi_1] &= e^{\frac{i(\Delta\phi+2n\pi)^2}{\Delta t}} \int \mathcal{D}\psi \, e^{i\int_{t_1}^{t_2} dt \frac{1}{2}\psi\partial_t{}^2\psi} \\
&= e^{\frac{i(\Delta\phi+2n\pi)^2}{\Delta t}} \det\left(\partial_t^2\right)^{-\frac{1}{2}} . \qquad (33)
\end{aligned}
$$

The determinant is computed as in the free non–relativistic particle case and the result is Δt. Thus, (33) is

$$
G_n[\phi_2, \phi_1] = \frac{1}{\sqrt{\Delta t}} e^{\frac{(\Delta\phi+2n\pi)^2}{\Delta t}} \qquad (34)
$$

Expression (34) is the propagation amplitude for a fixed homotopy class and, in consequence, the total propagation amplitude is

$$
G[\phi_2, \phi_1] = \sum_{n=-\infty}^{n=+\infty} \Xi_n \, G_n[\phi_2, \phi_1] \, , \qquad (35)
$$

where Ξ_n is a factor which has to be determined. By invoking completness and unitarity of the Green function *i.e.*

$$
\begin{aligned}
\Xi_n^* \, \Xi_m &= \Xi_{n+m} \, , \\
\Xi_n^* \, \Xi_n &= 1 \, ,
\end{aligned}
$$

we find that Ξ_n must be $e^{n\delta}$ where δ is a phase.

Using the identity

$$
\vartheta_3(\tau, z) = \sum_{n=-\infty}^{n=+\infty} e^{2inz+i\pi n^2\tau} = (-\tau)^{-1/2} \, e^{\frac{z^2}{i\pi\tau}} \, \vartheta_3\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) , \qquad (36)
$$

then, we find the final expression

$$
G[\phi_2, \phi_1] = \frac{1}{2\pi} \exp\left[i\frac{\delta\Delta\phi}{2\pi} - i\frac{\delta^2\Delta}{8\pi^2}\right] \vartheta_3\left(\frac{\Delta\phi}{2} - \frac{\Delta t\delta}{4\pi}, -\frac{\Delta t}{2\pi}\right) , \qquad (37)
$$

which is the result found by Schulman in 1968, althought the derivation given by him is slightly different.

Finally we will discuss briefly the formal derivation of the heuristic formula (16). Firstly, we should note the existence of two equivalent manifolds, $\mathcal{M}$ and their universal covering $\tilde{\mathcal{M}}$. $\mathcal{M}$ is a multiply connected manifold while $\tilde{\mathcal{M}}$ is simply connected. Both manifolds are related by

$$\mathcal{M} = \frac{\tilde{\mathcal{M}}}{G[x_2,\, x_1]}\,, \tag{38}$$

where $G[x_2,\, x_1]$ is a discrete group. By definition the quotient $\tilde{\mathcal{M}}/G[x_2,\, x_1]$ is the set of all homotopy classes, *i.e.*

$$\tilde{\mathcal{M}}/G[x_2,\, x_1] = \{[x],\, x \in \tilde{\mathcal{M}}\}\,. \tag{39}$$

Then, two points x and $\tilde{x}$ are equivalent under $G[x_2,\, x_1]$ if there exists an element $g \in G[x_2,\, x_1]$ such that

$$\tilde{x} = x \cdot g\,. \tag{40}$$

In the case considered above, $\mathcal{M} = S^1$ and $\tilde{M} = \Re$ and the relation (40) is

$$\tilde{x} = x + 2\,\pi\, n\,, \tag{41}$$

while $G[x_2,\, x_1] = \mathcal{Z}$, where $\mathcal{Z}$ is the integer group. Once this nomenclature is introduced we can define the path integral.

Let $\tilde{\psi}(\tilde{x})$ be the wave function on $\tilde{\mathcal{M}} = \Re$, this wave function is continuous and onevalued. Then (40) becomes

$$\tilde{\psi}(\tilde{x} \cdot g) = a(g) \cdot \tilde{\psi}(\tilde{x})\,, \tag{42}$$

$\forall g \in \mathcal{Z}$. If we impose the normalization of the wave function

$$|a(g)| = 1\,, \tag{43}$$

then $a(g)$ is a phase that satisfies the following property; let us consider the wave function with a well defined value. Then there exist the pre–image $\tilde{x} = p^{-1}(x)$ of x with $\psi(x) = \tilde{\psi}(\tilde{x}_0)$; after a complete turn around in S^1, $\psi(x)$ takes another value and the pre–image will be different, say $\tilde{x}.g_1$ then

$$\tilde{\psi}(\tilde{x}_0) \to \tilde{\psi}(\tilde{x}_0 \cdots g_1) = a(g_1) \cdot \tilde{\psi}(\tilde{x}_0) \,. \tag{44}$$

Giving another turn around we find

$$\tilde{\psi}(\tilde{x}_0 \cdot g_1) \to \tilde{\psi}(\tilde{x}_0 \cdot g_1\, g_2) = a(g_2)\, \tilde{\psi}(\tilde{x}_0 \cdot g_1) = a(g_1) \cdot a(g_2) \tilde{\psi}(\tilde{x}_0) \,, \tag{45}$$

and, as a consequence

$$\tilde{\psi}(\tilde{x}_0 \cdot g_1\, g_2) = a(g_1) \cdot a(g_2) \tilde{\psi}(\tilde{x}_0) \,. \tag{46}$$

Thus

$$a(g_1 \cdot g_2) = a(g_1) \cdot a(g_2) \,. \tag{47}$$

This last equation tells us, again, that $a(g)$ is a phase but also that there is a close relation between phase factors and the group. In the case at hand, the phase factor is an unitary irreducible representation of $\mathcal{Z}$.

The propagator in $\tilde{\mathcal{M}}$ is defined as usual,

$$\tilde{\psi}(\tilde{x}_1,\, \tilde{t}_1) = \int_{\tilde{\mathcal{M}}} d\tilde{x}\, \tilde{G}[\tilde{x}_2,\, \tilde{t}_2;\, \tilde{x}_1,\, \tilde{t}_1]\, \tilde{\psi}(\tilde{x}_2,\, \tilde{t}_2) \,, \tag{48}$$

where $\tilde{G}$ is the propagator for one–valued functions on $\mathcal{M}$ with a space made up of an infinite number of copies of $\mathcal{M}$. Assuming continuity on the one–valued functions, we can write (48) as

$$\tilde{\psi}(\tilde{x}_1,\, \tilde{t}_1) = \sum_{\Re} \int_{\Re} d\tilde{x}_1\, \tilde{G}[\tilde{x}_2,\, \tilde{t}_2;\, \tilde{x}_1,\, \tilde{t}_1]\, \psi(\tilde{x}_1 \cdot g),\, t') \,, \tag{49}$$

denoting the arbitrary point $\tilde{x}' \in \tilde{\mathcal{M}}$ by $\tilde{x}'_0 = \{\tilde{x}' g\}$ where x'_0 belongs to a copy on $\mathcal{M}$ for some fundamental domain $\mathcal{M}_0$. Thus

$$\tilde{\psi}(\tilde{x},\, \tilde{t}) = \sum_{g \in \mathcal{Z}} \int_{\tilde{\mathcal{M}}_0 \cdot g} d(\tilde{x}_0 \cdot g)\, \tilde{G}[\tilde{x},\, \tilde{t};\, \tilde{x}_0 \cdot g,\, \tilde{t}_1]\, \tilde{\psi}\, (\tilde{x}_0 \cdot g,\, t') \,, \tag{50}$$

but as the copies are identical, the integration on any copy is the same on the fundamental domain, then we write

$$\tilde{\psi}(\tilde{x},\, \tilde{t}) = \sum_{g\in\mathcal{Z}} \left[\int_{\tilde{\mathcal{M}}_0} d(\tilde{x})\, \tilde{G}[\tilde{x},\, \tilde{t};\, \tilde{x}_0\cdot g,\, \tilde{t}_1]\right] \tilde{\psi}(\tilde{x}_0\cdot g,\, t')\,, \qquad (51)$$

or

$$\tilde{\psi}(\tilde{x},\, \tilde{t}) = \int_{\tilde{\mathcal{M}}_0} d(\tilde{x}) \sum_{g\in\mathcal{Z}} \tilde{G}[\tilde{x}_2,\, \tilde{t}_2;\, \tilde{x}_0\cdot g,\, \tilde{t}_1]\, \tilde{\psi}(\tilde{x}_0\cdot g,\, t')\,, \qquad (52)$$

Following these arguments, if $\psi(x,t)$ is the wave function in the point x, then there exists a pre–image $\tilde{x}$ of x where $\tilde{\psi}(\tilde{x},t) = \psi(x,t)$. Furthermore, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are locally homeomorphic and, in consequence, $d\tilde{x} = dx$. Thus,

$$\psi(\tilde{x},\, \tilde{t}) = \int_{\mathcal{M}} dx\, G[\tilde{x},\, \tilde{t};\, x,\, t]\, \psi(x,\, t)\,, \qquad (53)$$

where

$$G[\tilde{x},\, \tilde{t};\, x,\, t] = \sum_{g\in\mathcal{Z}} \tilde{G}[\tilde{x}_0,\, \tilde{t}_0;\, \tilde{x},\, t']\, a(g)\,, \qquad (54)$$

with $x = p(\tilde{x}_0)$ and $x = p(x_0)$. Making $x_0 \to x_0 g^{-1}$ and $\tilde{x}_0 \to \tilde{x}_0 g^{-1}$ and afterwards $g \to g^{-1}$, we arrive finally to

$$G[\tilde{x},\, \tilde{t};\, x,\, t] = \sum_{g\in\mathcal{Z}} a(g^{-1})\, \tilde{G}[\tilde{x}_0,\, \tilde{t}_0;\, \tilde{x},\, t']\,, \qquad (55)$$

which is the standard formula for the propagator in a multiply connected manifold [2, 3]. Althought (55) was derived for a particular topology, it is a formula which is valid for general cases.

5. Applications

In this section we will apply the formulas derived in the above section to several problems such the Aharonov–Bohm effect including spin (and their relativistic extensions) and anyons.

The Aharonov–Bohm Effect

In the Aharonov–Bohm effect the propagation amplitude is

$$G[x_2,\, x_1] = \sum_n \Xi_n \int_{(n)} \mathcal{D}x e^{iS_{free}}. \tag{56}$$

In order to compute (56) it is convenient to discretize as follow

$$\begin{aligned} G[x_2,\, x_1] &= \lim_{n\to\infty} \left(\frac{\rho}{i\pi\Delta t}\right)^n \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k=1}^{n-1} dx_k\, dy_k \\ &\quad \times \exp\left[i\,\rho \sum_{j=1}^{n} \left(\frac{(x_j - x_{j-1})^2}{\Delta t} + \frac{(y_j - y_{j-1})^2}{\Delta t}\right)\right], \end{aligned} \tag{57}$$

where $\rho = m/2$, $\Delta t = (t_2 - t_1)/m$.

Using polar coordinates,

$$\begin{aligned} x_j &= r_j \cos\theta_j\,, \\ y_j &= r_j \sin\theta_j\,, \end{aligned} \tag{58}$$

and writing $dx_k dy_k = r_k dr_k d\theta_k$, and

$$(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2 = r_j^2 + r_{j-1}^2 - r_j\, r_{j-1}, \cos(\theta_j - \theta_{j-1})\,, \tag{59}$$

the equation (57) becomes

$$\begin{aligned} &G[x_2,\, x_1] \\ &= \lim_{n\to\infty} \left(\frac{\rho}{i\pi\Delta t}\right)^n \int_0^{\infty} \cdots \int_{-\pi}^{+\pi} \prod_{k=1}^{n-1} r_k\, dr_k\, d\theta_k \\ &\quad \times \exp\left[\frac{i\rho}{\Delta t} \sum_{j=1}^{n} \left(r_j^2 + r_{j-1}^2 - r_j\, r_{j-1}\, \cos(\theta_j - \theta_{j-1})\right)\right]. \end{aligned} \tag{60}$$

Now we impose that the electron can go around the solenoid. Technically this is equivalent to impose the constraint

$$\phi + 2\,\pi\, m - \sum_{j=1}^{n} (\theta_j - \theta_{j-1}) = 0\,. \tag{61}$$

Via a delta function, *i.e*

$$\begin{aligned}
&G[x_2,\, x_1] \\
&= \lim_{n\to\infty} \left(\frac{\rho}{i\pi\Delta t}\right)^n \int_0^\infty \cdots \int_{-\pi}^{+\pi} \prod_{k=1}^{n-1} r_k\, dr_k\, d\theta_k \\
&\quad \times \delta \left[\phi + 2\,\pi\, m - \sum_{j=1}^{n} (\theta_j - \theta_{j-1})\right] \\
&\quad \times \exp \left[\frac{i\rho}{\Delta t} \sum_{j=1}^{n} \left(r_j^2 + r_{j-1}^2 - r_j\, r_{j-1}\, \cos(\theta_j - \theta_{j-1})\right)\right] \,,
\end{aligned} \tag{62}$$

where ϕ is the angle between the source of electrons, the center of the solenoid and the screen (see fig. 3).

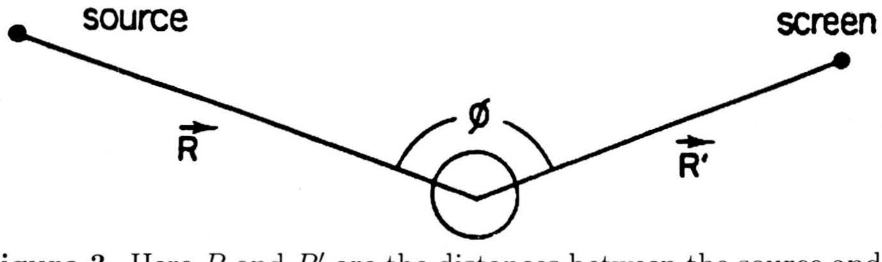

Figure 3. Here R and R' are the distances between the source and the screen to the centre of the solenoid.

Now, we can exponentiate the δ function and after a tedious calculation we find

$$\begin{aligned}
&G[x_2, x_1]_m \\
&= \lim_{n\to\infty} \left(\frac{\rho}{i\pi\Delta t}\right)^n \int_0^\infty \cdots \int_{-\pi}^{+\pi} \prod_{k=1}^{n-1} r_k\, dr_k\, \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda(\phi+2\pi m)}
\end{aligned}$$

$$\times \prod_{j=1}^{n} \exp\left[\frac{i\rho}{\Delta t}\,(r_j^2 + r_{j-1}^2)\right]$$
$$\times \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \prod_{l=1}^{n-1} d\chi_l\, e^{-\frac{2i\rho}{\Delta t}(r_j r_{j-1}\cos\chi_j - i\lambda\chi_j)}\,, \tag{63}$$

In order to compute these integrals we use the asymptotic formula

$$\int_{-\pi}^{+\pi} d\chi\, e^{i\lambda\chi + z\cos\chi} \;\rightarrow\; 2\,\pi\, I_{|\lambda|(z)}\,, \tag{64}$$

which is valid in the limit $z \longrightarrow \infty$.

Integrating in χ, and r

$$G[x_2,\, x_1]_m \;=\; \frac{\rho}{i\pi\Delta t}\, e^{i\frac{\rho}{\Delta t}(R^2+R'^2)} \times \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda(\phi+2\pi m)}\, I_{|\lambda|}\left(-2\,i\,\frac{RR'}{\Delta t}\right)\,, \tag{65}$$

which is the the propagator for the m–th homotopy class for the Aharonov–Bohm effect. This formula was first obtained by Inomata [8] and Gerry and Singh [9] in 1979 and recently was simplified by Shiek [10].

The total propagator is

$$G[x_2,\, x_1] = \sum_{n=-\infty}^{\infty} e^{2\pi i\alpha m}\, G[x_2,\, x_1]_m\,, \tag{66}$$

where α is the magnetic flux. Replacing (65) in (66)

$$G[x_2,\, x_1] \;=\; \frac{\rho}{i\pi\Delta t}\, e^{i\frac{\rho}{\Delta t}(R^2+R'^2)} \times \sum_{-\infty}^{\infty} (-i)^{|m+\alpha|}\, e^{-i(m+\alpha)\phi}\, J_{|m+\alpha|}\left(\frac{2RR'\rho}{\Delta t}\right). \tag{67}$$

Equation (67) has several "sub–applications." As was mentioned in section 2, the motion of two anyons is an example of it. In fact, let us consider the motion of 2 free particles in a plane. The Lagrangian is

$$L = \frac{1}{2}\dot{\vec{x}}_1^{\,2} + \frac{1}{2}\dot{\vec{x}}_2^{\,2} \,. \tag{68}$$

Defining relative and center of mass coordinates, as usual, we have

$$L = \frac{1}{2}\dot{\vec{x}}^{\,2} + \frac{1}{2}\dot{\vec{X}}_{CM}^{\,2} + \alpha\,\frac{d\Theta(\vec{x})}{dt} \,, \tag{69}$$

where $\vec{x} = \vec{x}_2 - \vec{x}_1$. The coordinate $\vec{X}_{CM}$ is the center of mass position, $d\Theta/dt$ is a topological invariant which has been added by hand and that, classically, does not contribute to the equation of motion and Θ is the relative angle between the particles.

The partition function for this system becomes (the motion of the center of mass is trivially decoupled)

$$Z = \frac{1}{2}\int d^2\vec{r}\,\left[\langle\vec{r}|e^{-\beta H_{rel}}|\vec{r}\rangle + \langle\vec{r}|e^{-\beta H_{rel}}|-\vec{r}\rangle\right] \,, \tag{70}$$

with

$$L_{rel} = \frac{1}{2}M\dot{\vec{r}}^{\,2} + \alpha\dot{\theta} \,, \tag{71}$$

The brackets appearing in (70) are just the definition of the Green function and were computed previously. However the propagator is divergent and we must regularize the expression

$$Z = \sum_{n=-\infty}^{\infty} \int_0^\infty dx\, e^{-x} I_{|n-\alpha|}(x) \,. \tag{72}$$

In order to regularize we replace e^{-x} by $e^{\varepsilon x}$ and take the limit $\varepsilon \to 0$ at the end of the calculation. The reader interested in the explicit calculation can see ref. [11]. The final result is

$$\begin{aligned} F_\nu(a) &= \int_0^\infty dx\, e^{-ax}\, I_\nu(x) \\ &= \frac{1}{\sqrt{a^2-1}}\left[a + \sqrt{a + \sqrt{a^2-1}}\right]^{-\nu} \,, \\ F_\nu(1+\varepsilon) &\to \frac{1}{\sqrt{2\varepsilon}}\left[1 + \sqrt{2}\,\varepsilon\right]^{-\nu} \,. \end{aligned} \tag{73}$$

With these expressions in mind we can compute the second virial coefficient

$$B(\alpha,\, T) = 2\,\lambda_T^2\, Z\,, \tag{74}$$

with $\lambda_T = (2\pi h^2/Mkt)^{1/2}$.

If we expand around the Fermi statistics $\alpha = 2j+1+\delta$ then we find

$$B(\alpha = 2\,j+1+\delta,\, T) = \frac{1}{4}\,\lambda_T^2 + 2\,\lambda_T^2\,. \tag{75}$$

More details about the calculation can be found, *e.g.*, in the book by Lerda [12].

5.2. The Relativistic Aharanov–Bohm Effect

The relativistic extension of the Aharonov–Bohm effect is straightforward, but firstly we must define the path integral for a relativistic particle [13].

A relativistic particle is defined by the following Lagrangian

$$L = \frac{1}{2N}\,\dot{x}^2 - \frac{1}{2}\,m^2\,N\,, \tag{76}$$

where N is the einbein.

The classical symmetries of (76) are

$$\begin{aligned} \delta x^\mu &= \epsilon\,\dot{x}^\mu\,, \\ \delta N &= \epsilon\,\dot{N}\,. \end{aligned} \tag{77}$$

The next step consists in computing the propagation amplitude associated to (76). However, this is not trivial because the relativistic particle is a generally covariant system and the propagator must be written á la Faddeev–Popov, *i.e.*,

$$G[x_2,\, x_1] = \int \mathcal{D}N\,\mathcal{D}x^\mu\, \det(N)^{-1}\,\delta(f(N))\,\det\left(\frac{\delta f(N)}{\delta\epsilon}\right)\, e^{iS}\,. \tag{78}$$

This expression deserves some explanations. Firstly, we have inserted the factor $\det(N)^{-1}$, by hand in, order to have a functional measure invariant under general coordinate transformations; secondly the remaining factors are the usual terms of the Faddeev–Popov procedure, being $f(N) = 0$ the gauge condition.

An appropiate gauge condition for this problem is $\dot{N} = 0$ (proper–time gauge) and having into account the causality principle (78) becomes

$$G[x_2, x_1] = \int_0^\infty dT \int \mathcal{D}x^\mu \, e^{i\int_1^2 d\tau\left(\frac{1}{2N(0)}\dot{x}^2 - \frac{1}{2}m^2 N(0)\right)}, \tag{79}$$

with $T = N(0)\Delta\tau$, and we have assumed boundary conditions

$$\begin{aligned} x^\mu(\tau_1) &= x_1^\mu, \\ x^\mu(\tau_2) &= x_2^\mu. \end{aligned} \tag{80}$$

The formula (79) was found by Schwinger in 1951.

In order to compute (79) we repeat the arguments given in the non–relativistic case. That is, we make the change of variables

$$x^\mu(\tau) = x_1^\mu + \frac{\Delta x^\mu}{\Delta\tau}(\tau - \tau_1) + y^\mu(\tau) = x_{cl}^\mu + y^\mu(\tau) \tag{81}$$

where x_{cl}^μ is the classical solution of the equation of motion and y^μ is a quantum fluctuation that satisfies

$$\begin{aligned} y^\mu(\tau_1) &= 0, \\ y^\mu(\tau_2) &= 0. \end{aligned} \tag{82}$$

Replacing (82) in (79) we find

$$G[x_2, x_1] = \int_0^\infty dT \, T^{-D/2} \, e^{i\frac{(\Delta x)^2}{2T} - i\frac{m^2}{2}T} = \int \frac{d^D p}{(2\pi)^D} \frac{e^{ip\cdot\Delta x}}{p^2 + m^2}, \tag{83}$$

which is the expected result.

The next step is to apply these results to the study of the relativistic Aharonov–Bohm effect. The main idea is simple, we write the propagation amplitude for a relativistic particle as it was discussed above and afterwards we separate the vector x^μ in components (x^0, x^1, x^2).

The main steps are the following:

1. Firstly, instead of (79), we write

$$
\begin{aligned}
G[x_2,\, x_1] \;&=\; \int_0^\infty dT\, e^{-i\frac{1}{2}m^2 N(0)} \\
&\times \int \mathcal{D}x^0\, e^{-i\int_1^2 d\tau \frac{1}{2N(0)}(\dot{x}^0)^2} \\
&\times \int \mathcal{D}x e^{i\int_1^2 d\tau (\frac{1}{2N(0)}\dot{x}^2)} \,.
\end{aligned}
\tag{84}
$$

   Then in (84) we can consider formally the integral in x_0 as an ordinary free non–relativistic particle with mass N_0^{-1} moving in a one dimensional space. The result of this integration is trivial

$$
\frac{1}{\sqrt{T}}\, e^{-i\frac{\Delta x_0{}^2}{2T}} \,. \tag{85}
$$

2. The integral in the spatial coordinates is more complicated but we can map this problem into a non–relativistic problem with formal mass N_0^{-1}. The final result is

$$
\begin{aligned}
G[x_2,\, x_1] \;&=\; \lim_{\epsilon\to 0} \sum_{n=-\infty}^{\infty} \frac{(-i)^{|n+\alpha|}}{|n+\alpha} e^{-in+\alpha\phi} \int \frac{d^3p}{(2\pi)^3}\, e^{ip\cdot\Delta x} \\
&\times \left(J_{|n+\alpha+1|}(\sqrt{RR'}\rho) + J_{|n+\alpha-1|}(\sqrt{RR'}\rho) \right) \\
&\times \left(K_{|n+\alpha+1|}(\sqrt{RR'}\rho) + K_{|n+\alpha-1|}(\sqrt{RR'}\rho) \right) \,,
\end{aligned}
\tag{86}
$$

with $\rho = \epsilon - (p^2 + m^2)$.

For details and other applications see [14].

6. Anyons in Two Dimensions

In this section we will discuss the idea of anyon from a more general point of view, but before let us consider a particular case known as Bose–Fermi Transmutation (BFT).

The idea, due to Polyakov [15], consists into take an spinning particle described by the action [16]

$$S = \int d\tau \left(m\sqrt{\dot{x}^2} - \frac{i}{2}\theta_\mu \dot{\theta}^\mu - \frac{i}{2}\theta_5\dot{\theta}_5 + \lambda\,\theta^\mu \dot{x}_\mu + \sqrt{\dot{x}^2}\,\lambda\,\theta_5 \right), \tag{87}$$

with θ_μ, θ_5 and λ, fermionic variables. Then, when we integrate the fermionic variables we find a bosonic description of a spinning particle or more precisely, an action like

$$S = \int d\tau \left(m\sqrt{\dot{x}^2} + \text{topological invariant} \right), \tag{88}$$

We will precise this result below.

In order to define appropriately the path integral we start by defining the gauge condition

$$\theta_5 = 0\,, \tag{89}$$

which is consistent with the constraint $\theta^\mu \dot{x}_\mu = 0$.

The next step consists in proposing the decomposition for the fermionic variable

$$\theta^\mu = n_1^\mu\,\kappa_1 + n_2^\mu\,\kappa_2 + e^\mu\,\kappa_e\,, \tag{90}$$

where n_1, n_2 and e are tri–vectors that satisfy

$$
\begin{aligned}
e^{\mu} &= \frac{\dot{x}_{\mu}}{\sqrt{\dot{x}^2}}\,, \\
n_1^{\mu} \cdot e_{\mu} &= 0\,, \\
n_i^{\mu} \cdot n_{\mu j} &= -\delta_{ij}\,.
\end{aligned} \tag{91}
$$

This decomposition is equivalent to choose a Frenet–Serret frame where the n's are the normal vectors and e is the vector tangent to the worldline.

The effective fermionic action is computed from

$$
e^{iS(x)} = \int \mathcal{D}\theta^{\mu}\,\mathcal{D}\theta_5\,\mathcal{D}\lambda\,\delta(\theta_5)\,e^{iS} = \exp\left[i\,m \int \tau\,\sqrt{\dot{x}^2}\right]\,\Phi\,, \tag{92}
$$

where Φ is the Polyakov spin factor defined as

$$
\begin{aligned}
\Phi &= \int \mathcal{D}\kappa_1\,\mathcal{D}\kappa_2 \\
&\quad \times \exp\left[\int d\tau\,\left(-\frac{1}{2}\,(\kappa_1\,\dot{\kappa}_1 + \kappa_2\,\dot{\kappa}_2) + (n_1 \cdot \dot{n}_2)\,\kappa_1\,\kappa_2\right)\right] \\
&= \det\left[\frac{d}{d\tau} + (n_1 \cdot \dot{n}_2)\right]\,.
\end{aligned} \tag{93}
$$

The calculation of the determinant is straightforward [17, 18]

$$
\det\left[\frac{d}{d\tau} + (n_1 \cdot \dot{n}_2)\right] = e^{ic\int d\tau (n_1 \cdot n_2)}\cos\left[\frac{1}{2}\int d\tau\,(n_1 \cdot n_2)\right]\,, \tag{94}
$$

where c parametrizes the different possible regularizations. If we impose invariance under the interchange of n_1 and n_2 we find that $c = 0$ and the spin factor becomes

$$
\Phi = \exp\left[\frac{i}{2}\int dt\,(n_1 \cdot \dot{n}_2)\right] + \exp\left[-\frac{i}{2}\int dt\,(n_1 \cdot \dot{n}_2)\right]\,, \tag{95}
$$

where the factors $\pm 1/2$ denotes the two possible spin states.

The next question is, how to generalize this result for other spins? The answer can be obtained from Chern–Simons theories. Let us start by considering a set of N relativistic particles minimally coupled to an Abelian Chern–Simons field. The action is

$$\begin{aligned} S &= \sum_{k=1}^{N} m \int d\tau \sqrt{\dot{x}_k^2} + \int d^3x\, J_\mu\, A^\mu \\ &+ \frac{1}{2\sigma} \int d^3x\, \epsilon^{\mu\nu\rho}\, A_\mu\, \partial_\nu A_\rho\,, \end{aligned} \tag{96}$$

where $J^\mu = \sum_{k=1}^{N} x^\mu \delta^{(3)}(x - x_k(\tau))$.

Then, we integrate the A_μ field and the result gives the effective action

$$S_{eff} = m \int d\tau \sqrt{\dot{x}^2} - \frac{\sigma}{2} \int d^3x\, d^3y\, J^\mu(x)\, K_{\mu\nu}(x,\, y)\, J^\nu(y)\,, \tag{97}$$

where $K_{\mu\nu}(x, y)$ is the inverse of the operator $\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$ and, of course, satisfies

$$\epsilon^{\mu\nu\rho}\, A_\mu\, \partial_\nu\, A_\rho\, K_{\rho\sigma}(x,\, y) = \delta^\mu_\sigma\, \delta(x - y)\,, \tag{98}$$

replacing J^μ in (97) we find that the non–local term becomes

$$-\frac{\sigma}{2} \sum_{i,j=1}^{N} I_{ij}\,, \tag{99}$$

where I_{ij} is

$$I_{ij} = \frac{1}{4\pi} \int dx_i^\mu\, dx_j^\nu\, \epsilon_{\mu\nu\rho}\, \frac{(x_i - x_j)^\rho}{|x_i - x_j|^3}\,. \tag{100}$$

For closed curves ij, I_{ij} becomes the linking number, while for $i = j$ there are additional contributions in the one particle sector. These diagonal terms are computed [19] by a regularization as a limit of non–diagonal terms. However, the result can be dependent of the regularization. In order to perform this calculation we consider two infinitesimally close curves. I become

$$I = \frac{1}{4\pi} \int d\tau \, d\tau' \lim_{\epsilon \to 0} \epsilon_{\mu\nu\rho} \frac{dx_\varepsilon^\mu}{d\tau} \frac{(x_\varepsilon(\tau) - x(\tau'))^\nu}{|x_\varepsilon(\tau) - x(\tau')|^3} \frac{dx_\varepsilon^\rho}{d\tau'}, \tag{101}$$

with

$$x_\varepsilon(\tau) = x(\tau) + \varepsilon n(\tau), \tag{102}$$

the non–conmutativity of the limit procedure $\varepsilon \to 0$ and the integration, implies

$$T = \lim_{\varepsilon \to 0} L_\varepsilon - I = \frac{1}{2\pi} \int d\tau \, \epsilon_{\mu\nu\rho} \, e^\rho \, n^\nu \, \dot{n}^\rho, \tag{103}$$

where $e^\mu = e^\mu/|e|$ is the normal principal vector. The quantity T is called the torsion of the curve and I is the self–linking of the curve.

The difference $T - L$ is denoted by $\mathcal{W}$ and is the writhing number or cotorsion. Thus, the effect of the Chern–Simons field is to produce the interaction lagrangian

$$L_{int} = s \, \mathcal{W}, \tag{104}$$

where $s = \sigma/4\pi$ is the spin of the system. In this way we see that the BFT procedure is a particular case of a more general formulation coming from of a Chern–Simons contruction.

7. Anyons in One–Dimension

In this section we will discuss the possibility of anyons in one dimension. This possibility can be analized in complete analogy with the two dimensional case. In two dimensions there are anyons because there are points which have been removed from the manifold. In one dimension we can repeat the same argument as follows. Let us consider two non–relativistic particles moving on a line. For this system the configuration space consists of two disjoint pieces (the real line minus the origin) because the point where the particles collide is singular. Classically the particles cannot go through each other, and they bounce elastically every time they meet. Thus, the action of this system is defined on the half–line [3, 6], *i.e.*

$$S = \frac{1}{2} \int_{t_1}^{t_2} dt \, \dot{x}^2 \,, \tag{105}$$

for $0 < x < \infty$, where x is the relative position of the two particles. As it is well known, the Hamiltonian associated to (105) is not self–adjoint on the naive Hilbert space because there is no conservation of probability at $x = 0$. The Hamiltonian for (105), however, can be made self–adjoint by adopting a class of boundary conditions for all the states in the Hilbert space of the form [20, 21]

$$\psi'(0) = \gamma \, \psi(0) \,, \tag{106}$$

where γ is an arbitrary real parameter.[1]

The computation of the propagator between an initial position x_1 and a final position x_2, for the above problem gives [22, 23]

$$\begin{aligned} G_\gamma[x(t_2), \, x(t_1)] \;\; &= \;\; G_0(x_2 - x_1) + G_0(x_2 + x_1) \\ &\quad -2\,\gamma \int_0^\infty d\lambda \, e^{-\gamma\lambda} \, G_0(x_2 + x_1 + \lambda) \,. \end{aligned} \tag{107}$$

G_0 is the Green function for a free non–relativistic particle, *i.e.*

$$G_0(x - y) = \frac{1}{\sqrt{2\pi i t}} \, e^{i(x-y)^2/2t} \,. \tag{108}$$

Although in one spatial dimension it is not possible to rotate particles, they can be exchanged and their "spin" and statistics can

[1]There is an alternative approach to this problem. In the classical configuration space we could have exchanged states, and $x < 0$ would have been also permitted. The resulting system is described by the same action as in (105) but with $x \neq 0$ instead of $x > 0$. In this configuration space the self–adjoint extension of the Hamiltonian imposes a condition that replaces (106), with two complex parameters $\gamma_\pm$ instead of only one γ. Here we shall not follow this approach. It is remarkable however that, even when in our approach particle interchange is not included *ab initio*, quantum mechanics brings it in at the end.

be determined by the (anti–)symmetry of the wave function. This (anti–)symmetry, in turn, depends on the values of the parameter γ. This last fact can be seen by taking the limits $\gamma = 0$ and $\gamma = \infty$ of (107) [23]

$$G_{\gamma=0,\,\infty} = G_0(x_2 - x_1) \pm G_0(x_2 + x_1)\,, \tag{109}$$

Under interchange of the positions of two particles in initial or final states, $G_{\gamma=0}$ has even parity and $G_{\gamma=\infty}$ has odd parity. Thus, for $\gamma = 0$ ($\gamma = \infty$) the particles behave as bosons (fermions). The cases $0 < \gamma < \infty$ give particles with fractional spin and statistics [24]. The propagator (107) can also be obtained in the path integral representation, summing over all paths $-\infty < x(t) < \infty$, but in the presence of a repulsive potential $\gamma\delta(x)$. This problem was considered in [22, 23] and the result is

$$G_\gamma[x(t_2),\, x(t_1)] = \int \mathcal{D}x(t)\, e^{iS}\,, \tag{110}$$

with

$$S = \int_{t_1}^{t_2} dt\, \left(\frac{1}{2}\,\dot{x}^2 + \gamma\,\delta(x(t))\right)\,, \tag{111}$$

Here $\mathcal{D}x(t)$ is the usual functional measure. The potential term $\gamma\,\delta(x(t))$ can be interpreted as a semi–transparent barrier at $x = 0$ that allows the possibility of tunneling to the other side of the barrier. This is just another way of expressing the possibility of interchanging the (identical) particles.

It is also interesting to note here that although in (1+1) dimensions the rotation group is discrete and the definition of the spin is a matter of convention, we may nevertheless view the one–dimensional motion on the half–line as a radial motion with orbital angular momentum $l = 0$ [25] in a central potential. This gives rise to another possible definition of spin by taking the following representation for the δ–function

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\sqrt{\epsilon}}{x^2 + \epsilon}. \tag{112}$$

Making a series expansion around $\epsilon = 0$, the leading term $\gamma\sqrt{\epsilon}/x^2$ is analogous to the centrifugal potential for the radial equation in a spherically symmetric system, with $\sqrt{\epsilon}\gamma$ playing the role of an squared intrinsic angular momentum. Thus the spin of the system (s) can be defined by

$$s^2 = \sqrt{\epsilon}\gamma. \tag{113}$$

For real s (113) makes sense only when $\gamma > 0$. This definition is consistent with the bosonic limit $\gamma = 0$. For the fermionic case, the limit $\gamma = \infty$ mentioned above is to be interpreted as simultaneous with the limit $\epsilon \to 0$, so that $\sqrt{\epsilon}\gamma = 1/4$. It is in this sense that the non–relativistic quantum mechanics on the half–line describes one–dimensional anyons. However, the normalization $s = 1/2$ for fermions is conventional. We can extend these results to relativistic anyons. The calculations are more involved and we will give only the final result for the propagator

$$\begin{aligned} G_\gamma[X(\tau_b), X(\tau_a)] &= G_0[X(\tau_b) - X(\tau_a)] + G_0[X(\tau_b) + X(\tau_a)] \\ &\quad -2\gamma \int_0^\infty d\lambda\, e^{-\gamma\lambda}\, G_0[X(\tau_b) + X(\tau_a) + \lambda]\,. \end{aligned} \tag{114}$$

The details are discussed in [26].

8. Conclusions

In these lectures we have discussed several aspects of quantum mechanics defined on non–trivial manifolds and, in particular, anyons in one and two dimensions.

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