VIRIAL THEOREM AND GRAVITATIONAL EQUILIBRIUM WITH A COSMOLOGICAL CONSTANT

Marek Nowakowski,
Juan Carlos Sanabria and **Alejandro García**

Departamento de Física, Universidad de los Andes
A. A. 4976, Bogotá, Colombia

Starting from the Newtonian limit of Einstein's equations in the presence of a positive cosmological constant, we obtain a new version of the virial theorem and a condition for gravitational equilibrium. Such a condition takes the form $\rho > \lambda\rho_{vac}$, where ρ is the mean density of an astrophysical system (*e.g.* galaxy, galaxy cluster or supercluster), λ is a quantity which depends only on the shape of the system, and ρ_{vac} is the vacuum density. We conclude that gravitational stability might be influenced by the presence of Λ depending strongly on the shape of the system.

1. Introduction

Around 1998 two teams (the Supernova Cosmology Project [**1**] and the High–Z Supernova Search Team [**2**]), by measuring distant type Ia Supernovae (SNIa), obtained evidence of an *accelerated* expanding universe. Such evidence brought back into physics Einstein's "biggest blunder", namely, a positive cosmological constant Λ, which would be responsible for speeding–up the expansion of the universe. However, as will be explained in the text below, the Λ term is not only of cosmological relevance but can enter also the domain of astrophysics. Regarding the application of Λ in astrophysics we note that, due to the small values that Λ can assume, its "repulsive" effect can only be appreciable at distances larger than about

1 Mpc. This is of importance if one considers the gravitational force between two bodies. On the other hand for extended bodies, if the astrophysical system is sufficiently diluted the "Λ–force" could overcome the Newtonian gravitational attraction leading to ask when such a system is gravitationally stable.

In order to address the question of whether a system can be in gravitational equilibrium or not, it is necessary to re–derive the virial theorem, in the Newtonian limit of Einstein's equations, with the presence of Λ different from zero. Some related astrophysical applications of the presence of $\Lambda > 0$ have been discussed in [**3**].

2. Newtonian limit of Einstein's equations with $\Lambda \neq 0$

In this section we will briefly address the problem of the Newtonian limit of Einstein's equation with the cosmological constant. More details can be found in [**4**]. We begin with Einstein's field equations

$$G_{\mu\nu} - \Lambda\, g_{\mu\nu} = -8\,\pi\, G\, T_{\mu\nu}\,, \tag{1}$$

where $G_{\mu\nu}$ is the usual Einstein tensor. We assume that the fields are weak enough so that the velocities of the bodies involved are much less than the speed of light, that is

$$|\Phi| \ll 1\,. \tag{2}$$

We also assume a nearly Lorentzian metric

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}\,, \tag{3}$$

such that, in agreement with eq. (2), we impose

$$|h_{\alpha\beta}| \ll 1\,. \tag{4}$$

It is then straightforward to show that in first order, eq. as [**5**]

$$\Phi(r) = -\frac{GM}{r} - \frac{1}{6}\,\Lambda\, r^2\,. \tag{5}$$

Since the general Newtonian limit has the form of a partial differential equation, it is clearly necessary to impose some boundary condition to find the solution. However, not every boundary condition is compatible with the condition (2). To see this point it is instructive to work out the constraints in more detail. To this end, it suffices to impose (2) on the spherically symmetric potential (6). It then follows readily that [**4**]

$$M_{max} = \frac{2\sqrt{2}}{3}\frac{1}{G\sqrt{\Lambda}} \gg M\,,$$

$$R_{max} \gg r \gg R_{min}\,, \tag{6}$$

where

$$\begin{aligned} R_{max} &\simeq \sqrt{\frac{6}{\Lambda}}\left(1-\frac{1}{3\sqrt{3}}\frac{M}{M_{max}}\right), \\ R_{min} &\simeq GM\left[1-\frac{1}{54}\left(\frac{M}{M_{max}}\right)^2\right]. \end{aligned}$$

Equations (7) show that the Newtonian limit is only valid if the mass M generating the gravitational field is much smaller than M_{max} and the distance we are allowed to consider in such a limit is restricted from below and above, as indicated.

If $\Lambda = 0$, both M_{max} and R_{max} tend to infinity. Hence, in principle, the Newtonian limit with $\Lambda \neq 0$ is quite different from the case $\Lambda = 0$.

Due to the existence of R_{max}, we can now easily see that the standard[1] Dirichlet boundary condition $\Phi|_{R\to\infty} = 0$ is not allowed. Instead, we can choose $\Phi|_R = 0$ for a certain finite radius R, which is of course smaller than R_{max}, but still quite sizable. This boundary condition is clearly consistent with (2). Then, the general solution of (4) can by obtained in the form

[1] *Standard* refers here to the case $\Lambda = 0$.

$$\Phi(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \frac{1}{6} \Lambda \, |\mathbf{x}|^2 \, d^3\mathbf{x}' + G \int G'(\mathbf{x},\, \mathbf{x}')\, \rho(\mathbf{x}')\, d^3\mathbf{x}' \,, \tag{7}$$

$$G'(\mathbf{x}, \mathbf{x}') = 4\,\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Y^*_{lm}(\theta', \phi') Y_{lm}(\theta, \phi)}{2l+1} \frac{r^l_< r^l_>}{R^{2l+1}} \,,$$

with $r_< = \min(|\mathbf{x}|, |\mathbf{x}'|)$, $r_> = \max(|\mathbf{x}|, |\mathbf{x}'|)$.

The above solution consists of three pieces: the internal Newtonian attractive potential, the external repulsive Λ part and a third term which is a direct effect of the boundary condition at a finite radius (an indirect effect of Λ). However, if we choose R to be one or two orders of magnitude smaller than R_{max} we are justified to neglect higher order contributions to Φ, which are suppressed by powers of $1/R^n$. Then, in a first approximation, it is legitimate to drop the last term in eq. (8). This leads to

$$\Phi(\mathbf{x}) = \Phi_N(\mathbf{x}) - \frac{\Lambda}{6} \, |\mathbf{x}|^2 \,, \tag{8}$$

$$\Phi_N(\mathbf{x}) = -G \int \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \, d^3\mathbf{x}' \,,$$

where the first term is the Newtonian gravitational potential and the second one corresponds to an "anti–gravity" force proportional to the distance, as mentioned before. Note again that the Λ–force acts as an *external* force. With our Λ–corrected potential we are ready to study its dynamical effects on large astronomical systems.

3. The new virial theorem from the collisionless Boltzman equation

As mentioned in the introduction, our main objective is to study gravitational equilibrium in the presence of the cosmological constant. The new form of the gravitational potential demands now a re–derivation of the virial theorem from the collisionless Boltzman equation.

A full description of the state of any collisionless system is given by specifying the number of stars or galaxies contained in a volume $d^3\mathbf{x}$ centered at $\mathbf{x}$ and with velocities in the range $d^3\mathbf{v}$ around $\mathbf{v}$. The system is under the influence of a potential $\Phi(\mathbf{x},\mathbf{t})$ (in this case the potential of eq. (9)). This quantity, denoted as $f(\mathbf{x},\mathbf{v},\mathbf{t})$, is called *phase–space density* or *distribution function.*

For the cases under consideration the number of stars or galaxies are obviously conserved and move smoothly through space, hence $f(\mathbf{x},\mathbf{v},\mathbf{t})$ must satisfy the Liouville theorem

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = 0\,, \tag{9}$$

where q_i are the spatial generalized coordinates and p_i the momentum coordinates. Taking into account that positions, velocities and accelerations are independent, eq. (9) becomes the collisionless Boltzman equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} \right) = 0\,. \tag{10}$$

Multiplying eq. (10) with v_i, integrating over all velocities and using the divergence theorem in the form

$$\frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i}\, d^3\mathbf{v} = -\frac{\partial \Phi}{\partial x_i} \int \delta_{ij} f\, d^3\mathbf{v} \tag{11}$$

we obtain

$$\frac{\partial(\rho \bar{v}_j)}{\partial t} + \frac{\partial(\rho \overline{v_i v_j})}{\partial x_i} + \rho \frac{\partial \Phi}{\partial x_j} = 0\,, \tag{12}$$

which is known as Jeans equation. In the last equation we have defined

$$\begin{aligned} \rho &= \int f d^3\mathbf{v}\,, \\ \bar{v}_i &= \frac{1}{\rho}\int f v_i d^3\mathbf{v}\,, \\ \overline{v_i v_j} &= \frac{1}{\rho}\int v_i v_j f d^3\mathbf{v} \end{aligned} \tag{13}$$

Here ρ has clearly the meaning of density. Now, multiplying eq. (13) with x_k and integrating over the spatial coordinates we get

$$\int x_k \frac{\partial(\rho\bar{v}_j)}{\partial t}\, d^3\mathbf{x} = -\int x_k \frac{\partial(\overline{v_i v_j})}{\partial x_i}\, d^3\mathbf{x} - \int x_k\, \rho\, \frac{\partial \Phi}{\partial x_j}\, d^3\mathbf{x}\,. \tag{14}$$

The first term on the right–hand side can be transformed again via the divergence theorem and in the second one we replace $\Phi(\mathbf{x})$ with the proper Newtonian limit, eq. (9). In this way one readily arrives at

$$\begin{aligned} \int x_k \frac{\partial(\rho\bar{v}_j)}{\partial t}\, d^3\mathbf{x} &= \int \rho\, \overline{v_k v_j}\, d^3\mathbf{x} - \int \rho\, x_k \frac{\partial \Phi_N}{\partial x_j}\, d^3\mathbf{x} \\ &+ \frac{\Lambda}{3}\int \rho\, x_k\, x_j\, d^3\mathbf{x}\,. \end{aligned} \tag{15}$$

At last, using the tensor definitions

$$\begin{aligned} K_{jk} &= \frac{1}{2}\int \rho\, \overline{v_j v_k}\, d^3\mathbf{x}\,, \\ W_{jk} &= -\int \rho\, x_k \frac{\partial \Phi_N}{\partial x_j}\, d^3\mathbf{x}\,, \\ I_{jk} &= \int \rho\, x_j\, x_k\, d^3\mathbf{x}\,, \end{aligned} \tag{16}$$

where K_{jk} corresponds to the kinetic energy tensor, W_{jk} to the Chandrasekhar potential energy tensor and I_{jk} to the moment of inertia tensor, eq. (16) reads

$$\int x_k \frac{\partial(\rho \bar{v}_j)}{\partial t}\, d^3\mathbf{x} = 2\, K_{jk} + W_{jk} + \frac{\Lambda}{3}\, I_{jk}\,. \tag{17}$$

The left–hand side of eq. (18) can be rewritten, due to the time independence of x_k, as

$$\begin{aligned} \int x_k \frac{\partial(\rho \bar{v}_j)}{\partial t}\, d^3\mathbf{x} &= \frac{1}{2}\frac{\partial}{\partial t}\int \rho\, (x_k\, \overline{v_j} + x_k\, \overline{v_j})\, d^3\mathbf{x} && (18)\\ &= \frac{1}{2}\frac{\partial}{\partial t}\int \rho\, (x_k\, \overline{v_j} + x_j\, \overline{v_k})\, d^3\mathbf{x}\,. && (19) \end{aligned}$$

Above we have used the fact that the tensors in the right–hand side of eq. (18) are symmetric. Recalling the definition of I_{jk} and taking the derivative with respect to time we get

$$\frac{\partial}{\partial t}\, I_{jk} = \int \frac{\partial \rho}{\partial t}\, x_i\, x_j\, d^3\mathbf{x}\,. \tag{20}$$

Now, because mass is obviously conserved, we can use the hydrodynamics *equation of continuity* in the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial\, (\rho \overline{v_i})}{\partial x_i} = 0\,. \tag{21}$$

Hence, replacing $\partial\rho/\partial t$ from eq. (22) into eq. (21) and using the divergence theorem we get

$$\begin{aligned} \frac{\partial I_{jk}}{\partial t} &= -\int \frac{\partial\, (\rho \overline{v_i})}{\partial x_i}\, x_j\, x_k\, d^3\mathbf{x} && (22)\\ &= \int \rho\, (x_k\, \overline{v_j} + x_j\, \overline{v_k})\, d^3\mathbf{x}\,. && (23) \end{aligned}$$

Comparing eqs. (24) and (20), eq. (18) can be cast into a more familiar form

$$\frac{1}{2}\frac{\partial^2 I_{jk}}{\partial t^2} = 2\, K_{jk} + W_{jk} + \frac{\Lambda}{3}\, I_{jk}\,, \tag{24}$$

which is the new tensor virial theorem. Taking the steady state condition, $\partial^2 I/\partial t^2 = 0$, eq. (25) transforms into the new scalar virial theorem for steady state

$$2K + W + \frac{\Lambda}{3} I = 0 \tag{25}$$

and the definitions (17) become

$$\begin{aligned} K &= \frac{1}{2} \int \rho \bar{v}^2 \, d^3\mathbf{x}\,, \\ W &= \frac{1}{2} \int \rho \, \Phi(\mathbf{x}) \, d^3\mathbf{x}\,, \\ I &= \int \rho \, |\mathbf{x}|^2 \, d^3\mathbf{x}\,. \end{aligned} \tag{26}$$

4. Applications

Once obtained our new virial theorem for an accelerated expanding universe, we can use it in a variety of applications. On one hand, given the mean velocity of a test object ($\bar{v}$ in definitions (27)) around a system, *i.e.* a star around a galaxy or a galaxy around a cluster or supercluster, one can infer the mean density, ρ, of the system. On the other hand, due to our assumption of equilibrium in the re–derivation of the virial theorem, one can find the conditions for such an equilibrium to exist, which are our main result. We concentrate here on astrophysical applications. A nice discussion of the use of the virial theorem with $\Lambda > 0$ to cosmology can be found in [**6**].

Since the kinetic energy term in eq. (25) is clearly positive, we transform eq. (25) into the inequality

$$\frac{\Lambda}{3} I + W \leq 0\,, \tag{27}$$

which is the equilibrium condition that we were looking for. In order to obtain some more insight into the meaning of this inequality, we will evaluate it for different shapes of astrophysical bodies, assuming some constant mean density (ρ) for the system. With $\rho = \text{const}$, the definitions (27) take the form

$$
\begin{aligned}
W &= -\frac{1}{2}\rho^2 G W' , \\
W' &= \int \left[\int \frac{d^3\mathbf{x}}{|\mathbf{x}-\mathbf{x}'|} \right] d^3\mathbf{x}' , \\
I &= \rho I' , \\
I' &= \int |\mathbf{x}|^2 \, d^3\mathbf{x} .
\end{aligned} \tag{28}
$$

Hence, substituting (29)into eq. (28), one gets

$$\rho \geq \lambda \, \rho_{vac} , \tag{29}$$

where ρ_{vac} refers to the vacuum density $\rho_{vac} = \Lambda/8\pi G$, and ρ is the system's mean density. For the geometric quantity λ we obtain

$$\lambda = \frac{16}{3} \pi \frac{I'}{W'} . \tag{30}$$

It should be noticed that the value of λ depends only on the shape of the system. Next, we focus our attention on computing λ for various geometries.

By evaluating the integrals and solving for ρ for the simplest case of a sphericallly symmetry system we get

$$\rho \geq 2 \, \rho_{vac} . \tag{31}$$

Now, for the more realistic case of an ellipsoid–like system with axes a, b and c, we have the following three possibilities (see [**7**]):

$$
\begin{aligned}
\lambda_{oblate} &= \frac{2}{3} e \left(\frac{a}{c}\right) \frac{\left[2 + \left(\frac{c}{a}\right)^2\right]}{\arcsin(e)} , \\
e &= \sqrt{1 - \left(\frac{c}{a}\right)^2} ,
\end{aligned}
$$

for $a = b > c$;

$$\lambda_{prolate} = \frac{4}{3}\left(\frac{c}{a}\right)^4\left[2\left(\frac{a}{c}\right)^2+1\right]\frac{e}{\ln\left(\frac{1+e}{1-e}\right)},$$
$$e = \sqrt{1-\left(\frac{a}{c}\right)^2},$$

for $a = b < c$; and

$$\lambda_{triaxial} = \frac{2}{3}\sqrt{a^2-c^2}\,\frac{(a^2+b^2+c^2)}{abc\,F(\theta,k)},$$
$$k = \sqrt{\frac{a^2-b^2}{a^2-c^2}}, \qquad (32)$$

for $a > b > c$, where $F(\theta, k)$ is the complete elliptic integral of the first kind, $\theta = \arccos(c/a)$.

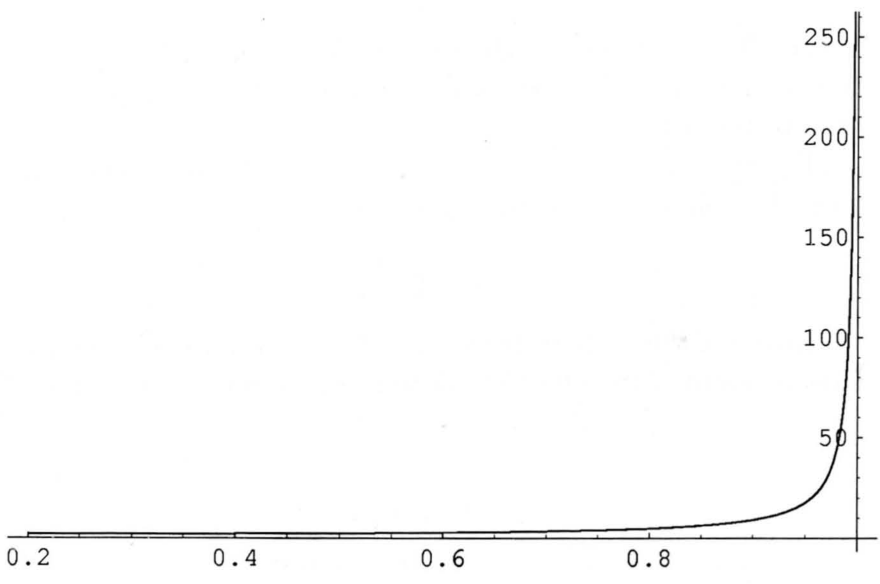

Figure 1. Plot of λ_{oblate} vs. eccentricity.

To see in general how the λ's behaves we recall the eccentricity of an ellipsoid $\epsilon = 1-\beta/\alpha$, where β and α are the minor and major axis respectively, and plot λ_{oblate} and $\lambda_{prolate}$ against the eccentricity (figures 1 and 2).

For the $\lambda_{triaxial}$ case, we define the ratios b/a and c/a and then plot $\lambda_{triaxial}$ against b/a for fixed values of c/a (figure 3).

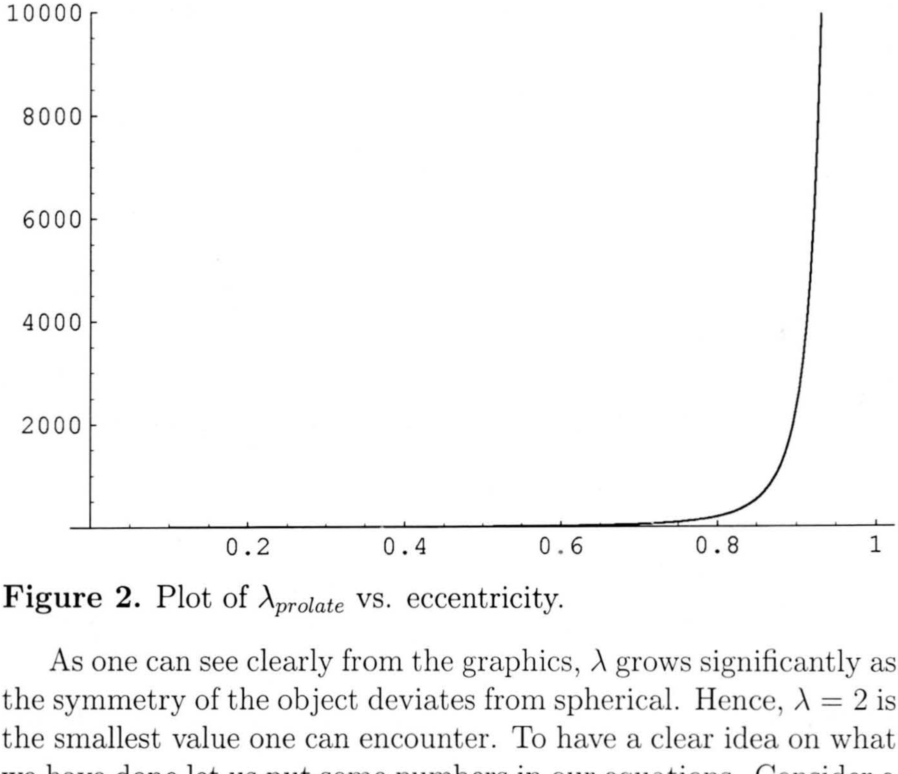

Figure 2. Plot of $\lambda_{prolate}$ vs. eccentricity.

As one can see clearly from the graphics, λ grows significantly as the symmetry of the object deviates from spherical. Hence, $\lambda = 2$ is the smallest value one can encounter. To have a clear idea on what we have done let us put some numbers in our equations. Consider a typical galaxy supercluster with mean density $\rho \sim 10^{-29}$ g cm^{-3} [**8**] (even values of the order of 10^{-30} g cm^{-3} or 10^{-3}g cm^{-3} are known to exist [**9**]). According to the last SN Ia measurements

$$
\begin{aligned}
\Omega_\Lambda &= \frac{\rho_{vac}}{\rho_{crit}} \sim (0.7 - 0.8)\,, \\
\rho_{crit} &= \frac{3H_0^2}{8\pi G}\,, \\
H_o &\simeq 100 h_0 \mathrm{km}^{-1}\mathrm{Mpc}^{-1}\,, \\
h_o &\simeq (0.6 - 0.8)\,, \qquad (33)
\end{aligned}
$$

we have $\rho_{vac} \sim 10^{-30}$ g cm^{-3}. Taking into account the fact that large galaxy superclusters tend to be very eccentric, one observes that the inequality (30) can be easily violated, in which case such superclusters cannot be gravitationally stable. For the case of typical elliptical galaxies one has $\rho \sim 10^{-26}$ g cm^{-3}. It follows that for very eccentric prolate galaxies, the inequality can, in principle, be violated.

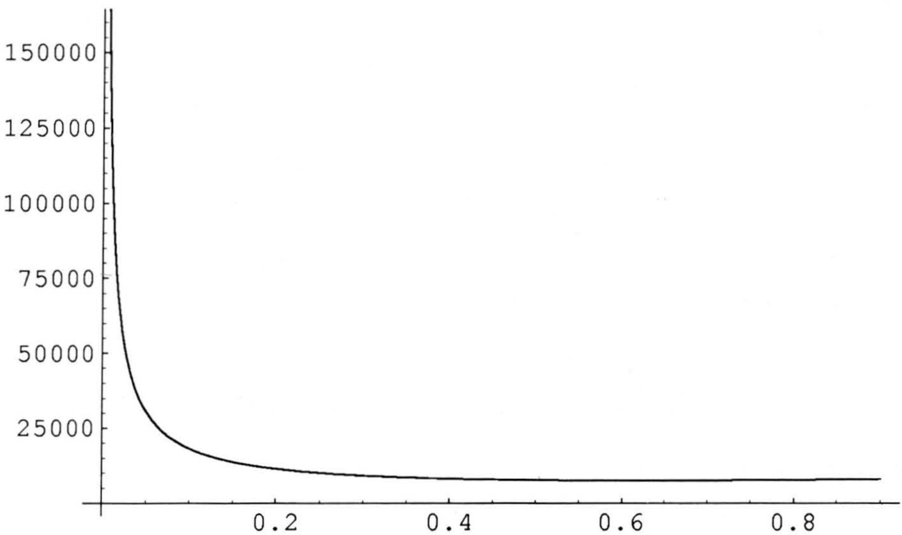

Figure 3. Plot of $\lambda_{triaxial}$ vs. b/a for a fixed value of $c/a = 0.001$.

5. Conclusions

We conclude that large galactic structures, like supercluster and possibly clusters with some deviations from spherical symmetry, may not be gravitationally stable. Hence, when $\Lambda \neq 0$, previously thought stable systems might no longer be considered in a steady state. We have also demonstrated how smaller systems, like galaxies eccentric enough, can also be affected by the presence of Λ. This may help explain why elliptical galaxies with eccentricities greater than 0.7 (in elliptical morphology E_7) have not been observed.[2] The last statement should be treated with some care since galaxy simulations also seem to indicate a low ellipticity in elliptic galaxies. However, it is obvious from our results that, assuming that galaxies are in gravitational equilibrium, the same result can be obtained from the point of view of the virial theorem alone, provided that Λ is non–zero.

[2] As, obviously, galaxies have reached their gravitational equilibrium.

References

1. S. Perlmutter *et al.* SNCP [Supernova Cosmology Project Collaboration], *Measurements of Ω and Λ form 42 high–red-shift supernovae,* LBNL–41801, *Cosmology from type Ia Supernovae*, January 1998 Meeting of the ASS, LBNL–42230; M. S. Turner, *Why Cosmologist believe the universe is accelerating,* astro–ph/990449. Supernova Cosmology Project web page http://www–supernova.lbl.gov/

2. B. P. Schmidt *et al.* HZSST [High–Z Supernova Search Team Collaboration], Ap. J. **507**, 46 (1998); astro–ph/9805200. HZSST web page: http://cfa–www.harvard.edu/cfa/oir /Research/supernova/HighZ.html/

3. M. Axenides, E. G. Floratos and L. Perivolaropoulos, Mod. Phys. Lett. A **15**, 1541 (2000), astro–ph/0004080; L. Bergström and U. Danielsson, astro–ph/0002152.

4. M. Nowakowski, *The consistent Newtonian limit of Einstein's gravity with a cosmological constant*, gr–qc/0004037, to appear in Int. J. Mod. Phys. D.

5. R. Adler, M. Bazin and M. Schiffer, *Introduction to General Relativity* (McGraw–Hill, 1965).

6. J. D. Barrow and G. Götz, Class. Quantum Grav. **6**, 1253 (1989).

7. J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, 1987).

8. M. V. Zombeck, *Handbook of Space Astrononmy and Astrophysics*, Second edition (Cambridge University Press, 1990).

9. C. D. Shane and C. A. Wirtanen, Publ. Lick Observatory **22** Part 1 (1967).