ANALYSIS OF SOME ASPECTS UPON THE MAGNETOSPHERE TERRESTRIAL

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Resumen

Usando un modelo simple se analiza cualitativamente la deformación que sufre el campo magnético terrestre debido al viento solar. Se estudiaron también las corrientes eléctricas generadas en el viento solar debido a la presencia del campo magnético de la Tierra.

Palabras claves: Magnetósfera, Viento Solar, Campo Magnético.

Abstract

Using a simple model we analyze qualitatively the deformation that terrestrial magnetic field suffers due to the presence of the solar wind; we also studied the electrical currents generated into the solar wind due to the presence of Earth's magnetic field.

Keywords: Magnetosphere, Solar Wind, Magnetic Field.

1 Introduction

Many problems of magnetohydrodynamics do not have analytic solutions and have to be approached using simple models that help to understand some important phenomena and to distinguish amongst the more and less influential ones.

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Terrestrial magnetosphere is the layer of the atmosphere related to terrestrial magnetic field, where interactions between the magnetic field and the wind coming from the Sun occur, deforming the magnetic field in the direction of solar wind flow. The solar wind is a "ionized gas" with a very large electrical conductivity, chiefly made up of protons and electrons of high velocities [1, 8, 9].

As the velocity of this fluid is so large, far away from the source of magnetic field it is not affected appreciable due to terrestrial magnetic field, however the magnetic field changes the trajectory of particles near the Earth, partly this is the cause that certain low energy particles remain trapped in the magnetic field oscillating and producing electromagnetic radiation, and in the poles their interaction with air molecules cause the know effect called northern dawn [1, 8].

This paper is organized as follows: In section 2 we derivate the Magnetohydrodynamic equations, in section 3 we describe the problem and calculate the deformation of terrestrial magnetic field due to the solar wind. Section 4 is devoted to analysis of the results and conclusions.

2 Magnetohydrodynamic

The laws of magnetohydrodynamics [2, 10] are deduced from Maxwell's equations:

$$\nabla \cdot \vec{E} = 4\pi \frac{\rho_e}{\epsilon}. \tag{2. 1}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{2. 2}$$

$$\nabla \cdot \vec{B} = 0 \tag{2. 3}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\mu \vec{j} + \frac{\epsilon\mu}{c}\frac{\partial \vec{E}}{\partial t} \tag{2. 4}$$

Introducing the Ohm's law for a conductive fluid, *i.e.* that the electric current circulating inside the fluid is proportional to the electric field, this is:

$$\vec{j'} = \sigma \vec{E'} \tag{2. 5}$$

where $\vec{j'}$ and $\vec{E'}$ are the electric current and the electric field as seen from the frame of reference of the fluid. By mean the Lorentz's transformations we pass from fluid reference frame to laboratory reference frame, in the non-relativistic approximation $v << c$.

$$\vec{j} = \vec{j'} + \rho_e \vec{v} \tag{2. 6}$$

$$\vec{E'} = \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \tag{2. 7}$$

Assuming a conductive fluid without net charge, *i.e.*, the charge density ρ_e is null. Substituting (2. 6) and (2. 7) into (2. 5) we obtain the final expression for the electric current:

$$\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right). \tag{2. 8}$$

If we replace (2. 8) in (2. 4), we obtain:

$$\nabla \times \vec{B} = \frac{4\pi}{c} \mu\sigma \left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right] + \frac{\epsilon\mu}{c} \frac{\partial \vec{E}}{\partial t} \tag{2. 9}$$

In this equation we observe that the second term on the righthand side can be despised, since its order of magnitude is very small in comparison to the first term as the fluid is a conductor (conductivity σ very large). With this consideration (2. 9) becomes:

$$\nabla \times \vec{B} = \frac{4\pi}{c} \mu\sigma \left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right]. \tag{2. 10}$$

We apply the rotational operator to (2. 10) and keeping in mind that the divergence of the field is always null, we have:

$$-\nabla^2 \vec{B} = \frac{4\pi}{c} \mu\sigma \left[\nabla \times \vec{E} + \nabla \times \left(\frac{\vec{v}}{c} \times \vec{B} \right) \right]. \tag{2. 11}$$

Considering Faraday's law in (2. 2), we arrive to a diffusion equation of magnetic field:

$$\frac{\partial \vec{B}}{\partial t} = \eta_m \nabla^2 \vec{B} + \nabla \times (\vec{v} \times \vec{B}) \tag{2. 12}$$

where $\eta_m = \frac{c^2}{4\pi\mu\sigma}$ is known as the magnetic viscosity. The diffusion equation must be accompanied with the movement equation (Newton's second law) and a continuity equation for the fluid [3, 5]:

$$\rho \frac{d\vec{v}}{dt} = -\nabla P + \frac{1}{c}\vec{j} \times \vec{B} + \eta \nabla^2 \vec{v}. \tag{2. 13}$$

$$\nabla \cdot (\rho \vec{v}) = -\frac{\partial \rho}{\partial t}. \tag{2. 14}$$

Equation (2. 14) is the continuity equation for mass, where ρ is the density of matter and $\vec{v}$ the velocity of the fluid; equation (2. 13) is called the Navier-Stokes's equation or movement equation for fluids. In this equation intervenes the pressure gradient, a term of magnetic force and a term due to the viscosity of the fluid. If we considered that the fluid is incompressible, equations (2. 12)-(2. 14) are a complete set describing the dynamics of the fluid and the magnetic field, on the other hand, if the fluid is considered compressible, in addition of equations (2. 12)-(2. 14) the state equation of the fluid should be in mind. Equations (2. 12)-(2. 14) are the magnetohydrodynamic equations.

3 Problem Description

The objective of this work is to analyze how the terrestrial magnetic field is modified by the presence of the solar wind in zones far away from Earth's magnetic field source. For this, we consider a conductive fluid whose velocity is constant, *i.e.* $\vec{v} = v_0 \hat{a_x}$. This approximation is valid in the sense that the velocity of solar wind particles is very high far away of Earth, and is not affected appreciably by the terrestrial magnetic field. Besides, we also consider an incompresible and irrotacional fluid, which is equivalent to exclude the dynamics related to the formation of vortices in the fluid, that generally form in the proximities of Earth, because its magnetic field is very strong. According to these considerations, the velocity of the fluid can be written as the gradient of a potential function of the form:

$$\vec{v} = -\nabla\phi_v. \tag{3. 1}$$

According to equation (2. 14), for an incompressible fluid its density, then we obtain:

$$\begin{aligned}
&\nabla \cdot (\rho\vec{v}) = 0 \\
&\rho(\nabla \cdot \vec{v}) + \nabla\rho \cdot \vec{v} = 0 \\
&\nabla \cdot \vec{v} = 0 \\
&\nabla^2\phi_v = 0.
\end{aligned} \tag{3. 2}$$

Another aspect that we consider is the stationary case in which the physical quantities remain constant in time. This approximation can be achieved if we take a constant velocity for the solar wind. Another factor might carry changes in Earth's magnetosphere is related to possible variations of Earth's magnetic dipole moment, however it has a very small effect during small intervals of time [4, 6, 7]. Accordingly, equation (2. 2) for the electric field can be expressed as:

$$\nabla \times \vec{E} = 0. \tag{3. 3}$$

In this model we also suppose that the fluid does not possess net electrical charge as, although the gas in consideration is composed by charged particles, to great scale the net charge is very small. Then Gauss's Law ec.(2. 1) is given by:

$$\nabla \cdot \vec{E} = 0. \tag{3. 4}$$

Taking into (2. 11), $\nabla \times \vec{E} = 0$, we obtain

$$-\nabla^2\vec{B} = \frac{4\pi}{c}\mu\sigma\nabla \times \left(\frac{\vec{v} \times \vec{B}}{c}\right)$$

the last expression is known as the diffusion equation for magnetic fields, which can be obtained also by considering that the electric field is null, *i.e.*, if we take a the trivial solution for equation (3. 4), then we have $\vec{E} = 0$. In our model obtaining a non-trivial solution for the diffusion equation (2. 12) can be a complicated calculation, as it also has to verify the condition $\nabla \cdot \vec{B} = 0$. By this reason we work with a vector magnetic potential such that:

$$\vec{B} = \nabla \times \vec{A}. \tag{3. 5}$$

from (3. 5) and (2. 10), using the vector identity

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}. \tag{3. 6}$$

we obtain the diffusion equation for the vector potential:

$$\nabla^2 \vec{A} = -\frac{4\pi}{c^2} \left[\vec{v} \times (\nabla \times \vec{A}) \right] + \nabla(\nabla \cdot \vec{A}). \tag{3. 7}$$

Using the vectorial identity:

$$\nabla(\vec{v} \cdot \vec{A}) = (\vec{v} \cdot \nabla)\vec{A} + (\vec{A} \cdot \nabla)\vec{v} + \vec{v} \times (\nabla \times \vec{A}) + \vec{A} \times (\nabla \times \vec{v}) \tag{3. 8}$$

and keeping in mind that the velocity is constant, expression (3. 8) is transformed into

$$\vec{v} \times (\nabla \times \vec{A}) = \nabla(\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \nabla)\vec{A}. \tag{3. 9}$$

By substituting (3. 9) in (3. 7) we obtain:

$$\nabla^2 \vec{A} = -\frac{4\pi}{c^2}\mu\sigma \left[\nabla(\vec{v} \cdot \vec{A}) - (\vec{v} \cdot \nabla)\vec{A} \right] + \nabla(\nabla \cdot \vec{A}). \tag{3. 10}$$

The constants that appear in the first term on the right hand side of (3. 10) correspond to the inverse of magnetic viscosity (η_m^{-1}). Then, the equation to solve is

$$\eta_m \nabla^2 \vec{A} = \nabla(\eta_m \nabla \cdot \vec{A} - \vec{v} \cdot \vec{A}) + (\vec{v} \cdot \nabla)\vec{A}. \tag{3. 11}$$

As the velocity is constant along the x axis, the equation is:

$$\eta_m \nabla^2 \vec{A} - (\vec{v} \cdot \nabla)\vec{A} = \nabla(\eta_m \nabla \cdot \vec{A} - v_0 A_x). \tag{3. 12}$$

To solve equation (3. 12) we use the condition $\nabla(\eta_m \nabla \cdot \vec{A} - v_0 A_x) = 0$, then, of the possible solutions that we obtain, we will find the solution for the potential vector that comply with this condition. So the equation to resolve for each component of the vector potential is:

$$\begin{aligned} \eta_m \nabla^2 A_x - (\vec{v} \cdot \nabla) A_x = 0 \\ \eta_m \nabla^2 A_y - (\vec{v} \cdot \nabla) A_y = 0 \\ \eta_m \nabla^2 A_z - (\vec{v} \cdot \nabla) A_z = 0. \end{aligned} \tag{3. 13}$$

It is well known that the velocity is proportional to the negative gradient of the potential function ϕ_v. Then we can write the last three equations as:

$$\eta_m \nabla^2 A_i - \nabla \phi_v \cdot \nabla A_i = 0, \quad con \;\; i = x, y, z \tag{3. 14}$$

hence the general equation to solve is:

$$\eta_m \nabla^2 B - \nabla \phi_v \cdot \nabla B = 0 \tag{3. 15}$$

where we call B each one of the components x, y and, z of the vector potential. To solve this problem we propose a solution of the form:

$$B = \phi_m B_0. \tag{3. 16}$$

Replacing this assumption in (3. 15) for the function B, we have that functions should satisfy:

$$\nabla \phi_m = -\frac{\phi_m}{2\eta_m} \nabla \phi_v. \tag{3. 17}$$

and

$$\nabla^2 B_0 - k^2 B_0 = 0 \tag{3. 18}$$

where,

$$k = \frac{v_0}{2\eta_m} = 2\pi\mu \frac{v_0 \sigma}{c^2}. \tag{3. 19}$$

The solution of equation (3. 18), is known as the modified Helmholtz's equation, which is given in terms of the spherical harmonics and the modified spherical Bessel polynomials, this is:

$$B_0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[C_{lm} i_l(kr) + D_{lm} n_l(kr)\right] Y_{lm}(\theta, \phi). \qquad (3.\ 20)$$

As we want a solution that converges in the infinite, because the magnetic field go to zero in the infinity, we should require that the $C_{lm} = 0$. Then the solution for B_0 is:

$$B_0 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[D_{lm} n_l(kr)\right] Y_{lm}(\theta, \phi). \qquad (3.\ 21)$$

The solution of (3. 17) can be obtained by direct integration, equalizing component by component in both sides of the equation, we obtain:

$$\phi_m = C_1 e^{-\frac{1}{2\eta_m}\phi_v} = C_1 e^{\frac{v_0}{2\eta_m}x}. \qquad (3.\ 22)$$

The integration constant C_1 is included into constant D_{lm}. Accordingly to (3. 16) we build a solution for the components of the potential vector $A = \phi_m B$:

$$A_x = e^{kx} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[D_{lm} n_l(kr)\right] Y_{lm}(\theta, \phi). \qquad (3.\ 23)$$

$$A_y = e^{kx} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[E_{lm} n_l(kr)\right] Y_{lm}(\theta, \phi). \qquad (3.\ 24)$$

and

$$A_y = e^{kx} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[F_{lm} n_l(kr)\right] Y_{lm}(\theta, \phi). \qquad (3.\ 25)$$

As in the proximities of the source the potential vector obtained must approach to the one produced by a dipole, it can be calculated applying Laplace's equation:

$$\begin{aligned} A_{xd} &= -\frac{m'}{r^2}\sin(\theta)\sin(\phi) = -\frac{m'y}{r^3} \\ A_{yd} &= \frac{m'}{r^2}\sin(\theta)\cos(\phi) = \frac{m'x}{r^3} \\ A_{zd} &= 0. \end{aligned} \tag{3. 26}$$

Where m' is the dipole moment. Studying the angular behavior of the solutions (3. 23), (3. 24) and (3. 25) we can consider a solution of the form:

$$\begin{aligned} A'_x &= -e^{-k(r-x)}\frac{my(1+kr)}{k^{3/2}r^3} \\ A'_y &= e^{-k(r-x)}\frac{mx(1+kr)}{k^{3/2}r^3} \\ A'_z &= 0. \end{aligned} \tag{3. 27}$$

Where m is not a dipole moment, as it does not have the right units, but it has a dependence with the Earth's dipole moment Of the form $m \propto \frac{m_{Earth'sdipole}}{Length^{3/2}}$.
However, in order to have a valid solution, we should require the divergence of A (3. 12) to be given by:

$$\eta_m \nabla\cdot\vec{A} - v_0 A_x = 0. \tag{3. 28}$$

This is the condition that we have been working. Therefore, it is necessary to add an extra term, coming from introducing the sum of (3. 24) into A'_y, and thus the final solution for the magnetic potential vector is:

$$\begin{aligned} A_x &= -\frac{my(1+kr)}{k^{3/2}r^3}e^{-k(r-x)} \\ A_y &= m\left[\frac{x(1+kr)}{k^{3/2}r^3} + \frac{1}{\sqrt{kr}}\right]e^{-k(r-x)} \\ A_z &= 0. \end{aligned} \tag{3. 29}$$

From (3. 5) we can determine the magnetic field which, for our case, is:

$$
\begin{aligned}
B_x &= \frac{e^{-k(r-x)} mz\left(3xr + k\left(3xr^2 + r^3\right) + k^2\left(xr^3 - r^4\right)\right)}{k^{3/2} r^6} \\
B_y &= \frac{e^{-k(r-x)} myz\left(3kr^2 + r\left(3 + k^2 r^2\right)\right)}{k^{3/2} r^6} \\
B_z &= \frac{e^{-k(r-x)} m\left(\left(3z^2 - r^2\right) r + kr^2\left(3z^2 - r^2\right)\right)}{k^{3/2} r^6} + \frac{e^{-k(r-x)} mk^2 z^2 r^3}{k^{3/2} r^6}
\end{aligned}
\tag{3. 30}
$$

It can be shown by means of direct calculation that this magnetic field satisfies Maxwell's equations and the diffusion equation (2. 12). Nevertheless, from the analytic expression is difficult to see the form that the field possesses, therefore it is convenient to observe a graphic of the magnetic field in the space. For doing this we should keep in mind that constant k contains the information of the solar wind, *i.e.* velocity and conductivity.

To see the influence of solar wind upon the magnetic field two graphics in the plane $y = 0$, $m = 1d$ are shown: figure 1a shows the case of $k = 0,005\ u^{-1}$ and figure 1b shows $k = 0,01\ u^{-1}$, where u is an adequate unit of length and d is an adequate unit for m.

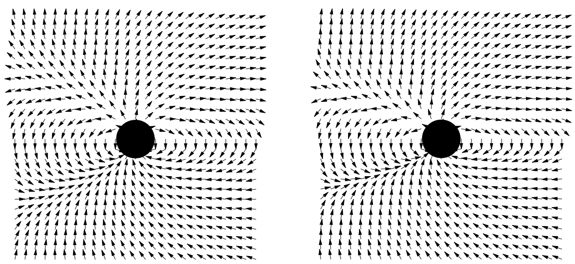

FIGURE 1. Terrestrial magnetic field modified by the presence of the solar wind; the solar wind comes from left to right. We have taken (a) $k = 0,005$ u^{-1} and $m = 1d$. (b) $k = 0,01\ u^{-1}$ and $m = 1d$.

Figures 1a and 1b show that the terrestrial magnetic field is dragged by the solar wind in the direction of its movement, compressing the field lines in the face of earth directed towards the sun.

In the other hand, as the magnetic field into magnetosphere is known, the current of charged particles of the solar wind travelling by the magnetosphere can be calculated using Ampere's law

$$\vec{j} = \frac{c}{4\pi}\nabla \times \vec{B}. \tag{3. 31}$$

and the current is

$$\begin{aligned} j_x &= 0 \\ j_y &= \frac{cme^{k(x-r)}\left(\left(x^2+y^2\right)\left(1-k^2z^2+kr\right)-z^2\left(2+k^2z^2+2kr\right)\right)}{2\pi\sqrt{k}r^5} \\ j_z &= \frac{cmyze^{k(x-r)}\left(3+3kr+k^2r^2\right)}{2\pi\sqrt{k}r^5}. \end{aligned} \tag{3. 32}$$

Figures $2a$ and $2b$ show the current in $2D$, using vectors to visualize its direction, just as we did with the magnetic field. It is clear from (2. 8) that the current is perpendicular to the velocity of the solar wind, this means that it does not possess a component in the $\hat{x}$ direction, by this reason we draw the components y and z. The values of k and m are the same as those for figure No. 1a; $m = 1d$, $k = 0,005\ u^{-1}$.

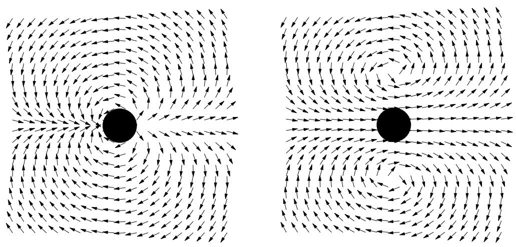

FIGURE 2. Induced current into the solar wind by the presence of the terrestrial magnetic field, the solar wind coming out of the page. We have taken (a) In the plane $x = 0$, $k = 0,005\ u^{-1}$ and $m = 1d$. (b) In the plane $x = 500$, $k = 0,005\ u^{-1}$ and $m = 1d$.

In (2. 13) we introduce the differential identity:

$$\frac{d}{dt} = (\vec{v}\cdot\nabla) + \frac{\partial}{\partial t}. \tag{3. 33}$$

Given that all temporary partial derivatives are zero, the equation (2. 13) can be written as

$$\rho(\vec{v}\cdot\nabla)\vec{v} = -\nabla P + \frac{1}{c}\vec{j}\times\vec{B} + \eta\nabla^2\vec{v}. \tag{3. 34}$$

The equation of movement (3. 34), according to our procedure should not be satisfied, due to the fact that it was taken a constant velocity in $\hat{x}$ direction for the solar wind. We must remind that (3. 34) tells us about the velocity of the fluid in all space, while that of our model is valid just far away of the planet. By this reason, in the equation of movement (3. 34) we cannot consider a constant velocity, since it would be a meaningless equation. It is convenient to consider small variations in the velocities, small enough to preserve the condition that far away from the planet velocities are almost constant. Later we will see that this consideration is correct as the magnetic force vanishes quickly if we move far away from Earth. It is difficult to find a solution of (3. 34), however we can center in the form of the force that the terrestrial magnetic field has upon the solar wind, for this we will consider the Lorentz's force:

$$\vec{F} = \frac{1}{c}\vec{j} \times \vec{B}. \tag{3. 35}$$

Substituting in (3. 35) the values of the magnetic field and of the current induced upon the solar wind, we obtain the force of the magnetic field of Earth upon the solar wind. In figure 3 we show a vector diagram of the forces in the plane $y = 0$. In this figure the effect of the magnetic field upon the fluid can be observed, trying to move it toward the "poles". In the $\hat{x}$ axis, which is the direction of the incident solar wind, the field tends to turn back the fluid, this force generates a strong pressure that, in a better model, would modify the velocity of the fluid making it to go around the magnetosphere, protecting the surface of the planet from high velocity ionized gases. In the same direction, the $\hat{x}$ axis, when the fluid has passed the planet it tends to brake but the braking force is not strong enough. To see the behavior of this forces as a function of the distance to the source, figure 4 shows the magnitude of forces, $|\vec{F}|$, using the same value of distance as in figure 1a. From this figure results that the force between the source and the fluid to those distances vanishes. Figures 1a and 1b have an interval in x and in y of $(-500u,\ 500u)$, while the magnitude of the force is obtained to be relevant in the distance interval $(-50u,\ 50u)$, still inside the planet as the circle representing Earth possesses a radius of $80u$. The force has been taken in adequate units of f.

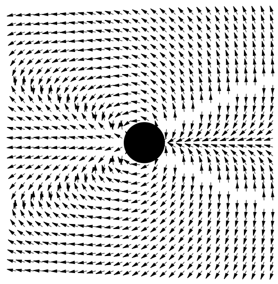

Figure 3. Force that does terrestrial magnetic field upon solar wind in the plane $y = 0$; the solar wind comes from left to right. We have taken $k = 0,005$ u^{-1} and $m = 1$.

Given the great velocity of incoming solar wind, far away from Earth it does not experience any deviation.

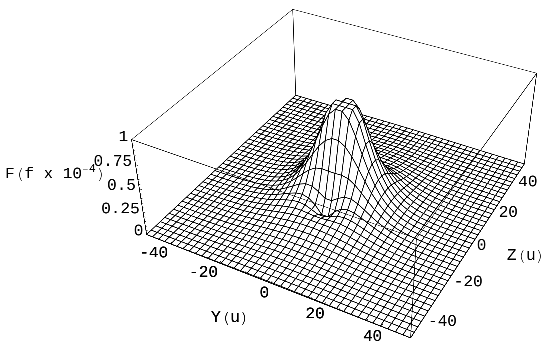

Figure 4. Magnitude of the force that does the terrestrial magnetic field upon the solar wind in the plane $x = 20u$; the solar wind comes upper downward. We have taken $k = 0,005$ u^{-1} and $m = 1d$.

4 Conclusions

We have shown that the terrestrial magnetosphere is affected by the solar wind, suffering a lengthening of the magnetic field lines. On the other hand, due to the presence of the terrestrial magnetic field, certain currents arise inside the solar wind, generating an additional magnetic field which superposes to the dipole magnetic field, creating the total field in the magnetosphere.

We have shown in this model that the force exerted by the terrestrial magnetic field upon the solar wind vanishes very quickly with distance, when we move away from Earth. As a consequence, our model is valid for determinate the magnetic field far from the Earth.

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