ORDER OF THE CHIRAL PHASE TRANSITION IN THE σ MODELS

F. A. Correa[1] y J. Morales[1,2]

[1] Departamento de Física, Universidad Nacional, Bogotá, Colombia.
[2] Centro Internacional de Física, Bogotá, Colombia.

Abstract

The *chiral phase transition* at a certain *critical temperature* is a restoration mechanism of the chiral symmetry, broken by the nonzero mass of quarks and mesons. The transition can be studied through several models, among which are the σ models. An analysis is made on the linear and nonlinear σ models with different approximations, and we show that the transition is of second order in both cases.

Keywords: Chiral phase transition, σ model

Resumen

La *transición en la fase quiral* a una cierta *temperatura crítica* es una restauración del mecanismo de la simetría quiral, rota debido a la masa finita de los quarks y los mesones. La transición puede ser estudiada a través de varios modelos, entre los cuales estan los modelos σ. Se hace un análisis sobre la base de los modelos σ lineales y no lineales con diferentes aproximaciones, y se muestra que la transición es de segundo orden en ambos casos.

Palabras claves: Transición de fase chiral, modelo σ.

Fabio Correa: e-mail: facorread@unal.edu.co

1 Introduction

A *chiral transformation* is a rotation in the *isospin* space of quarks. When a quark system is invariant under such a transformation, we say that the system is chirally symmetric. This is true in the case of massless quarks, but we know that actually they have mass. So with regard to the light quarks only (*up* and *down*), we can say that the chiral symmetry is an approximated symmetry of the strong interactions [9]. Nonzero mass of the corresponding mesons also breaks the symmetry.

There are two kinds of chiral transformations. The partially conserved Noether current is a vector in one case, and an axial vector in the other case. When working with the up and down quarks, the transformations can be studied through the group SU(2)×SU(2), which is isomorphic to O(4). This isomorphism is the reason for the Heisenberg magnet model being the unique description of the low energy dynamics of the QCD.

The σ models arise in this context. The σ model has the usual kinetic term and the potential

$$\frac{\lambda}{4}\left(\mathbf{\Phi}^2 - f_\pi^2\right)^2, \tag{1}$$

with a field $\mathbf{\Phi}$ having N components, where λ is a positive coupling constant, and f_π is the pion decay constant; this model is renormalizable. In the limit that $\lambda \to \infty$ the potential goes over to a δ-function constraint on the length of the field vector. This is the so-called nonlinear σ model. Both models describe a quark system in which the chiral symmetry is broken, in consistency with the nonzero mass of the light quarks and/or the light mesons.

If quarks are massless, QCD is expected to undergo a *chiral phase transition*, which may have implications for high energy nucleus-nucleus collisions. A pion gas with a σ meson undergoes the chiral phase transition. In this work, we use both the σ models to describe it and derive its pressure. The linear σ model is studied through a mean field approximation, while the nonlinear σ model is studied using chiral perturbation theory. The first and second term of the perturbation series are considered. In both studies, the transition

is of second-order and the critical temperature is the same, as we will show later.
The developments towards the calculation of the critical temperature follow closely those of Bochkarev and Kapusta[1].

2 Linear sigma model

The linear σ model Lagrangian is

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \mathbf{\Phi}^2\right)^2 - \frac{\lambda}{4}\left(\mathbf{\Phi}^2 - f_\pi^2\right)^2, \tag{2}$$

where λ is a positive coupling constant. The bosonic field $\mathbf{\Phi}$ has N components. Rather arbitrarily, we define that they are $N-1$ π fields and a σ field as the Nth component. The space spanned by these components has the usual Euclidean metric. Since the σ has the quantum number of the vacuum, and since the symmetry is broken at low temperatures, we immediately allow for a σ condensate v whose value depends on the temperature $1/\beta$ and is yet to be determined.
We write

$$\Phi_i(\mathbf{x},t) = \pi_i(\mathbf{x},t), \qquad i = 1,\ldots,N-1; \tag{3}$$

$$\Phi_N(\mathbf{x},t) = v + \sigma(\mathbf{x},t). \tag{4}$$

In terms of these fields the Lagrangian becomes [2,3]

$$\begin{aligned}\mathscr{L} = {} & \tfrac{1}{2}\left(\partial_\mu\pi\right)^2 + \tfrac{1}{2}\left(\partial_\mu\sigma\right)^2 - \tfrac{\lambda}{4}\left(v^2 - f_\pi^2\right)^2 \\ & - \tfrac{\lambda}{2}\left(v^2 - f_\pi^2\right)\left(2v\sigma + \sigma^2 + \pi^2\right) - \tfrac{\lambda}{4}\left(2v\sigma + \sigma^2 + \pi^2\right)^2\end{aligned} \tag{5}$$

We define the effective masses $\begin{cases}\bar{m}_\pi^2 = \lambda\left(v^2 - f_\pi^2\right) \\ \bar{m}_\sigma^2 = \lambda\left(3v^2 - f_\pi^2\right).\end{cases}$

Reorganizing terms, we get

$$\begin{aligned}\mathscr{L} = {} & -\tfrac{\lambda}{4}\left(f_\pi^2 - v^2\right)^2 + \tfrac{1}{2}\left[\left(\partial_\mu\pi\right)^2 - \bar{m}_\pi^2\pi^2 + \left(\partial_\mu\sigma\right)^2 - \bar{m}_\sigma^2\sigma^2\right] \\ & - \lambda v\left(v^2 - f_\pi^2\right)\sigma - \lambda v\sigma\left(\sigma^2 + \pi^2\right) - \tfrac{\lambda}{4}\left(\sigma^2 + \pi^2\right)^2\end{aligned} \tag{6}$$

At zero temperature, the potential is minimized when $v = f_\pi$. In this particular case,

$$\bar{m}_\pi^2 = 0, \qquad \bar{m}_\sigma^2 = \lambda\left(3v^2 - f_\pi^2\right) = 2\lambda f_\pi^2, \tag{7}$$

so the Goldstone theorem is satisfied.

The action at finite temperature is obtained by rotating to imaginary time, $\tau = it$, and integrating τ from 0 to $\beta = 1/T$. We keep the Minkowsi metric: $\partial_\mu = \partial/\partial x^\mu$ with $\partial_0 = \partial/\partial t = i\partial/\partial\tau$. Then the action is rewritten as

$$\begin{aligned} S = -\tfrac{\lambda}{4}\left(f_\pi^2 - v^2\right)^2 \beta V + \int \mathrm{d}^4x \Bigg(\tfrac{1}{2}\Big[(\partial_\mu\pi)^2 - \bar{m}_\pi^2\pi^2 \\ + (\partial_\mu\sigma)^2 - \bar{m}_\sigma^2\sigma^2\Big] - \lambda v\left(v^2 - f_\pi^2\right)\sigma \\ - \lambda v\sigma\left(\sigma^2 + \pi^2\right) - \tfrac{\lambda}{4}\left(\sigma^2 + \pi^2\right)^2\Bigg), \end{aligned} \tag{8}$$

where the abbreviation

$$\int \mathrm{d}^4x \equiv \int_0^\beta \mathrm{d}\tau \int_V \mathrm{d}^3x \tag{9}$$

is used.

At any temperature v is choosen such that $\langle\sigma\rangle = 0$. This eliminates any one-particle reducible diagrams in perturbation theory, leaving only one-particle irreducible (1PI) diagrams. We will allow for v to be temperature dependent. The simplest approximation at finite temperature is the mean field approximation; in this, we will neglect interactions among the particles or collective excitations.

The pressure of a free relativistic boson gas can be written in several ways:

$$P_0(T,m) = -\frac{1}{\beta}\int \frac{\mathrm{d}^3k}{(2\pi)^3} \ln\left(1 - e^{-\beta\omega}\right) \tag{10}$$

$$= \int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{k^2}{3\omega}\frac{1}{e^{\beta\omega} - 1}. \tag{11}$$

The pressure of the pion gas is given by

$$P = \frac{\mathrm{d}}{\mathrm{d}V}\frac{1}{\beta}\ln Z(\beta) = -\tfrac{\lambda}{4}\left(f_\pi^2 - v^2\right)^2 + VP_0(T, m_\sigma) + V(N-1)P_0(T, m_\pi) \quad (12)$$

The mean field approximation thus calculated has been a very simple, very good approximation, allowing to analyze lots of phenomena in finite temperature field theory. It was used in all the pioneering papers [4-6].

2.1 Second-order phase transition

We expect that the temperature rise tends to disorder the condensate until it dissapears. In the linear σ model we analyze this phenomena by expanding the free boson gas pressure about zero mass:

$$P_0(T, m) = \frac{\pi^2}{90}T^4 - \frac{m^2T^2}{24} + \frac{m^3T}{12\pi} + \dots \quad (13)$$

The efective masses are proportional to the square root of λ, and terms with m^3 or $\lambda^{3/2}$ are not part of the mean field approximation, so we neglect the last term of the expansion (13). The pion gas pressure becomes

$$P(T, v) \approx N\frac{\pi^2}{90}T^4 - \frac{\lambda}{2}v^2\left(f_\pi^2 - \frac{N+2}{12}T^2\right) - \frac{\lambda}{4}\left(v^4 + f_\pi^4 - \frac{N}{6}T^2 f_\pi^2\right) \quad (14)$$

Now we maximize the pressure with respect to v:

$$\frac{\mathrm{d}P}{\mathrm{d}v} = \lambda v\left(f_\pi^2 - \frac{N+2}{12}T^2\right) - \lambda v^3 = 0 \quad (15)$$

$$\therefore v^2 = \begin{cases} f_\pi^2 - \frac{N+2}{12}T^2 \\ 0 \end{cases} \quad (16)$$

The pressure must be a maximum because we expect the pion gas to be in a stable or metastable state. We verify the value of the second derivative for both solutions.

$$\frac{\mathrm{d}^2 P}{\mathrm{d}v^2} = \lambda\left(f_\pi^2 - \frac{N+2}{12}T^2\right) - 2\lambda v^2 \tag{17}$$

$$= \begin{cases} -\lambda v^2, \\ \lambda\left(f_\pi^2 - \frac{N+2}{12}T^2\right), \end{cases} \tag{18}$$

with respect to the cases of Eq. (16); the first case represents a maximum.
The condensate goes to zero at a critical temperature given by

$$T_c^2 = \frac{12}{N+2} f_\pi^2. \tag{19}$$

Above this temperature the condensate is zero, because thermal fluctuations are very large.
We evaluate Eq. (18) again:

$$\frac{\mathrm{d}^2 P}{\mathrm{d}v^2} = \begin{cases} -\lambda v^2, & v^2 = \frac{N+2}{12}\left(T_c^2 - T^2\right) \\ \lambda \frac{N+2}{12}\left(T_c^2 - T^2\right), & v^2 = 0. \end{cases} \tag{20}$$

In the second case the pressure is a minimum at $T < T_c$, so we show this result here [6-8].
Now we show that this is a second-order phase transition, calculating the pressure under the two temperature regimes.

$$P_<(T) = N\frac{\pi^2}{90}T^4 + \frac{\lambda}{2}v^2\frac{N+2}{12}\left(T_c^2 - T^2\right) - \frac{\lambda}{4}v^4 + \frac{\lambda}{4}f_\pi^2\left(\frac{N}{6}T^2 - f_\pi^2\right) \tag{21}$$

$$P_>(T) = N\frac{\pi^2}{90}T^4 + \frac{\lambda N}{48}f_\pi^2\left(2T^2 - T_c^2\right) - \frac{\lambda}{24}f_\pi^2 T_c^2 \tag{22}$$

Evaluating the pressure in the critical temperature,

$$P_<(T) = N\frac{\pi^2}{90}T_c^4 + \frac{\lambda N}{48}f_\pi^2 T_c^2 - \frac{\lambda}{24}f_\pi^2 T_c^2 = N\frac{\pi^2}{90}T_c^4 - \frac{\lambda N}{48}f_\pi^2 T_c^2 = P_>(T) \tag{23}$$

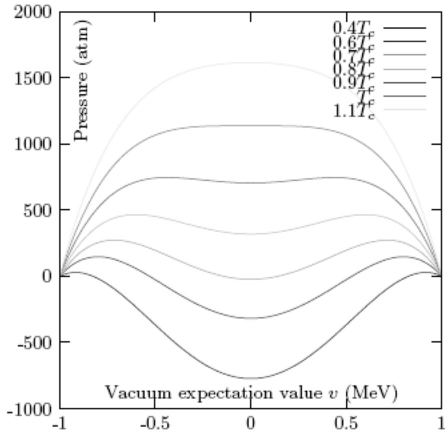

FIGURE 1. Above the critical temperature, there is only one pressure maximum at $\nu = 0$. Below T_C, there is a real maximum at $\nu \neq 0$

Evaluating the first derivative,

$$P'_<(T) = N\frac{2\pi^2}{45}T^3 + \lambda\left(\tfrac{N+2}{12}\right)^2\left(T^2 - T_c^2\right)T + \frac{\lambda N}{12}f_\pi^2 T$$

$$P'_>(T) = N\frac{2\pi^2}{45}T^3 + \frac{\lambda N}{12}f_\pi^2 T \tag{24}$$

$$P'_<(T_c) = N\frac{2\pi^2}{45}T_c^3 + \frac{\lambda N}{12}f_\pi^2 T_c = P'_>(T_c). \tag{25}$$

We just have shown that the pressure and its first derivative are continuous at the critical temperature; these tell us that the transition is neither zero- or first-order. Now, by showing that the second derivative is discontinuous, we come to the conclusion that the transition is of second order:

$$P''_<(T) = N\frac{2\pi^2}{15}T^2 + \lambda\left(\tfrac{N+2}{12}\right)^2\left(3T^2 - T_c^2\right) + \frac{\lambda N}{12}f_\pi^2$$

$$P''_>(T) = N\frac{2\pi^2}{15}T^2 + \frac{\lambda N}{12}f_\pi^2 \tag{26}$$

$$\begin{aligned} P''_<(T_c) &= N\frac{2\pi^2}{15}T_c^2 + \lambda\left(\tfrac{N+2}{12}\right)^2 2T^2 + \frac{\lambda N}{12}f_\pi^2 \\ &= P''_>(T_c) + \lambda\left(\tfrac{N+2}{12}\right)^2 2T^2. \end{aligned} \tag{27}$$

3 Nonlinear sigma model

This model is defined by the Lagrangian

$$\mathscr{L} = \tfrac{1}{2}\left(\partial_\mu \mathbf{\Phi}\right)^2 , \qquad (28)$$

together with the constraint

$$f_\pi^2 = \Phi^2(x,t) \qquad (29)$$

The length of the chiral field is fixed and cannot be changed by thermal fluctuations, so we say that chiral symmetry is built into this model and, therefore, there can be no chiral-symmetry restoring phase transition. Taking the limit $\lambda \to \infty$ constraints the length of the chiral field to be f_π just as in the nonlinear model [9,10]. The critical temperature, however, is independent of λ at least in the mean field approximation. So it would seem that the phase transition survives. If it is true, then we ought to be able to derive it entirely within the context of the nonlinear σ model. That is what we shall do, although it involves a lot more effort than the treatment of the linear model. Since the only parameter in the model is f_π, and we are interested in temperatures comparable to it, we cannot do an expansion in powers of T/f_π. The only parameter is N, the number of field components. This suggests an expansion in powers of $1/N$.
We begin by representing the field-constraining δ function by an integral.

$$Z = \int \mathscr{D}\Phi \mathscr{D}b' \exp \int \mathrm{d}^4x \,\left(\mathscr{L} + ib'\left(\Phi^2 - f_\pi^2\right)\right). \qquad (30)$$

The zero frequency, zero momentum condensate v is allowed. Following Polyakov, we also separate the zero frequency, zero momentum mode of the auxiliary field b'.
Integrating over all the other modes will give us an effective action involving the constant part of the fields. We will then minimize the free energy with respect to these constant parts, which is a saddle point approximation. Integrating over fluctuations about

the saddle point is a finite volume correction and of no consequence in the thermodynamic limit. The Fourier expansions are

$$\Phi_i(\mathbf{x},\tau) = \pi_i(\mathbf{x},\tau) = \sqrt{\frac{\beta}{V}} \sum_{\mathbf{p},n} e^{i(\mathbf{x}\cdot\mathbf{p}+\omega_n\tau)} \tilde{\pi}_i(\mathbf{p},n), \tag{31}$$

$$\Phi_N(\mathbf{x},\tau) = v + \sigma(\mathbf{x},\tau) \tag{32}$$

$$= v + \sqrt{\frac{\beta}{V}} \sum \mathbf{p}, n e^{i(\mathbf{x}\cdot\mathbf{p}+\omega_n\tau)} \tilde{\sigma}(\mathbf{p},n), \tag{33}$$

$$b'(\mathbf{x},\tau) = i\frac{m^2}{2} + b(\mathbf{x},\tau) \tag{34}$$

$$= i\frac{m^2}{2} + T\frac{\beta}{V} \sum_{\mathbf{p},n} e^{i(\mathbf{x}\cdot\mathbf{p}+\nu_n\tau)} \tilde{b}(\mathbf{p},n), \tag{35}$$

where we define the abbreviation

$$\sum_{\mathbf{p},n} \equiv \sum n \int \mathrm{d}^3 p (2\pi)^3. \tag{36}$$

The zero frequency and zero momentum modes have been excluded from the summations. The field Φ must be periodic in imaginary time for the usual reasons but there is no such requirement on b, hence $\omega_n = 2\pi nT$ and $\nu_n = \pi nT$. Since b has dimensions of inverse length squared we inserted another factor of T so as to make its Fourier amplitude dimensionless, as it is for the other fields. From (30), the action becomes

$$S = \int \mathrm{d}^4 x \left(\tfrac{1}{2} \left[(\partial_\mu \pi)^2 - m^2\pi^2 + (\partial_\mu \sigma)^2 - m^2\sigma^2 \right] \right.$$
$$\left. - ib(2v\sigma + \pi^2 + \sigma^2) \right) + \tfrac{1}{2} m^2 (f_\pi^2 - v^2)\beta V. \tag{37}$$

Please note that terms linear in the fields have been integrated to zero because

$$\langle \pi_i \rangle = \langle \sigma \rangle = \langle b \rangle = 0. \tag{38}$$

An effective action is derived by expanding e^S in powers of b. The term linear in b vanishes n account of $\tilde{b}(0,0) \propto \langle b \rangle = 0$. The term

proportional to b^2 is not zero and is exponentiated, thus summing a whole series of contributions. The term proportional to b^3 is not zero either and may also be exponentiated, summing an infinite series of higher-order terms left out of the order b^2 exponentiation. After the expansion is made, we perform a path integration over the pion and σ fields, only on the term quadratic in b^2. At last, the terms are scaled: $b \to b/\sqrt{2N}$.
We have

$$\begin{aligned} S_E = &-\frac{1}{2}\sum_{\mathbf{p},n}\left(\omega_j^2+\mathbf{p}^2+m^2\right)\Big[\tilde{\pi}(\mathbf{p},n)\cdot\tilde{\pi}(-\mathbf{p},-n) \\ &+\sigma(\mathbf{p},n)\sigma(-\mathbf{p},-n)\Big]-\frac{1}{2}\sum_{\mathbf{p},n}\Big[\Pi(\mathbf{p},\omega_n,T,m) \\ &+\frac{2}{N}\frac{v^2}{\omega_n^2+p^2+m^2}\Big]\tilde{b}(\mathbf{p},2n)\tilde{b}(-\mathbf{p},-2n) \\ &+\tfrac{1}{2}m^2(f_\pi^2-v^2)\beta V+\mathscr{O}(b^3/\sqrt{N}). \end{aligned} \tag{39}$$

The effective action is an infitine series in b. The coefficients are frequency and momentum dependent, arising from one-loop diagrams. In addition, each successive term is supressed by $1/\sqrt{N}$ compared to the previous one. This is the large N expansion.
The propagators for the pion and σ fields are of the usual form

$$\mathsf{D}_\phi^{-1}(\mathbf{p},\omega_n,m)=\omega_n^2+\mathbf{p}+m^2 \tag{40}$$

and the propagator for b is

$$\mathsf{D}_b^{-1}(\mathbf{p},\omega_n,m)=\Pi(\mathbf{p},\omega_n,T,m)+\frac{2}{N}\frac{v^2}{\omega_n^2+\mathbf{p}^2+m^2}, \tag{41}$$

with m and v to be determined. There appears the one-loop function

$$\begin{aligned} \Pi(\mathbf{p},\omega_n,T,m) = T\sum_i\int\frac{\mathrm{d}^3x}{(2\pi)^3}&\big((\omega_n-\omega_l)^2 \\ &+(\mathbf{p}-k)^2+m^2\big)^{-1}\big(\omega_l^2+k^2+m^2\big)^{-1}\big) \end{aligned} \tag{42}$$

Keeping only the terms up to order b^2 in S_E (the rest vanishes when $\to \infty$) allows us to obtain an explicit expression for the partition function and the pressure. This includes the next-to-leading order in N:

$$P = \frac{T}{V} \ln Z = \frac{1}{2} m^2 \left(f_\pi^2 - v^2 \right) - \frac{N}{2} T \sum_{\mathbf{p},n} \ln \left(\beta^2 (\omega_n^2 + p^2 + m^2) \right)$$
$$- \frac{T}{2} \sum_{\mathbf{p},n} \ln \left[\Pi(\mathbf{p}, \omega_n, T, m) + \frac{2}{N} \frac{v^2}{\omega_n^2 + p^2 + m^2} \right] \qquad (43)$$

The second term under the last logarithm should and will be set to zero at this order. It may be needed at higher order in the $1/N$ theory to regulate infrared divergences.
The pressure is extremized with respect to m^2:

$$\frac{\partial P}{\partial m^2} = \frac{T}{V} \frac{1}{2} \left\langle -\pi^2 - \sigma^2 + f_\pi^2 - v^2 \right\rangle = - \left\langle \pi^2 \right\rangle - \left\langle \sigma^2 \right\rangle + f_\pi^2 - v^2 = 0 \qquad (44)$$

This condition is equivalent to the thermal average of the constraint:

$$f_\pi^2 = v^2 + \left\langle \pi^2 \right\rangle + \left\langle \sigma^2 \right\rangle = \left\langle \mathbf{\Phi}^2 \right\rangle . \qquad (45)$$

If an approximation to the exact partition function is made, such as the large N expansion, this constraint should still be satisfied. In fact, it will isolate a preferred value for m.
To leading order in N we may neglect the term involving Π entirely. The pressure is then

$$P = \frac{1}{2} m^2 (f_\pi^2 - v^2) + N P_0(T, m). \qquad (46)$$

This must be a maximum with respect to v:

$$\frac{\partial P}{\partial v} = -m^2 v = 0, \qquad (47)$$

which is equivalent to the condition $\langle \sigma \rangle = 0$ by means of (35). There are two possibilities:

1. $m = 0$: There are massless particles, or Goldstone bosons, and the value of the condensate v is determined by the thermally averaged constraint. This is the *asymmetric* phase.

2. $v = 0$: The thermally averaged constraint (45) is satisfied by a nonzero T-dependent mass. There are no Goldstone bosons. This is the *symmetric*, or *symmetry-restored* phase.

Evidently, there is a chiral symmetry-restoring phase transition.
In the leading order of the $1/N$ approximation, the particles are represented by free fields with a potentially T-dependent mass m. For any free bosonic field ϕ,

$$\frac{\partial P_0}{\partial m^2}(T,m) = \left\langle \mathbf{\Phi}^2 \right\rangle = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\omega}\frac{1}{e^{\beta\omega}-1}, \tag{48}$$

with $\omega^2 = p^2 + m^2$. Thus, extremizing the pressure with respect to m^2 is equivalent to satisfying the thermally averaged constraint:

$$\begin{aligned} &\frac{\partial}{\partial m^2}\left(-\frac{1}{2}T\sum_{\mathbf{p},n}\ln\left(\beta(w_n^2+p^2+m^2)\right)\right) \\ &\quad = -\frac{1}{2}T\int\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\beta}{2\omega}\left(1+2\frac{1}{e^{\beta\omega}-1}\right) \end{aligned} \tag{49}$$

We renormalize the expression:

$$= -\frac{1}{2}T\int\frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{\beta}{\omega}\frac{1}{e^{\beta\omega}-1} = -\frac{1}{2}\left\langle \mathbf{\Phi}^2 \right\rangle \tag{50}$$

Then we have

$$\frac{\partial P_0}{\partial m^2}(T,m) = -\frac{1}{2}\left\langle \mathbf{\Phi}^2 \right\rangle \tag{51}$$

$$\therefore \frac{\partial P}{\partial m^2} = \frac{1}{2}\left(f_\pi^2 - v^2\right) - \frac{N}{2}\left\langle \mathbf{\Phi}^2 \right\rangle = 0 \tag{52}$$

$$\therefore f_\pi^2 = v^2 + \left\langle \pi^2 \right\rangle + \left\langle \sigma^2 \right\rangle = \left\langle \mathbf{\Phi}^2 \right\rangle \tag{53}$$

We consider now the two different phases:

- The mass is zero in the *asymmetric* phase. The constraint is satisfied by a temperature-dependent condensate:

$$v^2(T) = f_\pi^2 - \frac{NT^2}{12} \tag{54}$$

This condensate goes to zero at a critical temperature of

$$T_C^2 = \frac{12}{N} f_\pi^2. \tag{55}$$

At exactly T_C, the thermally averaged constraint is satisfied by the fluctuations of N massless degrees of freedom without the help of a condensate.

- In the symmetric phase the condensate is zero. The constraint is satisfied by thermal fluctuations alone:

$$f_\pi^2 = N \left\langle \mathbf{\Phi}^2 \right\rangle = N \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\omega} \frac{1}{e^{\beta\omega} - 1}. \tag{56}$$

At fixed T, thermal fluctuations decrease with increasing mass. The constraint is only satisfied by massless excitations at *one single* temperature, namely T_C. At temperatures $T > T_C$ the mass must be greater than zero. Near T_C the mass should be small, and the fluctuations may be expanded about $m = 0$ as

$$f_\pi^2 \approx NT^2 \left[\frac{1}{12} - \frac{m}{4\pi T} - \frac{m^2}{8\pi^2 T^2} \ln \frac{m}{4\pi T} - \frac{m^2}{16\pi^2 T^2} \right] \tag{57}$$

$$T_C^2 = \frac{12}{N} f_\pi^2 = 12T^2 \left(\frac{1}{12} - \frac{m}{4\pi T} - \ldots \right) = T^2 - \frac{3mT}{\pi} \tag{58}$$

As T approaches T_C from above, the mass approaches zero like

$$m = \frac{\pi}{3T} \left(T^2 - T_C^2 \right). \tag{59}$$

This is a second-order phase transition since there is no possibility of metastable supercooled or superheated states. We will evaluate the order of this transition in the cases

a) $m = 0$,

b) $v = 0$.

Function evaluation Given $m \gg 1$ in the neighborhood of T_C we have for any T, and in particular, for $T = T_C$,

$$P_<(T) = NP_0(T,0) = N\frac{\pi^2}{90}T^4 \tag{60}$$

$$P_>(T) = \frac{1}{2}\frac{\pi^2}{9T^2}\left(T^2 - T_C^2\right)^2 f_\pi^2 + NP_0\left(T, \frac{\pi}{3T}\left(T^2 - T_C^2\right)\right) \tag{61}$$

$$= \frac{N}{24}\frac{\pi^2}{9}\left(T^2 - T_C^2\right)^2\left(\frac{T_C}{T}\right)^2 + N\left(\frac{\pi^2}{90}T^4 - \frac{T^2}{24}\frac{1}{2}\frac{\pi^2}{9T^2}\left(T^2 - T_C^2\right)^2\right) \tag{62}$$

$$= N\frac{\pi^2}{90}T^4 + \frac{N}{24}\frac{\pi^2}{9}\left(T^2 - T_C^2\right)^2\left(\frac{T_C}{T}\right)^2 - \frac{1}{2}\frac{N}{24}\frac{\pi^2}{9}\left(T^2 - T_C^2\right)^2 \tag{63}$$

First derivative evaluation

$$P'_>(T_C) = N\frac{\pi^2}{90}4T_C^3 = P'_<(T_C) \tag{64}$$

Second derivative evaluation

$$P''_>(T_C) = N\frac{\pi^2}{90}12T_c^2 - 4\frac{N}{24}\frac{\pi^2}{9}\left(3T_C^2 - T_C^2\right) \tag{65}$$

$$= P''_<(T_C) - \frac{N}{3}\frac{\pi^2}{9}T_C^2 \tag{66}$$

Again, we have continuity in the pressure and its first derivative, but there is a discontinuity in the second derivative: a second-order phase transition.

The mass must grow faster than the temperature at very high temperatures in order to keep the field fluctuations fixed and equal to f_π^2. Asymptotically, the particles move nonrelativistically. This allows us to compute the fluctuations analytically. We get

$$f_\pi^2 = N\left(\frac{T}{2\pi}\right)^{3/2}\sqrt{m}e^{-m/T} \tag{67}$$

This is a trascendental equation for m. It can also be written as

$$m = T \ln \left(\frac{NT}{2\pi f_\pi} \sqrt{\frac{mT}{2\pi f_\pi^2}} \right) \sim T \ln \frac{T^2}{T_C^2} \tag{68}$$

It is rather amusing that, in leading order of the large N approximation, the elementary excitations are massless below T_C, become massive above T_C, and at asymptotically high temperatures move nonrelativistically.
The result to first order in $1/N$ provides a good insight into the nature of the two-phase structure of the nonlinear σ model, but it is not quite satisfactory for two reasons:

1. It predicts N massless Goldstone Bosons in the broken symmetry phase when we know there ought to be $N-1$.
2. In this model, $T_C^2 = 12 f_\pi^2/N$, and in the linear σ model it is $T_C^2 = 12 f_\pi^2/(N+2)$. We expect them to be the same in the limit $\lambda \to \infty$.

Both these problems can be rectified by inclusion of the second order in $1/N$; this is, the inclusion of the b field.
It is natural to expect that the b field will contribute essentially one negative degree of freedom to the T^4 term in the pressure so as to give $N-1$ Goldstone bosons in the low temperature phase. Therefore, we move one of the N degrees of freedom and put it together with the b contribution.

$$\begin{aligned} P = \frac{T}{V} \ln Z = \frac{1}{2} m^2 \left(f_\pi^2 - v^2 \right) \\ - \frac{N-1}{2} T \sum_{\mathbf{p},n} \ln \left(\beta^2 (\omega_n^2 + p^2 + m^2) \right) \\ - \frac{T}{2} \sum_{\mathbf{p},n} \ln \left[\beta^2 (\omega_n^2 + p^2 + m^2) \Pi(\mathbf{p}, \omega_n, T, m) \right] \end{aligned} \tag{69}$$

After the procedure described by Bochkarev and Kapusta[1], we do an expansion in m/T as before.

$$P = (N-1) \frac{\pi^2}{90} T^4 - \tfrac{N+2}{24} m^2 T^2 + \tfrac{1}{2} m^2 (f_\pi^2 - v^2) + \tfrac{N}{12\pi} m^3 T \tag{70}$$

We maximize in the high temperature phase where $v = 0$:

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\tfrac{N+2}{12} m T^2 + m f_\pi^2 + \tfrac{N}{4\pi} m^2 T = 0 \tag{71}$$

$$f_\pi^2 = \tfrac{N+2}{12} T^2 - \tfrac{N}{4\pi} m T \tag{72}$$

$$= T^2 \left[\tfrac{N+2}{12} - \tfrac{N}{4\pi} \tfrac{m}{T} \right]. \tag{73}$$

This gives the same critical temperature as in the mean field treatment of the linear σ model:

$$T_C^2 = \frac{12}{N+2} f_\pi^2. \tag{74}$$

The mass approaches zero from above like

$$m(T) = \frac{\pi(N+2)}{3NT} (T^2 - T_C^2) \tag{75}$$

We will evaluate the order of this transition in this order of the χPT, in the cases

a) $m = 0$,

b) $v = 0$.

Function evaluation Given $m \gg 1$ in the neighborhood of T_C we have for any T, and in particular, for $T = T_C$,

$$P_a(T) = (N-1) P_0(T,0) = (N-1) \frac{\pi^2}{90} T^4 \tag{76}$$

$$\begin{aligned} P_b(T) = (N-1)\frac{\pi^2}{90} T^4 - \frac{\pi^2}{24} \frac{(N+2)^3}{9N^2} \left(T^2 - T_C^2\right)^2 \\ + \frac{\pi^2 (N+2)^2}{18 N^2 T^2} \left(T^2 - T_C^2\right)^2 f_\pi^2 \\ + \frac{\pi^2}{27} \frac{(N+2)^3}{12 T^2 N^2} \left(T^2 - T_C^2\right)^3 \end{aligned} \tag{77}$$

First derivative evaluation

$$P_b'(T_C) = (N-1) \frac{\pi^2}{90} 4 T_C^3 = P_a'(T_C) \tag{78}$$

Second derivative evaluation

$$P_b''(T_C) = N\frac{\pi^2}{90}12T_c^2 - \frac{\pi^2}{24}\frac{(N+2)^3}{9N^2}4\left(3T^2 - T_C^2\right) \tag{79}$$

$$= P_a''(T_C) - \frac{\pi^2}{3}\frac{(N+2)^3}{9N^2}T_C^2 \tag{80}$$

This is a second order phase transition.

4 Concluding remarks

The chiral phase transition has been studied using three approximations: mean field approximation on the linear σ model, first order of χPT, and first and second order of χPT. We have shown in the three approximations that the transition is of second order, however, the critical temperatures are not the same, as pointed out for the first order of χPT. This made the inclusion of the second order necessary.

References

[1] Alexander Bochkarev and Joseph Kapusta, Phys. Rev. D **54** (1996), no. 6, 4066-4079, available at `arXiv:hep-ph/0210398`.

[2] Claude Itzykson and Jean Bernard Zuber, *Quantum Field Theory*, First ed., International Series in Pure and Applied Physics, McGraw-Hill, Singapore, 1980.

[3] David Bailin and Alexander Love, *Introduction to Gauge Field Theory*, First ed., Graduate Student Series in Physics, IOP, London, 1986.

[4] Fl. Stancu, *Group Theory in Subnuclear Physics*, First ed., Oxford Studies in Nuclear Physics, Oxford University Press, Oxford, 1996.

[5] S. Scherer, *Introduction to Chiral Perturbation Theory*, ArXiv High Energy Physics - Phenomenology e-prints (October 2002), available at `arXiv:hep-ph/0210398`.

[6] Ashok Das, *Finite Temperature Field Theory*, World Scientific, Singapore, 1997.

[7] N. P. Landsman and C. G. van Weert, *Real and Imaginary Time Field Theory at Finite Temperature and Density*, Phys. Rept. **145** (1987), 141.

[8] Michel Le Bellac, *Thermal Field Theory*, Cambridge University Press, Cambridge, 1996.

[9] Volker Koch, *Aspects of Chiral Symmetry*, ArXiv Nuclear Physics - Theory e-prints (2004), 53, available at `arXiv:nucl-th/9706075v2`.

[10] J. F. Donoghue, E. Golowich, and Barry R. Holstein, *Dynamics of the standard model*, Vol. 2, 1992.