BASES FOR QUANTUM ALGEBRAS AND SKEW POINCARÉ-BIRKHOFF-WITT EXTENSIONS

BASES PARA ÁLGEBRAS CUÁNTICAS Y EXTENSIONES TORCIDAS DE POINCARÉ-BIRKHOFF-WITT

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Abstract

Considering quantum algebras and skew Poincaré-Birkhoff-Witt (PBW for short) extensions defined by a ring and a set of variables with relations between them, we are interesting in finding a criteria and some algorithms which allow us to decide whether an algebraic structure, defined by variables and relations between them, can be expressed as a skew PBW extension, so that the base of the structure is determined. Finally, we illustrate our treatment with examples concerning quantum physics.

Keywords: Quantum algebras, skew Poincaré-Birkhoff-Witt extensions, diamond lemma.

Resumen

Para las álgebras cuánticas y las extensiones torcidas de Poincaré-Birkhoff-Witt definidas por un anillo y un conjunto de variables con relaciones entre ellas, estamos interesados en establecer un criterio y algunos algoritmos que nos permitan decidir si una estructura algebraica, definida en términos de generadores y

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relaciones, puede expresarse como una extensión torcida de Poincaré-Birkhoff-Witt, de manera que se determine la base de la misma. Ilustramos nuestro tratamiento con diversas álgebras de la física cuántica.

Palabras clave: Álgebras cuánticas, extensiones torcidas de Poincaré-Birkhoff-Witt, lema del diamante.

Introduction

Historically, the importance of quantum algebras has been considered for several authors in the context of quantum mechanics, see [1] and [2]. For instance, in [3] it was presented a purely algebraic formulation of quantum mechanics which does not require the specification of a space of state vectors; rather, the required vector spaces can be identified as substructures in the algebra of dynamical variables (suitably extended for bosonic systems). As we can see, this formulation of quantum mechanics captures the undivided wholeness characteristic of quantum phenomena, and provides insight into their characteristic nonseparability and nonlocality. In fact, and like the authors say in [3], "the formalism we present fulfils Dirac's aim of working with the algebra of quantum mechanics alone. Furthermore, this approach addresses Dirac's interpretational difficulty, since it can be interpreted in terms of a "process" approach to quantum theory".

Now, from a philosophical point of view, it is very important the new relationships between physics and mathematics that emerge with Heisenberg's discovery of matrix mechanics and its development in the work of Born, Jordan, and Heisenberg himself. Precisely, this is the Einstein's view of "the Heisenberg method", as "a purely algebraic method of description of nature". In [4], chapter 4, it is examined the shift from geometry to algebra in quantum mechanics as a reversal of the philosophy that governed classical mechanics by grounding it mathematically in the geometrical description of the behavior of physical objects in space and time (Heisenberg's matrix mechanics abandons any attempts to develop this type of description and instead offers essentially algebraic machinery for predicting the outcomes of experiments observed in measuring instruments).

One of the fundamental objects in quantum theory is the Heisenberg algebra (see [5] for a detailed exposition of this quantum algebra). This algebra and its generalizations - *deformations* - have recently become of interest in both theoretical physics and mathematics, where it is regarded as a fundamental object and as a suitable model for checking various physical and mathematical ideas and constructions (c.f. [6–15], and others). For example, in [15] it is discussed representations of the Heisenberg relation in various mathematical structures; in [12], it is investigated the structure of two-sided ideals - a key concept in noncommutative algebra - in the q-deformed Heisenberg algebras and the relationships of this algebra with the quantum plane, and its realizations are of primary importance to studying the dynamics of a q-deformed quantum system (see [11] for an exposition of the q-deformed Heisenberg algebra and its relation with the origin of q-calculus).

Actually, and following [14], "algebraic methods have long been applied to the solution of a large number of quantum physical systems. In the last decades, quantum algebras appeared in the framework of quantum integrable one-dimensional models and have ever since been applied to many physical phenomena [...] It was found that it could be generalized leading to the concept of deformed Heisenberg algebras [16], that have been used in many areas, as nuclear physics, condensed matter, atomic physics, etc". Indeed, the algebraic approach in theoretical physics has been also considered in a possible reconciliation of the quantum mechanics with general relativity theory, where the gravity does not need to be quantized [17].

With this in mind, several families of algebras have been defined with the purpose of studying mathematical and physical properties of different algebraic systems. One of them are the skew Poincaré-Birkhoff-Witt extensions (PBW for short) introduced in [18]. These extensions have been studied in several papers ([18–28], and others), and the PhD Thesis [29], where the first author studied ring and module theoretical properties of these algebras.

Skew PBW extensions are defined by a ring and a set of variables with relations between them, (analogously to the definition of several quantum algebras, see [6, 8–10, 13, 30–32], and others). In the study of these algebras it is important to specify one basis for every one of them, since this allows us to characterize several properties with physical meaning. This can be appreciated in several works: in [33] it was considered the PBW theorem for quantized universal enveloping algebras; in [34] it was established the quantum PBW theorem for a wide class of associative algebras; in [35], it was studied the PBW bases for quantum groups using the notion of Hopf algebra, and in [36] it was considered this theorem for diffusion algebras. Following this idea, in this article we present a criteria and some algorithms which decide whether a given ring with some variables and relations can be expressed as a skew PBW extension with a basis in the sense of Definition 2.1. With this objective, our techniques used here are fairly standard and follow the same path as other text on the subject (see [37] and [29]). The results presented are new for skew PBW extensions and all they are similar to others existing in the literature (cf. [12, 15, 33, 35], and others).

The paper is organized as follows. Section 1 contains the criteria and algorithms of our treatment. Section 2 is dedicated to definition and some properties of skew PBW extensions. Section 3 presents two examples of quantum algebras which illustrate the results established in Section 1 (other examples can be found in [29]). Finally, we present some conclusions about this topic and a future work.

1. Diamond Lemma and PBW Bases

Bergman's Diamond Lemma [37] provides a general method to prove that certain sets are bases of algebras which are defined in terms of generators and relations. For instance, the Poincaré-Birkhoff-Witt theorem, which appeared at first for universal enveloping algebras of finite dimensional Lie algebras (see [30] for a detailed treatment) can be derived from it. PBW theorems have been considered several classes of commutative and noncommutative algebras (see [33–36], and others). With this in mind, and since skew PBW extensions are defined by a ring and a set of variables with relations between them (Definition 2.1), in this section we establish a criteria and some algorithms which decide whether a given ring with some variables and relations can be expressed as a skew PBW extension. This answer is obtained following the original ideas presented in [37] and the treatment developed in [29].

Definition 1.1. (i) Let X be a non-empty set and denote by $\langle X\rangle$ and $R\langle X\rangle$ the free monoid on X and the free associative R-ring on X, respectively. A subset $Q \subseteq \langle X\rangle \times R\langle X\rangle$ is called a *reduction system* for $R\langle X\rangle$. An element $\sigma = (W_\sigma, f_\sigma) \in Q$ has components W_σ a word in $\langle X\rangle$ and f_σ a polynomial in $R\langle X\rangle$. Note that every reduction system for $R\langle X\rangle$ defines a factor ring $A = R\langle X\rangle/I_Q$, with I_Q the two-sided ideal of $R\langle X\rangle$ generated by the polynomials $W_\sigma - f_\sigma$, with $\sigma \in Q$.

(ii) If σ is an element of a reduction system Q and $A, B \in \langle X\rangle$, the R-linear endomorphism $r_{A\sigma B} : R\langle X\rangle \to R\langle X\rangle$, which fixes all elements in the basis $\langle X\rangle$ different from $AW_\sigma B$ and sends this particular element to $Af_\sigma B$ is called a *reduction* for Q. If r is a reduction and $f \in R\langle X\rangle$, then f and $r(f)$ represent the same element in the R-ring $R\langle X\rangle/I_Q$. Thus, reductions may be viewed as rewriting rules in this factor ring.

(iii) A reduction $r_{A\sigma B}$ acts trivially on an element $f \in R\langle X\rangle$ if $r_{A\sigma B}(f) = f$. An element $f \in R\langle X\rangle$ is said to be *irreducible* under Q if all reductions act trivially on f. Note that the set $R\langle X\rangle_{\rm irr}$ of all irreducible elements of $R\langle X\rangle$ under Q is a left submodule of $R\langle X\rangle$.

(iv) Let f be an element of $R\langle X\rangle$. We say that f *reduces* to $g \in R\langle X\rangle$, if there is a finite sequence $r_1, \dots, r_n$ of reductions such that $g = (r_n \cdots r_1)(f)$. We will write $f \to_Q g$. A finite sequence of reductions $r_1, \dots, r_n$ is said to be *final* on f, if $(r_n \cdots r_1)(f) \in R\langle X\rangle_{\rm irr}$.

(v) An element $f \in R\langle X\rangle$ is said to be *reduction-finite,* if for every infinite sequence $r_1, r_2, \dots$ of reductions there exists some positive integer m such that r_i acts trivially on the element $(r_{i-1}\cdots r_1)(f)$, for every $i > m$. If f is reduction-finite, then any maximal sequence of reductions $r_1,\ldots,r_n$ such that r_i acts non-trivially on the element $(r_{i-1}\cdots r_1)(f)$, for $1\leq i\leq n$, will be finite. Thus, every reduction-finite element reduces to an irreducible element. We remark that the set of all reduction-finite elements of $R\langle X\rangle$ is a left submodule of $R\langle X\rangle$.

(vi) An element $f\in R\langle X\rangle$ is said to be *reduction-unique* if it is reduction-finite and if its images under all final sequences of reductions coincide. This value is denoted by $r_Q(f)$.

Proposition 1.2 ([29], Lemma 3.1.2)**.** (i) *The set $R\langle X\rangle_{\mathrm{un}}$ of reduction-unique elements of $R\langle X\rangle$ is a left submodule, and $r_Q : R\langle X\rangle_{\mathrm{un}} \to R\langle X\rangle_{\mathrm{irr}}$ becomes an R-linear map.* (ii) *If $f,g,h\in R\langle X\rangle$ are elements such that ABC is reduction-unique for all terms A,B,C occurring in respectively f,g,h, then fgh is reduction-unique. Moreover, if r is any reduction, then $fr(g)h$ is reduction-unique and $r_Q(fr(g)h) = r_Q(fgh)$.*

Proof. (i) Consider $f,g\in R\langle X\rangle_{\mathrm{un}}$, $\lambda\in R$. We know that $\lambda f + g$ is reduction-finite. Let $r_1,\ldots,r_m$ be a sequence of reductions (note that it is final on this element), and $r := r_m\cdots r_1$ for the composition. Using that f is reduction-unique, there is a finite composition of reductions r' such that $(r'r)(f) = r_Q(f)$, and in a similar way, a composition of reductions r'' such that $(r''r'r)(g) = r_Q(g)$. Since $r(\lambda f+g)\in R\langle X\rangle_{\mathrm{irr}}$, then $r(\lambda f+g) = (r''r'r)(\lambda f+g) = \lambda(r''r'r)(f) + (r''r'r)(g) = \lambda r_Q(f) + r_Q(g)$. Hence, the expression $r(\lambda f + g)$ is uniquely determined, and $\lambda f + g$ is reduction-unique. In fact, $r_Q(\lambda f + g) = \lambda r_Q(f) + r_Q(g)$, and therefore (i) is proved.
(ii) From (i) we know that fgh is reduction-unique. Consider $r = r_{D\sigma E}$, for $\sigma\in Q$, $D,E\in\langle X\rangle$. The idea is to show that $fr(g)h$ is reduction-unique and $r_Q(fr(g)h) = r_Q(fgh)$. Note that if f,g,h are terms A,B,C, then $r_{AD\sigma EC}(ABC) = Ar_{D\sigma E}(B)C$, that is, $Ar_{D\sigma E}(B)C$ is reduction-unique with $r_Q(ABC) = r_Q(Ar_{D\sigma E}(B)C)$. Now, more generally, $f=\sum_i \lambda_i A_i$, $g=\sum_j \mu_j B_j$, $h=\sum_k \rho_k C_k$, where the indices i,j,k run over finite sets, with λ_i,μ_j,ρ_k, and where A_i,B_j,C_k are terms such that $A_iB_jC_k$ is reduction unique for

every i, j, k. In this way, $fr(g)h = \sum_{i,j,k} \lambda_i \mu_j \rho_k A_i r(B_j) C_k$. Finally, since $A_i r(B_j) C_k$ is reduction-finite for every i, j, k, and we have $r_Q(A_i r(B_j) C_k) = r_Q(A_i B_j C_k)$ from (i), $fr(g)h$ is reduction-unique and $r_Q(fr(g)h) = r_Q(fgh)$. □

Proposition 1.3 ([29], Proposition 3.1.3)**.** *If every element $f \in R\langle X\rangle$ is reduction-finite under a reduction system Q, and I_Q is the ideal of $R\langle X\rangle$ generated by the set $\{W_\sigma - f_\sigma \mid \sigma \in Q\}$ then $R\langle X\rangle = R\langle X\rangle_{irr} \oplus I_Q$ if and only if every element of $R\langle X\rangle$ is reduction-unique.*

Proof. Suppose that $R\langle X\rangle = R\langle X\rangle_{\text{irr}} \oplus I_Q$ and consider $f \in R\langle X\rangle$. Note that if $g, g' \in R\langle X\rangle$ are elements for which f reduces to g and g', then $g - g' \in R\langle X\rangle \cap I_Q = \{0\}$, that is, f is reduction-unique. Conversely, if every element of $R\langle X\rangle$ is reduction-unique under Q, then $r_Q : R\langle X\rangle \to R\langle X\rangle_{\text{irr}}$ is a R-linear projection. Consider $f \in \ker(r_Q)$, that is, $r_Q(f) = 0$. Then $f \in I_Q$, whence the $\ker(r_Q) \subseteq I_Q$, but in fact, $\ker(r_Q)$ contains I_Q: for every $\sigma \in Q, A, B \in \langle X\rangle$, we have $r_Q(A(W_\sigma - f_\sigma)B) = r_Q(AW_\sigma B) - r_Q(Af_\sigma B) = 0$ from Proposition 1.2, when $r = r_{1\sigma 1}$. □

Under the previous assumptions, $A = R\langle X\rangle / I_Q$ may be identified with the left free R-module $R\langle X\rangle_{irr}$ with R-module structure given by the multiplication $f * g = r_Q(fg)$.

Definition 1.4. An *overlap ambiguity* for Q is a 5-tuple of the form (σ, τ, A, B, C), where $\sigma, \tau \in Q$ and $A, B, C \in \langle X\rangle \setminus \{1\}$ such that $W_\sigma = AB$ and $W_\tau = BC$. This ambiguity is *solvable* if there exist compositions of reductions r, r' such that $r(f_\sigma C) = r'(Af_\tau)$. Similarly, a 5-tuple (σ, τ, A, B, C) with $\sigma \neq \tau$ is called an *inclusion ambiguity* if $W_\tau = B$ and $W_\sigma = ABC$. This ambiguity is solvable if there are compositions of reductions r, r' such that $r(Af_\tau B) = r'(f_\sigma)$.

Definition 1.5. A partial monomial order $\leq$ on $\langle X\rangle$ is said to be *compatible* with Q if f_σ is a linear combination of terms M with $M < W_\sigma$, for all $\sigma \in Q$.

Proposition 1.6 ([29], Proposition 3.1.6)**.** *If $\leq$ is a monomial partial order on $\langle X\rangle$ satisfying the descending chain condition and* *compatible with a reduction system Q, then every element $f \in R\langle X\rangle$ is reduction-finite. In particular, every element of $R\langle X\rangle$ reduces under Q to an irreducible element.*

Let $\leq$ be a monoid partial order on $\langle X\rangle$ compatible with the reduction system Q. Let M be a term in $\langle X\rangle$ and write Y_M for the submodule of $R\langle X\rangle$ spanned by all polynomials of the form $A(W_\sigma - f_\sigma)B$, where $A, B \in \langle X\rangle$ are such that $AW_\sigma B < M$. We will denote by V_M the submodule of $R\langle X\rangle$ spanned by all terms $M' < M$. Note that $Y_M \subseteq V_M$.

Definition 1.7. An overlap ambiguity (σ, τ, A, B, C) is said to be *resolvable* relative to $\leq$ if $f_\sigma C - Af_\tau \in Y_{ABC}$. An inclusion ambiguity (σ, τ, A, B, C) is said to be *resolvable* relative to $\leq$ if $Af_\tau C - f_\sigma \in Y_{ABC}$.

If r is a finite composition of reductions, and f belongs to V_M, then $f - r(f) \in Y_M$. Hence, $f \in Y_M$ if and only if $r(f) \in Y_M$ ([19], Proposition 3.1.8).

Proposition 1.8 (Bergman's Diamond Lemma [37]; [29], Theorem 3.21)**.** *Let Q be a reduction system for the free associative R-ring $R\langle X\rangle$, and let $\leq$ be a monomial partial order on $\langle X\rangle$, compatible with Q and satisfying the descending chain condition. The following conditions are equivalent:* (i) *all ambiguities of Q are resolvable;* (ii) *all ambiguities of Q are resolvable relative to $\leq$;* (iii) *all elements of $R\langle X\rangle$ are reduction-unique under Q;* (iv) $R\langle X\rangle = R\langle X\rangle_{\text{irr}} \oplus I_Q$.

1.1. Algorithms

Throughout this section we will consider the lexicographical degree order $\preceq_{\text{deglex}}$ to be defined on the variables $x_1, \ldots, x_n$. For more details about these orders, see [18], section 3.

Definition 1.9. A reduction system Q for the free associative R-ring $R\langle x_1, \ldots, x_n\rangle$ is said to be a $\preceq_{\text{deglex}}$-*skew reduction system* if the following conditions hold: (i) $Q = \{(W_{ji}, f_{ji}) \mid 1 \leq i < j \leq n\}$; (ii) for every $j > i$, $W_{ji} = x_j x_i$ and $f_{ji} = c_{i,j} x_i x_j + p_{ji}$, where $c_{i,j} \in R \setminus \{0\}$ and $p_{ji} \in R\langle x_1, \ldots, x_n\rangle$; (iii) for each $j > i$, $\text{lm}(p_{ji}) \preceq_{\text{deglex}} x_i x_j$. We will denote $(Q, \preceq_{\text{deglex}})$ this type of reduction systems.

Note that if $0 \neq p \in \sum_{r_\alpha} x^\alpha$, $r_\alpha \in R$, we consider its *Newton diagram* as $\mathcal{N}(p) := \{\alpha \in \mathbb{N} \mid r_\alpha \neq 0\}$. Let $\exp(p) := \max \mathcal{N}(p)$. In this way, by Proposition 1.6 every element $f \in R\langle x_1, \dots, x_n\rangle$ reduces under Q to an irreducible element. Let I_Q be the two-sided ideal of $R\langle x_1, \dots, x_n\rangle$ generated by $W_{ji} - f_{ji}$, for $1 \leq i < j \leq n$. If $x_i + I_Q$ is also represented by x_i, for each $1 \leq i \leq n$, then we call *standard terms* in A. Proposition 1.11 below shows that any polynomial reduces under Q to some standard polynomial and hence standard terms in A generate this algebra as a left free R-module.

Proposition 1.10 ([29], Lemma 3.2.2). *If $(Q, \preceq_{\text{deglex}})$ is a skew reduction system, then the set $R\langle x_1, \dots, x_n\rangle_{irr}$ is the left submodule of $R\langle x_1, \dots, x_n\rangle$ consisting of all standard polynomials $f \in R\langle x_1, \dots, x_n\rangle$.*

Proof. It is clear that every standard term is irreducible. Now, let us see that if a monomial $M = \lambda x_{j_1} \cdots x_{j_s}$ is not standard, then some reduction will act non-trivially on it. If $s < 2$ the monomial is clearly standard. This is also true if $j_k \leq j_{k+1}$, for every $1 \leq k \leq s - 1$. Let $s \geq 2$. There exists k such that $j_k > j_{k+1}$ and $M = Cx_jx_iB = CW_{ji}B$ where $j = j_k$, $i = j_{k+1}$ and where C and B are terms. Then $CW_{ji}B \to_Q Cf_{ji}B$ acts non trivially on M. □

Proposition 1.11 ([29], Proposition 3.2.3). *If $(Q, \preceq_{\text{deglex}})$ is a skew reduction system for the set $R\langle x_1, \dots, x_n\rangle$, then every element of $R\langle x_1, \dots, x_n\rangle$ reduces under Q to a standard polynomial. Thus the standard terms in $A = R\langle x_1, \dots, x_n\rangle/I_Q$ span A as a left free module over R.*

Proof. It follows from Proposition 1.10 and Proposition 1.6. □

Next, we present an algorithm to reduce any polynomial in $R\langle x_1, \dots, x_n\rangle$ to its standard representation modulo I_Q. The basic step in this algorithm is the reduction of terms to polynomials of smaller leading term. In the proof of Proposition 1.10 we can choose k to be the least integer such that $j_k > j_{k+1}$, thus yielding a procedure to define for every non-standard monomial λM a reduction denoted red that acts non-trivially on M. In this way, the linear map red : $R\langle x_1,\ldots,x_n\rangle \to R\langle x_1,\ldots,x_n\rangle$ depends on M. However, the following procedure is an algorithm.

Algorithm: Monomial reduction algorithm

INPUT: $M = \lambda x_{j_1}\cdots x_{j_r}$ a non standard monomial.
OUTPUT: $p = \mathrm{red}(M)$, a reduction under Q of the monomial M
INITIALIZATION: $k = 1, C = \lambda$
 WHILE $j_k \leq j_{k+1}$ **DO**
 $C = Cx_{j_k}$
 $k = k+1$
  IF $k+2 \leq r$ **THEN**
   $B = x_{j_{k+2}}\cdots x_{j_r}$
  ELSE
   $B = 1$
   $j = j_k,\ i = j_{k+1}$
   $p = Cf_{j,i}B.$

An element $f \in R\langle x_1,\ldots,x_n\rangle$ is called *normal* if $\deg(X_t) \preceq_{\mathrm{deglex}} \deg(\mathrm{lt}(f))$, for every term $X_t \neq \mathrm{lt}(f)$ in f. (In Definition 2.4 we will see that elements of skew PBW extensions are normal).

Proposition 1.12 ([29], Proposition 3.2.4)**.** *Let* $(Q, \preceq_{\mathrm{deglex}})$ *be a skew quantum reduction system. There exists a R-linear map* $\mathrm{stred}_Q : R\langle x_1,\ldots,x_n\rangle \to R\langle x_1,\ldots,x_n\rangle_{\mathrm{irr}}$ *satisfying the following conditions:* (i) *for every* $f \in R\langle x_1,\ldots,x_n\rangle$*, there exists a finite sequence* $r_1,\ldots, r_m$ *of reductions such that* $\mathrm{stred}_Q(f) = (r_m\cdots r_1)(f)$*;* (ii) *if f is normal, then* $\mathrm{mdeg}(\mathrm{lm}(f)) = \mathrm{mdeg}(\mathrm{lm}(\mathrm{stred}_Q(f)))$.

From the proof of Proposition 1.12 we obtain the next algorithm. Remark 1.13 and Theorem 1.14 are the key results connecting this section with skew PBW extensions.

Algorithm: Reduction to standard form algorithm

INPUT: f a non-standard polynomial.
OUTPUT: $g = \mathrm{stred}_Q(f)$ a standard reduction under Q of f
INITIALIZATION: $g = 0$
 WHILE $f \neq 0$ **DO**
  IF $\mathrm{lm}(f)$ is standard **THEN**
   $f = f - \mathrm{lm}(f)$
   $g = g + \mathrm{lm}(g)$
  ELSE
   $f = f - \mathrm{lm}(f) + \mathrm{red}(\mathrm{lm}(f)).$

Remark 1.13. *A free left R-module A is a skew PBW extension with respect to $\preceq_{\text{deglex}}$ if and only if it is isomorphic to the quotient $R\langle x_1, \dots, x_n\rangle / I_Q$, where Q is a skew reduction system with respect to $\preceq_{\text{deglex}}$.*

By Proposition 1.8, the set of all standard terms forms a R-basis for $A = R\langle x_1, \dots, x_n\rangle / I_Q$. We have the following key result:

Theorem 1.14 ([29], Theorem 3.2.6)**.** *Let $(Q, \preceq_{\text{deglex}})$ be a skew reduction system on $R\langle x_1, \dots, x_n\rangle$ and let $A = R\langle x_1, \dots, x_n\rangle / I_Q$. For $1 \leq i < j < k \leq n$, let g_{kji}, h_{kji} be elements in $R\langle x_1, \dots, x_n\rangle$ such that $x_k f_{ji}$ (resp. $f_{kj} x_i$) reduces to g_{kji} (resp. h_{kji}) under Q. The following conditions are equivalent:*

(i) *A is a skew PBW extension of R;*

(ii) *the standard terms form a basis of A as a left free R-module;*

(iii) *$g_{kji} = h_{kji}$, for every $1 \leq i < j < k \leq n$;*

(iv) $\operatorname{stred}_Q(x_k f_{ji}) = \operatorname{stred}_Q(f_{kj} x_i)$, *for every $1 \leq i < j < k \leq n$.*

Moreover, if A is a skew PBW extension, then $\operatorname{stred}_Q = r_Q$ and A is isomorphic as a left module to $R\langle x_1, \dots, x_n\rangle_{\text{irr}}$ whose module structure is given by the product $f * g := r_Q(fg)$, for every $f, g \in R\langle x_1, \dots, x_n\rangle_{\text{irr}}$.

Proof. The equivalence between (i) and (ii) as well between (i) and (iii) is given by Proposition 1.8. The equivalence between (i) and (iv) is obtained from Proposition 1.8 and Proposition 1.12. The remaining statements are also consequences of Proposition 1.8. □

Theorem 1.14 gives an algorithm to check whether the algebraic structure $R\langle x_1, \dots, x_n\rangle / I_Q$ is a skew *PBW* extension since $\operatorname{stred}_Q(x_k f_{ji})$ and $\operatorname{stred}_Q(f_{kj} x_i)$ can be computed by means of Algorithm "Reduction to standard form algorithm".

Remark 1.15. *In [38], it was also investigated the problem of determining if one quantum algebra have a PBW basis, and more especifically, if the algebra is a skew PBW extension, using different tools. In this sense, our Theorem 1.14 establishes an analogous result to [38], Theorem 2.4.*

2. Skew Poincaré-Birkhoff-Witt extensions

Skew PBW extensions introduced in [18] include many algebras of interest for modern mathematical physicists. As examples of these extensions, we mention the following: (a) the enveloping algebra of any finite-dimensional Lie algebra; (b) any differential operator formed from commuting derivations; (c) any Weyl algebra; (d) those differential operator rings $V(B, L)$ where L is a Lie algebra which is also a finitely generated free B-module equipped with a suitable Lie algebra map to derivations on B; (e) the twisted or smash product differential operator ring involving finite-dimensional Lie algebras acting on a ring by derivations together with Lie 2-cocycles; (f) group rings of polycyclic by finite groups; (g) Ore algebras of injective type; (h) operator algebras; (i) diffusion algebras; (j) some quantum algebras; (k) quadratic algebras in 3 variables; (l) some types of Auslander-Gorenstein rings; (m) some skew Calabi-Yau algebras; (n) quantum polynomials, (o) some quantum universal enveloping algebras. A detailed list of examples of skew PBW extensions is presented in [29], [20] and [24].

Definition 2.1 ([18], Definition 1)**.** Let R and A be rings. We say that A *is a skew PBW extension of* R (also called a *σ-PBW extension of* R), if the following conditions hold:

(i) $R \subseteq A$;

(ii) there exist elements $x_1, \dots, x_n \in A$ such that A is a left free R-module, with basis the basic elements $\mathrm{Mon}(A) := \{x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$.

(iii) For each $1 \le i \le n$ and any $r \in R \setminus \{0\}$, there exists an element $c_{i,r} \in R \setminus \{0\}$ such that $x_i r - c_{i,r} x_i \in R$.

(iv) For any elements $1 \le i, j \le n$, there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n$.

Under these conditions, we write $A := \sigma(R)\langle x_1, \dots, x_n \rangle$.

Proposition 2.2 ([18], Proposition 3)**.** *Let A be a skew PBW extension of R. For each $1 \le i \le n$, there exists an injective* *endomorphism* $\sigma_i : R \to R$ *and an* σ_i*-derivation* $\delta_i : R \to R$ *such that* $x_i r = \sigma_i(r)x_i + \delta_i(r)$*, for each* $r \in R$*.*

Two particular cases of skew PBW extensions are considered in the following definition.

Definition 2.3 ([18], Definition 4)**.** Let A be a skew PBW extension of R. (a) A is called *quasi-commutative* if the conditions (iii) and (iv) in Definition 2.1 are replaced by (iii'): for each $1 \leq i \leq n$ and all $r \in R \setminus \{0\}$ there exists $c_{i,r} \in R \setminus \{0\}$ such that $x_i r = c_{i,r} x_i$; (iv'): for any $1 \leq i, j \leq n$ there exists $c_{i,j} \in R \setminus \{0\}$ such that $x_j x_i = c_{i,j} x_i x_j$; (b) A is called *bijective* if σ_i is bijective for each $1 \leq i \leq n$, and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.

Definition 2.4 ([18], Definition 6)**.** Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 2.2.

(i) For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sigma^{\alpha} := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. If $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$; then $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

(ii) For $X = x^{\alpha} \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ and $\deg(X) := |\alpha|$. The symbol $\succeq$ will denote a total order defined on $\mathrm{Mon}(A)$ (a total order on $\mathbb{N}_0^n$). For an element $x^{\alpha} \in \mathrm{Mon}(A)$, $\exp(x^{\alpha}) := \alpha \in \mathbb{N}_0^n$. If $x^{\alpha} \succeq x^{\beta}$ but $x^{\alpha} \neq x^{\beta}$, we write $x^{\alpha} \succ x^{\beta}$. Every element $f \in A$ can be expressed uniquely as $f = a_0 + a_1 X_1 + \cdots + a_m X_m$, with $a_i \in R \setminus \{0\}$, and $X_m \succ \cdots \succ X_1$. With this notation, we define $\mathrm{lm}(f) := X_m$, the *leading monomial* of f; $\mathrm{lc}(f) := a_m$, the *leading coefficient* of f; $\mathrm{lt}(f) := a_m X_m$, the *leading term* of f; $\exp(f) := \exp(X_m)$, the *order* of f; and $E(f) := \{\exp(X_i) \mid 1 \leq i \leq t\}$. Note that $\deg(f) := \max\{\deg(X_i)\}_{i=1}^t$. Finally, if $f = 0$, then $\mathrm{lm}(0) := 0$, $\mathrm{lc}(0) := 0$, $\mathrm{lt}(0) := 0$. We also consider $X \succ 0$ for any $X \in \mathrm{Mon}(A)$. Again, for a detailed description of monomial orders in skew PBW extensions, see [18], Section 3.

3. Examples

In this section we present two examples of skew PBW extensions which illustrate the results of Section 1.1. Our aim is to show that several rings have a PBW basis in the sense of Definition 2.1. Other well known examples for quantum physics (Weyl algebras, quantum Weyl algebras, dispin algebras, Woronowicz algebra, skew polynomial rings, q-Heisenberg algebra, etc) can be realized following the ideas presented in this paper (see [29] for a detailed description of each one of these algebras).

Hayashi algebra

With the purpose of obtaining bosonic representations of the Drinfield-Jimbo quantum algebras, Hayashi considered in [39] the $\mathcal{A}_q^-$ algebra. Let us see its construction (we follow [34], Example 2.7.7). Let $\mathbf{U}$ be the algebra generated by the indeterminates $\omega_1, \dots, \omega_n, \psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*$, with the relations

$$\begin{aligned} \psi_j\psi_i - \psi_i\psi_j = \psi_j^*\psi_i^* - \psi_i^*\psi_j^* = \omega_j\omega_i - \omega_i\omega_j = \psi_j^*\psi_i - \psi_i\psi_j^* = 0, &\qquad 1 \le i < j \le n,\\ \omega_j\psi_i - q^{-\delta_{ij}}\psi_i\omega_j = \psi_j^*\omega_i - q^{-\delta_{ij}}\omega_i\psi_j^* = 0, &\qquad 1 \le i, j \le n,\\ \psi_i^*\psi_i - q^2\psi_i\psi_i^* = -q^2\omega_i^2, &\qquad 1 \le i \le n. \end{aligned} \tag{3.1}$$

Let $x_1 := \omega_1, \dots, x_n := \omega_n, x_{n+1} := \psi_1, \dots, x_{2n} := \psi_n, x_{2n+1} := \psi_1^*, \dots, x_{3n} := \psi_n^*$. The relations (3.1) are equivalent to

$$\begin{aligned} x_jx_i - x_ix_j = x_{n+j}x_{n+i} - x_{n+i}x_{n+j} = x_{2n+j}x_{2n+i} - x_{2n+i}x_{2n+j} = 0, &\qquad 1 \le i < j \le n,\\ x_{n+i}x_j - q^{\delta_{ij}}x_jx_{n+i} = x_{2n+j}x_i - q^{-\delta_{ij}}x_ix_{2n+j} = 0, &\qquad 1 \le i, j \le n,\\ x_{2n+j}x_{n+i} = x_{n+i}x_{2n+j}, &\qquad 1 \le i < j \le n,\\ x_{2n+i}x_{n+i} = q^2x_{n+i}x_{2n+i} - q^2x_i^2, &\qquad 1 \le i \le n. \end{aligned}$$

Again, consider $x_1 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots \prec x_{2n} \prec x_{2n+1} \prec x_{3n}$. Then $(Q, \preceq_{\text{deglex}})$ is a skew reduction system, and we obtain the following cases:

- $1 \le i < j < k \le n$: $\text{stred}_Q(x_kf_{ji}) = x_kx_ix_j = x_ix_kx_j = x_ix_jx_k$, $\text{stred}_Q(f_{kj}x_i) = x_jx_kx_i = x_jx_ix_k = x_ix_jx_k$;
- $n+1 \le i < j < k \le 2n$: $\text{stred}_Q(x_kf_{ji}) = x_kx_ix_j = x_ix_kx_j = x_ix_jx_k$, $\text{stred}_Q(f_{kj}x_i) = x_jx_kx_i = x_jx_ix_k = x_ix_jx_k$;
- $2n+i \le i < j < k \le 3n$: $\text{stred}_Q(x_kf_{ji}) = x_kx_ix_j = x_ix_kx_j = x_ix_jx_k$, $\text{stred}_Q(f_{kj}x_i) = x_jx_kx_i = x_jx_ix_k = x_ix_jx_k$;

- $1 \leq i < j \leq n$ and $n+1 \leq k \leq 2n$:

  if $k = n+i$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = q x_i x_k x_j = q x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j q x_i x_k = q x_i x_j x_k$;

  if $k \neq n+i,\ n+j$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = x_i x_k x_j = x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = x_i x_j x_k$;

  if $k = n+j$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = x_i x_k x_j = x_i q x_j x_k = q x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = q x_j x_k x_i = q x_j x_i x_k = q x_i x_j x_k$;

- $1 \leq i < j \leq n$ and $2n+1 \leq k \leq 3n$:

  if $k = 2n+i$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = q^{-1} x_i x_k x_j = q^{-1} x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j q^{-1} x_i x_k = q^{-1} x_i x_j x_k$;

  if $k \neq 2n+i,\ 2n+j$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = x_i x_k x_j = x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = x_i x_j x_k$;

  if $k = 2n+j$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = x_i x_k x_j = x_i q^{-1} x_j x_k = q^{-1} x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = q^{-1} x_j x_k x_i = q^{-1} x_j x_i x_k = q^{-1} x_i x_j x_k$;

- $1 \leq i \leq n,\ n+1 \leq j \leq 2n,\ 2n+1 \leq k \leq 3n$:

  if $j = n+i$ and $k = 2n+i$, then, $\text{stred}_Q(x_k f_{ji}) = x_k q_x i x_j = q q^{-1} x_i x_k x_j = x_i(q^2 x_j x_k - q^2 x_i^2) = q^2 x_i x_j x_k - q^2 x_i^3$,
  $\text{stred}_Q(f_{kj} x_i) = (q^2 x_j x_k - q^2 x_i^2) x_i = q^2 x_j x_k x_i - q^2 x_i^3 = q^2 x_i x_j x_k - q^2 x_i^3$;

  if $j = n+i$ and $k \neq 2n+i$, then, $\text{stred}_Q(x_k f_{ji}) = x_k q x_i x_j = q x_i x_k x_j = q x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = q x_i x_j x_k$;

  if $j \neq n+i$ and $k = 2n+i$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = q^{-1} x_i x_k x_j = q^{-1} x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j q^{-1} x_i x_k = q^{-1} x_i x_j x_k$;

  if $j \neq n+i$ and $k \neq 2n+i$, then, $\text{stred}_Q(x_k f_{ji}) = x_k x_i x_j = x_i x_k x_j = x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = x_i x_j x_k$;

- $1 \leq i \leq n$ and $n+1 \leq j < k \leq 2n$:

  if $j = n+i$, then, $\text{stred}_Q(x_k f_{ji}) = x_k q x_i x_j = q x_i x_k x_j = q x_i x_j x_k$,
  $\text{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = q x_i x_j x_k$;

if $j, k \neq n+i$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k x_i x_j = x_i x_k x_j = x_i x_j x_k$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = x_i x_j x_k$;

if $k = n+i$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k x_i x_j = q x_i x_k x_j = q x_i x_j x_k$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j q x_i x_k = q x_i x_j x_k$;

- $1 \leq i \leq n$ and $2n+1 \leq j < k \leq 3n$,

if $j = 2n+i$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k q^{-1} x_i x_j = q^{-1} x_i x_k x_j = q^{-1} x_i x_j x_k$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = q^{-1} x_i x_j x_k$;

if $j, k \neq 2n+1$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k x_i x_j = x_i x_k x_j = x_i x_j x_k$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j x_i x_k = x_i x_j x_k$;

if $k = 2n+1$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k x_i x_j = q^{-1} x_i x_k x_j = q^{-1} x_i x_j x_k$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_i = x_j q^{-1} x_i x_j = q^{-1} x_i x_j x_k$;

- $n+1 \leq i' := n+i \leq 2n$ and $2n+i \leq j < k \leq 3n$:

if $j = 2n+1$, then, $\mathrm{stred}_Q(x_k f_{ji'}) = x_k(q^2 x_{i'} x_j - q^2 x_i^2) = q^2 x_k x_{i'} x_j - q^2 x_k x_i^2 = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_k$,
$\mathrm{stred}_Q(f_{kj} x_{i'}) = x_j x_k x_{i'} = x_j x_{i'} x_k = (q^2 x_{i'} x_j - q^2 x_i^2) x_k = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_k$;

if $j, k \neq 2n+i$, then, $\mathrm{stred}_Q(x_k f_{ji'}) = x_k x_{i'} x_j = x_{i'} x_k x_j = x_{i'} x_j x_k$,
$\mathrm{stred}_Q(f_{kj} x_{i'}) = x_j x_k x_{i'} = x_j x_{i'} x_k = x_{i'} x_j x_k$;

if $k = 2n+i$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k x_{i'} x_j = (q^2 x_{i'} x_k - q^2 x_i^2) x_j = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_j$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_{i'} = x_j(q^2 x_{i'} x_k - q^2 x_i^2) = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_j$;

- $n+1 \leq i' := n+i < j \leq 2n$ and $2n+1 \leq k \leq 3n$:

if $k = 2n+i$, then, $\mathrm{stred}_Q(x_k f_{ji'}) = x_k x_{i'} x_j = (q^2 x_{i'} x_k - q^2 x_i^2) x_j = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_j$,
$\mathrm{stred}_Q(f_{kj} x_i') = x_j x_k x_{i'} = x_j(q^2 x_{i'} x_k - q^2 x_i^2) = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_j$;

if $k \neq 2n+i, 2n+j$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k x_{i'} x_j = x_{i'} x_k x_j = x_{i'} x_j x_k$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_{i'} = x_j x_{i'} x_k = x_{i'} x_j x_k$;

if $k = 2n+1$, then, $\mathrm{stred}_Q(x_k f_{ji}) = x_k x_{i'} x_j = (q^2 x_{i'} x_k - q^2 x_i^2) x_j = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_j$,
$\mathrm{stred}_Q(f_{kj} x_i) = x_j x_k x_{i'} = x_j(q^2 x_{i'} x_k - q^2 x_i^2) = q^2 x_{i'} x_j x_k - q^2 x_i^2 x_j$.

As we have seen, $\mathrm{stred}_Q(x_k f_{ji}) = \mathrm{stred}_Q(f_{kj} x_i)$, for every $1 \leq i < j < k \leq 3n$, so $\{\omega_1, \dots, \omega_n, \psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*\}$ form a $\Bbbk$-basis of $\mathbf{U}$. Now, to obtain the Hayashi algebra $\mathcal{A}_q^-$, we take the field of the complex numbers and consider the multiplicative monoid $\mathbf{S}$ generated by $\omega_1, \dots, \omega_n$. Since $\mathbf{S}$ is a regular Ore set and the localization $\mathbf{S}^{-1}\mathbf{U}$ exists, then $\mathcal{A}_q^-$ is $\mathbf{S}^{-1}\mathbf{U}$ modulo the ideal generated by $\psi_i\psi_i^* - q^2\psi_i^*\psi_i\omega_i^{-2}$, for $i = 1, \dots, n$ (see [20], section 3.8, for localizations in skew PBW extensions).

Non-Hermitian realization of a Lie deformed, non-canonical Heisenberg algebra

In [6], it was studied the non-Hermitian realization of a Lie deformed, a non-canonical Heisenberg algebra, considering the case of operators $A_j,\ B_k$ which are non-Hermitian (i.e., $\hbar = 1$)

$$\begin{aligned} A_j(1 + i\lambda_{jk})B_k - B_k(1 - i\lambda_{jk})A_j &= i\delta_{jk} \\ [A_j, B_k] &= 0 \quad (j \neq k) \\ [A_j, A_k] &= [B_j, B_k] = 0, \end{aligned} \tag{3.2}$$

and,

$$\begin{aligned} A_j^+(1 + i\lambda_{jk})B_k^+ - B_k^+(1 - i\lambda_{jk})A_j^+ &= i\delta_{jk} \\ [A_j^+, B_k^+] &= \hat{A}0 \ (j \neq k), \\ [A_j^+, A_k^+] &= [B_j^+, B_k^+] = 0 \end{aligned} \tag{3.3}$$

where $A_j \neq A_j^+,\ B_k \neq B_k^+$ $(j, k = 1, 2, 3)$. If the operators $A_j,\ B_k$ are in the form $A_j = f_j(N_j+1)a_j,\ B_k = a_k^+ f_k(N_k+1)$, where $a_j,\ a_j^+$ are leader operators of the usual Heisenberg-Weyl algebra, with N_j the corresponding number operator $(N_j = a_j^+ a_j,\ N_j \mid n_j\rangle = n_j|n_j\rangle)$, and the structure functions $f_j(N_j + 1)$ complex, then it is showed in [6] that A_j and B_k are given by

$$\begin{aligned} A_j &= \sqrt{\frac{i}{1 + i\lambda_j}} \left(\frac{[(1 - i\lambda_j)/(1 + i\lambda_j)]^{N_j+1} - 1}{(1 - i\lambda_j)/(1 + i\lambda_j) - 1} \frac{1}{N_j + 1} \right)^{\frac{1}{2}} a_j \\ B_k &= \sqrt{\frac{i}{1 + i\lambda_k}} a_k^+ \left(\frac{[(1 - i\lambda_k)/(1 + i\lambda_k)]^{N_k+1} - 1}{(1 - i\lambda_k)/(1 + i\lambda_k) - 1} \frac{1}{N_k + 1} \right)^{\frac{1}{2}}. \end{aligned}$$

Next, we show that this algebra is a skew PBW extension of a field $\Bbbk$. Let $x_1 := B_1,\ x_2 := B_2,\ x_3 := B_3,\ x_4 := A_1,\ x_5 := A_2$, and $x_6 := A_3$. Under these identifications, the relations (3.2) are equivalent to the following:

$$
\begin{aligned}
&x_2x_1 = x_1x_2, && x_3x_1 = x_1x_3, && x_4x_1 = \frac{1-i\lambda_{11}}{1+i\lambda_{11}}x_1x_4 + i,\\
&x_5x_1 = x_1x_5, && x_6x_1 = x_1x_6, && x_5x_2 = \frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_5 + i\\
&x_3x_2 = x_2x_3, && x_4x_2 = x_2x_4, && x_6x_3 = \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_6 + i\\
&x_5x_3 = x_3x_5, && x_6x_2 = x_2x_6, && x_4x_3 = x_3x_4\\
&x_5x_4 = x_4x_5, && x_6x_4 = x_4x_6, && x_6x_5 = x_5x_6.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\operatorname{stred}_Q(x_3f_{21}) &= x_3x_1x_2 = x_1x_3x_2 = x_1x_2x_3\\
\operatorname{stred}_Q(f_{32}x_1) &= x_2x_3x_1 = x_2x_1x_3 = x_1x_2x_3\\
\operatorname{stred}_Q(x_4f_{32}) &= x_4x_2x_3 = x_2x_4x_3 = x_2x_3x_4\\
\operatorname{stred}_Q(f_{43}x_2) &= x_3x_4x_2 = x_3x_2x_4 = x_2x_3x_4\\
\operatorname{stred}_Q(x_4f_{21}) &= x_4x_1x_2 = \left(\frac{1-i\lambda_{11}}{1+i\lambda_{11}}x_1x_4 + i\right)x_2\\
&= \frac{1-i\lambda_{11}}{1+i\lambda_{11}}x_1x_4x_2 + ix_2 = \frac{1-i\lambda_{11}}{1+i\lambda_{11}}x_1x_2x_4 + ix_2\\
\operatorname{stred}_Q(f_{42}x_1) &= x_2x_4x_1 = x_2\left(\frac{1-i\lambda_{11}}{1+i\lambda_{11}}x_1x_4 + i\right)\\
&=\hat{A}x_2x_1\frac{1-i\lambda_{11}}{1+i\lambda_{11}}x_4 + ix_2 = \frac{1-i\lambda_{11}}{1+i\lambda_{11}}x_1x_2x_4 + ix_2\\
\operatorname{stred}_Q(x_5f_{43}) &= x_5x_3x_4 = x_3x_5x_4 = x_3x_4x_5\\
\operatorname{stred}_Q(f_{54}x_3) &= x_4x_5x_3 = x_4x_3x_5 = x_3x_4x_5\\
\operatorname{stred}_Q(x_5f_{32}) &= x_5x_2x_3 = \left(\frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_5 + i\right)x_3\\
&= \frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_5x_3 + ix_3 = \frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_3x_5 + ix_3\\
\operatorname{stred}_Q(f_{53}x_2) &= x_3x_5x_2 = x_3\left(\frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_5 + i\right)\\
&= \frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_3x_2x_5 + ix_3 = \frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_3x_5 + ix_3
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{stred}_Q(x_5f_{21}) &= x_5x_1x_2 = x_1x_5x_2 = x_1\left(\frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_5 + i\right)\\
&= \frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_1x_2x_5 + ix_1\\
\mathrm{stred}_Q(f_{52}x_1) &= \left(\frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_5 + i\right)x_1 = \frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_2x_5x_1 + ix_1\\
&=\hat{A}\frac{1-i\lambda_{22}}{1+i\lambda_{22}}x_1x_2x_5 + ix_1\\
\mathrm{stred}_Q(x_6f_{54}) &= x_6x_5x_4 = x_4x_6x_5 = x_4x_5x_6\\
\mathrm{stred}_Q(f_{65}x_4) &= x_5x_6x_4 = x_5x_4x_6 = x_4x_5x_6\\
\mathrm{stred}_Q(x_6f_{43}) &= x_6x_3x_4\left(\frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_6 + i\right)x_4 = \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_6x_4 + ix_4\\
&= \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_4x_6 + ix_4,\\
\mathrm{stred}_Q(f_{64}x_3) &= x_4x_6x_3 = x_4\left(\frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_6 + i\right) = \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_4x_3x_6 + ix_4\\
&= \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_4x_6 + ix_4\\
\mathrm{stred}_Q(x_6f_{32}) &= x_6x_2x_3 = x_2x_6x_3 = x_2\left(\frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_6 + i\right)\\
&= \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_2x_3x_6 + ix_2\\
\mathrm{stred}_Q(f_{63}x_2) &= \left(\frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_6 + i\right)x_2 = \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_3x_6x_2 + ix_2\\
&= \frac{1-i\lambda_{33}}{1+i\lambda_{33}}x_2x_3x_6 + ix_2\\
\mathrm{stred}_Q(x_6f_{21}) &= x_6x_1x_2 = x_1x_6x_2 = x_1x_2x_6\\
\mathrm{stred}_Q(f_{62}x_1) &= x_2x_6x_1 = x_2x_1x_6 = x_1x_2x_6.
\end{aligned}
$$

Since $\mathrm{stred}_Q(x_kf_{ji}) = \mathrm{stred}_Q(f_{kj}x_i)$, for every $1 \leq i < j < k \leq 6$, then the elements $B_1^{a_1}, B_2^{a_2}, B_3^{a_3}, A_1^{a_4}, A_2^{a_5}$ and $A_3^{a_6}$, $a_i \in \mathbb{N}$, for every i, form a basis of the Lie-deformed Heisenberg algebra, and from (3.2), we can see that this algebra is a skew PBW extension over the complex numbers.

Conclusions and future work

In this paper, we have presented a criteria to determine whether an algebra defined by generators and relations can be expressed as a skew PBW extension. Nevertheless, since the limited size of the paper, there are a lot of remarkable algebras of the theoretical physics which are skew PBW extensions and were not illustrated here (see [29] for more examples). As a future work, we will investigate a theory of PBW bases for another kinds of quantum algebras more general than skew PBW extensions over fields. The techniques to be used will concern noncommutative differential geometry (see [27]) with the aim of characterizing algebras arising in geometries of noncommutative spaces and their interactions with quantum physics, in the sense of [40], [41], and others.

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References

[1] P. A. M. Dirac, Proc. Camb. Phil. Soc. **23**, 412 (1926).

[2] H. Weyl, *Gruppentheorie und Quantenmechanik (English Edition)* (Dover, 1932).

[3] N. A. M. Monk and B. J. Hiley, Found. Phys. Lett. **11**, 371 (1998).

[4] A. Plotnitsky, *Epistemology and Probability. Bohr, Heisenberg, Schrödinger, and the Nature of Quantum-Theoretical Thinking. Fundamental Theories of Physics 161* (Springer, 2010).

[5] V. S. Varadarajan, *Reflections on Quanta, Symmetries, and Supersymmetries* (Springer, 2011).

[6] A. Jannussis, A. Leodaris, and R. Mignani, Phys. Lett. A **197**, 187 (1995).

[7] A. Turbiner, Funct. Anal. Appl. **29**, 291 (1995).

[8] M. Pillin, Commun. Math. Phys **180**, 23 (1996).
[9] H. Y. Pan and Z. S. Zhao, Phys. Lett. A **282**, 251 (2001).
[10] H. Y. Pan and Z. S. Zhao, Phys. Lett. A **312**, 1 (2003).
[11] P. Narayama, Physica A **328**, 145 (2003).
[12] L. Hellstrom and S. Silvestrov, Expo. Math. **23**, 99 (2005).
[13] J. Douari, Int. J. Theor. Phys **45**, 893 (2006).
[14] E. M. F. Curado, Y. Hassouni, M. A. Rego-Monteiro, and L. M. C. S. Rodrigues, Phys. Lett. A **372**, 3350 (2008).
[15] L. Zhe, Sci. China Math. **541**, 2427 (2011).
[16] M. Arik and D. D. Coon, J. Math. Phys. **17**, 524 (1976).
[17] D. A. Slavnov, Theoret. Math. Phys. **171**, 848 (2012).
[18] C. Gallego and O. Lezama, Commun. Algebra **39**, 50 (2011).
[19] A. Reyes, Rev. Col. Mat. **47**, 95 (2013).
[20] O. Lezama and A. Reyes, Commun. Algebra **42** (1200-1230).
[21] A. Reyes, Rev. Col. Mat. **48**, 79 (2014).
[22] A. Reyes, Rev. Integr. Temas Mat. **33**, 173 (2015).
[23] O. Lezama, J. P. Acosta, and A. Reyes, Rev. Un. Mat. Argentina **56**, 39 (2015).
[24] H. Suárez, O. Lezama, and A. Reyes, Ciencia en Desarrollo **6**, 205 (2015).
[25] A. Reyes and H. Suárez, Bol. Mat. **23**, 71 (2016).
[26] A. Reyes and H. Suárez, Rev. Integr. Temas Mat. **34**, 147 (2016).
[27] A. Reyes and H. Suárez, Ciencia en Desarrollo **7**, 99 (2016).
[28] H. Suárez and A. Reyes, Far East J. Math. Sci. (FJMS) **101**, 301 (2017).
[29] A. Reyes, *Ring and Module Theoretical Properties of Skew PBW Extensions*, Ph.D. thesis, Universidad Nacional de Colombia, Bogotá - Colombia (2013).
[30] J. Dixmier, *Enveloping Algebras* (Graduate Studies in Mathematics, American Mathematical Society, 1996).
[31] H. Gausterer, H. Grosse, and L. E. Pittner, *Geometry and Quantum Physics, Lecture Notes in Physics, Proceedings, Schladming, Austria* (Springer-Verlag, 2000).

[32] M. Rosenbaun and J. D. Vergara, Gen. Relativ. Gravit. **38**, 607 (2006).

[33] H. A. Yamane, Publ. RIMS Kyoto Univ. **25**, 503 (1989).

[34] R. Berger, Comm. Math. Physics **143**, 215 (1992).

[35] C. M. Ringel, J. Reine Angew. Math. **470**, 51 (1996).

[36] O. Hinchcliffe, *Diffusion Algebras*, Ph.D. thesis, University of Sheffield, Sheffield, South Yorkshire – England (2005).

[37] G. M. Bergman, Adv. Math. **29**, 178 (1978).

[38] J. P. Acosta and O. Lezama, Alg. Dis. Math. **20**, 1 (2015).

[39] T. Hayashi, Commun. Math. Phys **127**, 129 (1990).

[40] H. Gausterer, H. Grosse, and L. E. Pittner, *Geometry and Quantum Physics, Lecture Notes in Physics, Proceedings of the 38. Internationale Universitätswochen für Kern-und Teilchenphysik, Schladming, Austria, January* (Springer-Verlag Berlin Heidelberg, 200).

[41] M. Marcolli and D. E. Parashar, *Quantum Groups and Noncommutative Spaces, Perspectives on Quantum Geometry* (Springer-Verlag Berlin Heidelberg, 2011).