https://doi.org/10.15446/recolma.v55n2.102513

Revista Colombiana de Matemáticas
Volumen 55(2021)2, páginas 139-147

On frames that are iterates of a multiplication operator

Sobre marcos que son iteraciones de un operador de multiplicación

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Abstract. A result from the recent paper of the first named author on frame properties of iterates of the multiplication operator $T_\varphi f = \varphi f$ implies in particular that a system of the form $\{\varphi^n\}_{n=0}^{\infty}$ cannot be a frame in $L_2(a,b)$. The classical exponential system shows that the situation changes drastically when one considers systems of the form $\{\varphi^n\}_{n=-\infty}^{\infty}$ instead of $\{\varphi^n\}_{n=0}^{\infty}$. This note is dedicated to the characterization of all frames of the form $\{\varphi^n\}_{n=-\infty}^{\infty}$ coming from iterates of the multiplication operator T_φ. It is shown in this note that this problem can be reduced to the following one:

Problem. *Find (or describe a class of) all real-valued functions α for which $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(a,b)$.*

In this note we give a partial answer to this problem.

To our knowledge, in the general statement, this problem remains unanswered not only for frame, but also for Schauder and Riesz basicity properties and even for orthonormal basicity of systems of the form $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{+\infty}$.

Key words and phrases. Dynamical sampling, Operator orbit, frame, Schauder bases, system of powers, Lebesgue spaces.

2020 Mathematics Subject Classification. 42C15, 46E30, 46B25, 46B15.

Resumen. Un resultado reciente por parte del primer autor del artículo acerca de marcos muestra que para las iteraciones del operador multiplicativo $T_\phi(f) = \phi f$ un sistema de la forma $\{\phi^n\}_{n=0}^{\infty}$ no puede ser un marco para $L_2(a,b)$. La situación cambia radicalmente cuando se consideran sistemas de la forma $\{\phi^n\}_{n=-\infty}^{\infty}$ en vez de $\{\phi^n\}_{n=0}^{\infty}$. El objetivo de este artículo es caracterizar marcos de la forma $\{\phi^n\}_{n=-\infty}^{\infty}$ que son iteraciones del operador multiplicativo $T_\phi(f)$. En esta nota probamos que el problema se reduce al siguiente:

Problema. *Caracterice la clase de funciones α para las cuales $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{n=\infty}$ es un marco de $L_2(a,b)$.*

En este artículo damos una respuesta parcial al problema. Hasta donde sabemos, en el caso general el problema sigue abierto, no sólo para marcos, sino también para determinar cuándo la familia $\{\phi^n\}_{n=-\infty}^{\infty}$ es una base de Schauder y de Riesz e inclusive cuándo es una base ortonormal.

Palabras y frases clave. Muestreo dinámico, marcos, órbitas de operadores, bases de Schauder, sistema de potencias, espacios de Lebesgue.

1. Introduction

Studying frame properties for families of elements obtained by iterates of operators is one of the central problems in dynamical sampling, a relatively new research topic in applied harmonic analysis. This new research field has attracted considerable attention in recent years (see, for example, [1, 2, 3],[5],[14], [7, 8, 9, 13, 10] and the bibliography therein).

Understanding basicity properties (completeness, Schauder basicity, frameness, etc.) of iterates of operators is problematic even in the case of well known "standard" operators (see, for example, [11] and [4].)

It was recently shown in [19] that the iterates $\{T_\varphi^n f\}_{n=0}^{\infty}$ of the multiplication operator $T_\varphi f = \varphi f$ cannot be a frame in $L_2(a,b)$ for any measurable function φ and square summable function f. This fact shows in particular that a system of the form $\{\varphi^n\}_{n=0}^{\infty}$ cannot be a frame in $L_2(a,b)$ for any measurable function φ. Concerning the basis, pseudo-basis and frame properties of systems of the form $\{\varphi^n\}_{n=0}^{\infty}$ (which differs from the system $\{\varphi^n\}_{n=-\infty}^{\infty}$ by indexing) we refer to the papers [15, 16, 17, 18, 19].

As, for example, the classical exponential system shows, unlike the systems of the form $\{\varphi^n\}_{n=0}^{\infty}$, existence of frames of the form $\{\varphi^n\}_{n=-\infty}^{\infty}$ is evident.

This note is dedicated to the characterization of all frames of the form $\{\varphi^n\}_{n=-\infty}^{\infty}$. It is shown in this note that this problem can be reduced to the following one:

Problem. *Find all real-valued functions α for which $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(a,b)$.*

In this note we give a partial answer to this problem.

2. Auxiliary notions and propositions

We begin by recalling some notions.

Definition 2.1. ([20], [21]) A sequence $\{x_n\}_{n=0}^{\infty}$ in a Banach space $\{X, \|\|\}$ is said to be a Schauder basis for $\{X, \|\|\}$ if to each vector x in the space there corresponds a unique sequence of scalars $\{\alpha_n\}_{n=0}^{\infty}$ such that

$$\lim_{N\to\infty} \|x - \sum_{n=0}^{N} \alpha_n x_n\| = 0.$$

Definition 2.2. ([20]) A sequence $\{x_n\}_{n=0}^{\infty}$ in a Banach space $\{X, \|\|\}$ with $x_n \neq 0$ is said to be a pseudo-basis (or a system of representation) for $\{X, \|\|\}$ if to each vector x in the space there corresponds a sequence of scalars $\{\alpha_n\}_{n=0}^{\infty}$ such that

$$\lim_{N\to\infty} \|x - \sum_{n=0}^{N} \alpha_n x_n\| = 0.$$

Definition 2.3. ([6], [21]) A sequence $\{x_n\}_{n=0}^{\infty}$ in a Hilbert space H is a frame for H if there exist constants $A, B > 0$ such that

$$A\|x\|^2 \leq \sum_{n=0}^{\infty} |(x, x_n)|^2 \leq B\|x\|^2$$

for all $x \in H$.

We will rely on the following propositions, first of which is an immediate consequence of the definition of a frame.

Lemma 2.4. *If $\{x_n\}_{n\in\mathbb{N}}$ is a frame of a Hilbert space H, then it is bounded.*

Proof. It follows directly from the definition that $|(x, x_n)|^2 \leq B\|x\|^2$ for any $n = 0, 1, 2, \ldots$. Now taking $x = x_n$ in the latter inequality we arrive at the inequalities $\|x_n\|^2 \leq B$ which proves the lemma. ☑

Lemma 2.5. *Let Φ be a measurable function on (a, b). Then $\{\Phi^n\}_{n=0}^{\infty}$ is $L_2(a, b)$ - bounded sequence if and only if $|\Phi(\cdot)| \leq 1$ a.e. on (a, b).*

Proof. The "if" part of the proposition is evident.

Now, suppose that $\{\Phi^n\}_{n=0}^{\infty}$ is $L_2(a, b)$ - bounded sequence.

Let $E = \{x : |\Phi(x)| > 1\}$. Then $E = \bigcup_{k=1}^{\infty} E_k$, where $E_k = \{x : |\Phi(x)| > a_k\}$ and $a_k = 1 + \frac{1}{k}$.

Assume, in order to get a contradiction, that $\operatorname{mes} E > 0$. Then there is a number k_0 such that $\operatorname{mes} E_{k_0} > 0$. This, along with the boundedness of $\{\Phi^n\}_{n=0}^{\infty}$, implies the existence of a constant number C such that

$$C \geq \int_a^b |\Phi(x)|^{2n} dx \geq \int_{E_{k_0}} |\Phi(x)|^{2n} dx \geq \int_{E_{k_0}} a_{k_0}^{2n} dx = a_{k_0}^{2n} \operatorname{mes} E_{k_0};$$

this is a contradiction since $\lim_{n\to\infty}(a_{k_0}^{2n} \operatorname{mes} E_{k_0}) = \infty$. So, $\operatorname{mes} E = 0$ and the lemma is proved. ☑

***Remark* 2.6.** This proof of Lemma 2.5 was provided by the reviewer; the authors thank him/her for this proof which is shorter and simpler than the authors' initial proof.

We will make use of the following observation.

Lemma 2.7. *Let $[p,q] \subset [0,2\pi]$. Then the classical exponential system $\{e^{int}\}_{-\infty}^{\infty}$ is a 1-tight frame in $L_2(p,q)$ space.*

Proof. Take an arbitrary function $f \in L_2(p,q)$. Extend the definition of the function f to $[0,2\pi] \setminus [p,q]$ by defining it to be equal to zero on it. Denote the obtained function by Φ.

It is evident that

$$\int_p^q f(x)e^{inx}dx = \int_0^{2\pi} \Phi(x)e^{inx}dx.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |\int_p^q f(x)e^{inx}dx|^2 = \sum_{n=-\infty}^{\infty} |\int_0^{2\pi} \Phi(x)e^{inx}dx|^2 = \|\Phi\|^2_{L_2(0,2\pi)},$$

last equality being valid because the classical exponential system is complete orthonormal system in $L_2(0,2\pi)$. Since $\|f\|_{L_2(p,q)} = \|\Phi\|_{L_2(0,2\pi)}$, we obtain from here that

$$\sum_{n=-\infty}^{\infty} |\int_p^q f(x)e^{inx}dx|^2 = \|f\|^2_{L_2(p,q)}.$$

Since f is assumed to be arbitrary function from $L_2(p,q)$, the last equality proves the lemma. ☑

Let ξ be an absolutely continuous strictly increasing function on $[p,q]$ such that $\xi(p) = a$ and $\xi(q) = b$. The following proposition, which is a substitution rule for Lebesgue integral, is valid (see, for example, [12, p.244]).

Proposition 2.8. *Let f be Lebesgue integrable function on $[a,b]$ and ξ be a function satisfying above mentioned conditions. Then*

$$\int_a^b f(x)dx = \int_p^q f(\xi(t))\xi'(t)dt.$$

3. Main results

One of the main results of this note is the following theorem.

Theorem 3.1. *Let φ be a measurable function on (a,b). If $\{\varphi^n\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(a,b)$, then $|\varphi(\cdot)| = 1$ a.e. in (a,b), i.e. the function φ is an exponential function of the form $\varphi(\cdot) = e^{i\alpha(\cdot)}$, where α is a real-valued function.*

***Proof*.** Since $\{\varphi^n\}_{n=-\infty}^{+\infty}$ is assumed to be a frame in $L_2(a,b)$, we find by Lemma 2.4 that it is bounded. We consider $\{\varphi^n\}_{n=-\infty}^{0}$ and $\{\varphi^n\}_{n=0}^{+\infty}$ halves of the initial sequence separately. Both these parts are bounded as well.

Consider the first half, $\{\varphi^n\}_{n=-\infty}^{0}$. Take $\Phi = \frac{1}{\varphi} = \varphi^{-1}$ in Lemma 2.5. Since $\{\varphi^n\}_{n=-\infty}^{0}$ is bounded, we find by Lemma 2.5 that $|\varphi(\cdot)| \geq 1$ a.e. on (a,b).

Now consider the second half: $\{\varphi^n\}_{n=0}^{+\infty}$. By taking $\Phi = \varphi$ in Lemma 2.5 we find that $|\varphi(\cdot)| \leq 1$ a.e. on (a,b) since $\{\varphi^n\}_{n=0}^{\infty}$ is bounded.

By combining last two arguments we obtain that $|\varphi(\cdot)| = 1$ a.e. on (a,b).

The remaining part is a direct consequence of the latter fact. The theorem is proved. ☑

This proposition shows that the characterization of frames of the form $\{\varphi^n\}_{n=-\infty}^{+\infty}$ in $L_2(a,b)$ space is equivalent to the determination of the class of all real-valued functions α for which the system $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(a,b)$.

Following this idea, we arrive at the following problem:

Problem. *Find all real valued functions α for which $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(a,b)$.*

In this note we give a partial answer to this problem by providing a sufficient condition on the function α under which the system $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(a,b)$.

Theorem 3.2. *Let α be an invertible function defined on the interval $[a,b]$ whose inverse $\xi : [p,q] \to [a,b]$ satisfies*

1) ξ is absolutely continuous, strictly increasing function on $[p,q]$;

2) $\xi(p) = a$ and $\xi(q) = b$;

3) there are constants $A, B > 0$ such that $A \leq \xi'(t) \leq B$ for all $t \in [p,q]$;

4) $[p,q] \subset [0, 2\pi]$;

Then the system $\{e^{in\alpha(\cdot)}\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(a,b)$.

Note that first two conditions of this theorem enables the use of substitution rule (Proposition 2.8). As for the third and fourth conditions, after the proof of the theorem we provide examples which show their importance.

***Proof*.** Take arbitrary function $f \in L_2(a,b)$. By Proposition 2.8, we can write

$$\int_a^b f(x)e^{in\alpha(x)}dx = \int_p^q f(\xi(t))\xi'(t)e^{int}dt.$$

Therefore,

$$\sum_{n=-\infty}^{\infty} |\int_a^b f(x)e^{in\alpha(x)}dx|^2 = \sum_{n=-\infty}^{\infty} |\int_p^q f(\xi(t))\xi'(t)e^{int}dt|^2. \qquad (1)$$

It follows from Proposition 2.8 and condition 3) that $f(\xi(\cdot))\xi'(\cdot) \in L_2(p,q)$. Then, taking the condition 4) into account, we find by Lemma 2.7 that

$$\sum_{n=-\infty}^{\infty} |\int_p^q f(\xi(t))\xi'(t)e^{int}dt|^2 = \|f(\xi(\cdot))\xi'(\cdot)\|^2_{L_2(p,q)}.$$

Thus, (1) can be written as follows:

$$\sum_{n=-\infty}^{\infty} |\int_a^b f(x)e^{in\alpha(x)}dx|^2 = \|f(\xi(\cdot))\xi'(\cdot)\|^2_{L_2(p,q)}. \tag{2}$$

Since

$$\|f(\xi(\cdot))\xi'(\cdot)\|^2_{L_2(p,q)} = \int_p^q |f(\xi(t))|^2\xi'(t)\xi'(t)dt,$$

we find from the condition 3) posed on the function ξ that

$$A\int_p^q |f(\xi(t))|^2\xi'(t)dt \le \|f(\xi(\cdot))\xi'(\cdot)\|^2_{L_2(p,q)} \le B\int_p^q |f(\xi(t))|^2\xi'(t)dt.$$

Taking into account Proposition 2.8, these inequalities can be rewritten as follows

$$A\int_a^b |f(x)|^2dx \le \|f(\xi(\cdot))\xi'(\cdot)\|^2_{L_2(p,q)} \le B\int_a^b |f(x)|^2dx,$$

$$A\|f\|^2_{L_2(a,b)} \le \|f(\xi(\cdot))\xi'(\cdot)\|^2_{L_2(p,q)} \le B\|f\|^2_{L_2(a,b)}.$$

Hence, by the relation (2)

$$A\|f\|^2_{L_2(a,b)} \le \sum_{n=-\infty}^{\infty} |\int_a^b f(x)e^{in\alpha(x)}dx|^2 \le B\|f\|^2_{L_2(a,b)}.$$

The theorem is proved. ☑

***Remark* 3.3.** Notice that the function α of Theorem 3.2 is continuous, being the inverse of a continuous and strictly monotonic function defined on an interval of R, and as a consequence, is measurable.

Example 1. Let $a = 0, b = 2\pi$ and $\alpha(x) = \sqrt{x+1}$. Then, one can verify easily that all conditions posed on the function $\alpha(x)$ are fulfilled. Therefore, the system $\{e^{in\sqrt{x+1}}\}_{n=-\infty}^{+\infty}$ is a frame in $L_2(0,2\pi)$.

The following example shows the importance of the condition 3) in Theorem 3.2.

Example 2. The system $\{e^{in\sqrt{x}}\}_{-\infty}^{\infty}$, for which the condition 3) fails, is not a frame in $L_2(0,2\pi)$.

Proof. In this case, $a = 0, b = 2\pi$ and $\alpha(x) = \sqrt{x}$. Therefore $\xi(t) = t^2$ and $p = 0, q = \sqrt{2\pi}$. Conditions 1) and 2) are fulfilled and hence, we can use the substitution formula for Lebesgue integration in this example.

Take an arbitrary natural number m. Denote

$$f_m(x) = \begin{cases} 1, & \text{if } x \in (0, \frac{1}{m}); \\ 0, & \text{if } x \in (\frac{1}{m}, 2\pi). \end{cases}$$

The equality (2) shows that

$$\sum_{n=-\infty}^{\infty} |\int_0^{2\pi} f_m(x) e^{in\alpha(x)} dx|^2 = \|f_m(\xi(\cdot))\xi'(\cdot)\|^2_{L_2(0,\sqrt{2\pi})},$$

and hence

$$\sum_{n=-\infty}^{\infty} |\int_0^{2\pi} f_m(x) e^{in\alpha(x)} dx|^2 = \int_0^{\sqrt{2\pi}} |f_m(\xi(t))|^2 \xi'(t)\xi'(t) dt,$$

According to the definition of function $f_m(x)$ we find from here that

$$\sum_{n=-\infty}^{\infty} |\int_0^{2\pi} f_m(x) e^{in\alpha(x)} dx|^2 = \int_0^{\frac{1}{\sqrt{m}}} |f_m(\xi(t))|^2 \xi'(t)\xi'(t) dt.$$

Since $\xi'(t) = 2t$, last equality implies that

$$\sum_{n=-\infty}^{\infty} |\int_0^{2\pi} f_m(x) e^{in\alpha(x)} dx|^2 \leq \frac{2}{\sqrt{m}} \int_0^{\frac{1}{\sqrt{m}}} |f_m(\xi(t))|^2 \xi'(t) dt.$$

Again, using the definition of $f_m(x)$, this inequality can be written as follows

$$\sum_{n=-\infty}^{\infty} |\int_0^{2\pi} f_m(x) e^{in\alpha(x)} dx|^2 \leq \frac{2}{\sqrt{m}} \int_0^{\sqrt{2\pi}} |f_m(\xi(t))|^2 \xi'(t) dt. \tag{3}$$

Using substitution rule for Lebesgue integration we arrive at

$$\int_0^{\sqrt{2\pi}} |f_m(\xi(t))|^2 \xi'(t) dt = \int_0^{2\pi} |f_m(x)|^2 dx = \|f_m(x)\|^2_{L_2(0,2\pi)}.$$

From here, using (3), we find that

$$\sum_{n=-\infty}^{\infty} |\int_0^{2\pi} f_m(x) e^{in\alpha(x)} dx|^2 \leq \frac{2}{\sqrt{m}} \|f_m\|^2_{L_2(0,2\pi)}$$

for arbitrary natural number m. These inequalities show that the system $\{e^{in\sqrt{x}}\}_{-\infty}^{\infty}$ is not a frame in $L_2(0, 2\pi)$. ☑

The next example shows the importance of the condition 4) in the theorem.

Example 3. The system $\{e^{in(2x+1)}\}_{-\infty}^{\infty}$, for which the condition 4) fails, is not complete and hence is not a frame in $L_2(0, 2\pi)$.

Acknowledgements. The authors are grateful to Professor B.T.Bilalov for encouraging discussion. The authors also thank the reviewer for his/her valuable comments.

The first named author thanks Professor O. Christensen and Professor A. Aldroubi for kind communications. He also thanks N. Guliyev for useful assistance and A. Huseynli for discussions.

The first-named author is supported by TUBITAK - ANAS I (19042020) grant.

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(Recibido en marzo de 2020. Aceptado en agosto de 2020)

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