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Periodic orbits in a seasonal $SIRS$ model with both incidence and treatment generalized rates

Órbitas periódicas en un modelo $SIRS$ estacional con tasas generalizadas de incidencia y tratamiento

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Abstract. In this work, we prove that a seasonal-dependent $SIRS$ model with general incidence and treatment rates has periodic solutions. This generalized model is analyzed using Leray-Schauder degree theory to prove the existence of a periodic solution. Finally, numerical simulations are shown to illustrate the theoretical results.

Key words and phrases. Leray-Schauder degree, SIRS models, periodic orbits, basic reproduction number.

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Resumen. En este trabajo, nosotros probamos que un modelo SIRS estacional denso-dependiente con tasas generalizadas de incidencia y tratamiento tiene soluciones periódicas. Este modelo generalizado es analizado usando teoría de grado de Leray-Schauder para probar la existencia de órbitas periódicas. Finalmente, se muestran simulaciones numéricas para ilustrar los resultados teóricos.

Palabras y frases clave. Grado Leray-Schauder, modelos SIRS, órbitas periódicas, número de reproducción básico.

1. Introduction

Forecasting the evolution of an infectious disease has been a very important goal in mathematical epidemiology. The construction of epidemiological models helps not only to understand the way that a disease spreads in a population but also to design public health strategies, which are intended either to control or to eradicate an epidemic outbreak. In mathematical epidemiology, there are the so called compartmental epidemic models. In these models, population is divided into classes. Each class represents a group of individuals that have some demographical or epidemiological characteristic. In some cases, population is divided into age groups. In other situations, population can be divided into epidemiological categories. For example, when a SIR model is constructed, population is divided into susceptible, infectious, and recovered individuals, which are denoted by $S(t)$, $I(t)$ and $R(t)$ respectively. The $SIQR$ and $SITR$ models are obtained from a SIR model by adding isolated individuals (denoted Q(t)) and treated individuals (denoted $T(t)$), respectively.

Some autonomous compartmental epidemic models may show periodic solutions when some conditions over the parameters are satisfied (as in [2, 9, 11, 14, 15, 17, 19, 28]). In these cases, existence of periodic solutions in these compartmental epidemic models is related to changes in the disease prevalence because of variations in the number of contacts between susceptible and infectious individuals. However, even though seasonality is so ubiquitous in nature, mechanisms producing seasonal diseases are little known . This ubiquity is the main reason why identifying the principal seasonality environmental drivers becomes extremely difficult.

There is a wide variety of seasonal infections in human population: measles and chickenpox, influenza and respiratory infections or malaria and dengue. For these infectious diseases, there are different seasonal drivers and these can be either environmental or social drivers.

Several compartmental epidemic models have been used to describe the dynamics of infectious diseases using different incidence functions. For example; Van den Driessche and Watmough [24] studied the effect over a population due to an incidence rate of the form $\beta I(1+\alpha I^n)S$, and Capasso and Serio [4] introduced a saturated incidence function $\frac{\beta SI}{(1+\alpha I)}$ in an epidemic model to study the cholera epidemic in 1973. Also, Xiao and Ruan [29] proposed a non-monotonic incidence function $\frac{\beta SI}{(1+\alpha I^2)}$ in an *SIRS* epidemic model. Particularly, this incidence function decreases due to the psychological effect that is induced by the presence of infectious individuals in the population. The examples shown above are generalizations of either the mass-action law or the standard incidence. Other general saturation incidence functions are $\frac{\beta I^k S}{(1+\alpha I^n)}$, $\frac{\beta IS}{(1+\alpha_1 I+\alpha_2 I^2)}$ and $\beta e^{-mI}IS$, which were studied in [6, 16, 20, 21]. The incidence functions shown above may describe certain epidemiological mechanism such as crowding effects, media impact, an increased likelihood of infection from multiple exposures, or behavioral changes in the susceptible population as a function of the fraction of infective individuals.

In all the examples shown above, the infection rate β is assumed constant. However, the use of a periodic function $\beta(t)$ may help to understand the reasons behind sustained oscillations. For example, Dushoff et al. [9] showed that oscillations could be sustained by the resonance of the natural periodicity in the influenza infection rate and the endogenous damped oscillations of compartmental models. Jódar and coauthors [15] studied the existence of periodic solutions in the dynamics of respiratory syncytial virus in a seasonal epidemic *SIRS* model. Both papers used an incidence function given by $\beta(t)SI$. In [2] and [19], a seasonal epidemic model using $\frac{\beta(t)SI}{1+kI}$ as the incidence function was studied.

Also, in mathematical epidemiology, different kinds of functions to describe a per-capita treatment rate are used. These treatment rates is a way to explore the impact of medical treatments to control the number of infectious individuals. Examples of per-capita treatment functions, which appear in the mathematical epidemiological literature, are rI, $(r_1+r_2I)I$, $(r_2e^{-r_3I})I$, $(r_1+r_2e^{-r_3I})I$, $\frac{r_1I}{1+kI^n}$ or piecewise linear functions; see [7, 10, 11, 23, 26, 27, 30, 31]. Notice that, the recovery rates showed before could describe several health system characteristics and hospitalization conditions. For example: the carrying capacity of hospitals, number of hospitals, doctors and nurses and even the cost of public health policies and treatments effectiveness.

Knowing the evolution of an infectious disease can be possible when some epidemiological quantities are calculated. The most famous epidemiological parameter is the so called basic reproduction number, which is denoted by R_0. The basic reproduction number is defined as the number of secondary infections that result from the introduction of a single infectious individual into an entirely susceptible population during his infectious period [1]. Then, if R_0 is less than 1, the infectious population approaches the disease-free state, and if R_0 is greater than 1, there is an epidemic outbreak. This definition of R_0 is appropriate for a nonseasonal infection for which the infectious rate is assumed to be constant. However, when modeling seasonal diseases, this definition has serious limitations because the number of secondary infections is not constant in the time.

For epidemiological models with a seasonal infectious rate, the basic reproduction number is defined by

$$\bar{R}_0 = D\int_0^1 \beta(t)dt. \tag{1}$$

In expression (1), D denotes the infectious average time, and $\beta(t)$ denotes the seasonal infectious rate at time t. Thus, $\bar{R}_0$ is defined as the average number of secondary infections that results from the random introduction at the time of a single infectious individual into a completely susceptible population [12, 22]. In this case, the condition $\bar{R}_0 < 1$ is not enough to stop an epidemic outbreak. However, it is sufficient and necessary for the infectious population to approach the disease-free state in the long term. Notice that, the control policy of bringing $\bar{R}_0$ below 1 fails to prevent the epidemic outbreak because transmission sequences can appear if $D\beta(t) > 1$. This scenario can occur in the high incidence season.

In epidemiological mathematical literature, there are models that present periodic solutions when the infection force is modeled by general non-linear functions with a seasonal infection rate [13]. In that scenario, most of the seasonal models with non linear infection force assume that the number of recovered individuals is proportional to the number of infectious individuals. In some cases, the term that describes the number of recovered individuals shows a saturation phenomenon [11]. However, it is of paramount importance to know if periodic solutions can be avoided when, in the modeling process, a general recovery function is used to describe changes in a recovered population due to different phenomenological mechanisms. In this work, we show that a periodic orbit can exist in a $SIRS$ epidemic model for a wide variety of incidence and treatment functions.

To do this, we propose a general per-capita treatment function that is given by

$$T(I) := \frac{(r_1 + r_2 e^{-r_3 I})I}{1 + kI^n}. \tag{2}$$

The parameters r_1, r_2, r_3 and k are non-negative constants and $n \geq 1$. Notice that $T(I)$ generalizes some treatment functions mentioned paragraphs above.

Also, we assume that the general incidence function $\beta(t)Sf(I)$ satisfies:

i) $f(I) = If_1(I)$ with $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$ a C^1-function,

ii) $f(0) = 0$ and $f(I) > 0$ for all $I > 0$,

iii) $f_1'(I) \leq 0$ for all $I \geq 0$.

***Remark* 1.1.** similar conditions on f for autonomous systems were considered in [5, 18].

In Section 2, we propose a seasonal $SIRS$ model with both incidence and treatment generalized rates. In Section 3, we prove the existence of a periodic orbit using Leray-Schauder degree theory (see [3, 17]). Also, proper adjustments for the general incidence function are developed using a Hurwitz condition to control the roots of the characteristic polynomial associated to the model. We propose a second homotopy to deal with the general treatment function. In Section 4, we present numerical simulations of the solutions of the model to illustrate the theoretical results. Finally, in Section 5, we provide a few concluding remarks.

2. The model

The $SIRS$ model with the incidence and treatment rate mentioned above is given by .

$$\begin{aligned} S' &= \mu(N-S) - \beta(t)Sf(I) + \eta R, \\ I' &= \beta(t)Sf(I) - (\gamma+\mu)I - T(I), \\ R' &= \gamma I - (\mu+\eta)R + T(I), \end{aligned} \tag{3}$$

where $N := S + I + R$ denotes the total population. In the proposed model, there are an inflow of newborns in the susceptible class at rate μN and deaths in each class at rates μS, μI and μR. Observe that the births balance the deaths. Then, the population size N is constant.

Usually, the seasonal infection rate is described with a sinusoidal function as follows

$$\beta(t) = \beta(1 + \sigma\cos(2\pi t)). \tag{4}$$

In (4), σ is related to the strength of seasonal forcing. That is, σ is the amplitude of the seasonal variation in the transmission of the disease. $\beta(t)$ describes the oscillations around a baseline rate β. The other model parameters are considered positive constants, with the following interpretation: μ is the birth-death rate, γ is the natural disease recovery rate, η is the rate of waning of immunity, that is, η is the rate at which the treatment wears off.

3. Analysis of the model

For model (3), $N'(t) = \mu N - \mu N = 0$. Then, $N(t)$, the population size, is constant, and for simplicity, we assume that $N = 1$. Then, the set

$$\Omega := \{(S, I; R) : S \geq 0, I \geq 0, R \geq 0, S + I + R \leq 1\} \tag{5}$$

is positively invariant for (3). Using the fact that $R = 1 - S - I$, model (3) is reduced to

$$\begin{aligned} &S' = \eta(1-S-I) + \mu(1-S) - \beta(t)Sf(I), \\ &I' = \beta(t)Sf(I) - (\gamma+\mu)I - \tfrac{(r_1+r_2e^{-r_3I})I}{1+kI^n}. \end{aligned} \tag{6}$$

Therefore, we will analyze the dynamics of system (6) in

$$\Gamma := \{(S, I) : S \geq 0, I \geq 0, S + I \leq 1\}. \tag{7}$$

To do this, we restrict our attention to such a region of feasibility Γ.

The disease-free equilibrium for model (6) is $(S_0, I_0) = (1, 0)$. Observe that, the trivial equilibrium exists for all values of the parameters of the model. Using the next generation matrix proposed in [25], we calculate the basic reproduction number for the seasonal epidemic model given in (6), which is given by

$$\mathcal{R}_0 := \frac{\overline{\beta} f'(0)}{\gamma + \mu + r_1 + r_2},$$

with $\overline{\beta} := \frac{1}{T}\int_0^T \beta(t)dt$. Notice that R_0 is not affected by the parameters r_3, k, α and η.

We express the periodic function $\beta(t)$ as

$$\beta(t) = \overline{\beta} + \beta_0(t), \text{ where } \int_0^T \beta_0(t)dt = 0.$$

We construct two homotopies to prove that model (6) shows at least a periodic solution. For this, we analyze a particular case of model (6) for which we construct a first homotopy. After that, we analyze the general case using a second homotopy and the result will be proved.

For the particular case, let $k = r_3 = 0$. Then, we propose the homotopy

$$\begin{aligned} S' &= -\eta I + (\mu + \eta)(1 - S) - \beta_\lambda S f(I), \\ I' &= \beta_\lambda S f(I) - (\gamma + \mu + r_1 + r_2) I. \end{aligned} \tag{8}$$

for $\lambda \in [0, 1]$ and $\beta_\lambda := \overline{\beta} + \lambda \beta_0(t)$.

Then, the following result will be proved.

Theorem 3.1. *Let $k = r_3 = 0$. If $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$, then there is at least one T-periodic orbit of model* (8) *whose components are positive.*

Theorem 3.1 will be proved using Leray-Schauder degree theory; see [17]. For this, the proof of Theorem 3.1 will start by introducing the following Banach spaces. Let

$$\mathcal{C}_T^l := \{(S, I) : S, I \in C^l(\mathbb{R}, \mathbb{R}), S(t+T) = S(t), I(t+T) = I(t)\}; \text{where } l = 0, 1.$$

Now, the homotopy (8) will be reformulated as a functional problem defined on the Banach spaces where periodic solutions correspond to the zeros of convenient operators.

Motivated by (8), the operators $L : \mathcal{C}_T^1 \to \mathcal{C}_T^0$ and $N_\lambda : \mathcal{C}_T^0 \to \mathcal{C}_T^0$ are defined by

$$L(S, I) := (S' + (\mu + \eta)S, I' + (\gamma + \mu + r_1 + r_2)I),$$

and

$$N_\lambda(S, I) := (-\eta I + (\mu + \eta) - \beta_\lambda S f(I), \beta_\lambda S f(I)).$$

Then, system (8) can be written as $L(S,I) = N_\lambda(S,I)$. Also, because L is invertible, the above equation can be rewritten as

$$F_\lambda(S,I) := (S,I) - L^{-1} \circ N_\lambda(S,I) = 0. \tag{9}$$

As $\mathcal{C}_T^1$ is compactly embedded in $\mathcal{C}_T^0$, the operator L^{-1} from $\mathcal{C}_T^0$ to $\mathcal{C}_T^0$ exists; therefore, $L^{-1} \circ N_\lambda : \mathcal{C}_T^0 \to \mathcal{C}_T^0$ is a compact operator. In a similar way, $F_\lambda : \mathcal{C}_T^0 \to \mathcal{C}_T^0$ is considered. Thus, equation (9) is a functional reformulation of system (8).

In the following, an open bounded subset on the Banach space must be constructed such that the family of operators does not support zeros over the boundary of such open set.

Consider the open sets

$$D := \{(S,I) \in \mathcal{C}_T^0 \ : \ S > 0, I > 0, S + I < 1\}$$

and

$$U := \{(S,I) \in D \ : \ \min_{[0,T]} S(t) < \delta\},$$

with $0 < \delta < 1$ to be fixed.

Next, we prove the following lemma:

Lemma 3.2. *If $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$, then for any $\lambda \in [0,1]$ there are no solutions (S,I) of (8) entirely contained in ∂U.*

Proof. First, choose δ such that $\frac{1}{\mathcal{R}_0} < \delta \frac{f_1(1)}{f_1(0)}$. Notice that $(S_0, I_0) = (1,0)$ is the unique solution of (8) entirely contained in ∂D for any $\lambda \in [0,1]$, i.e., there are no solutions different from $(S_0, I_0) = (1,0)$ that remain on the boundary of D for all time t.

Assume that $(S,I) \in \partial U$, then $(S,I) \notin \partial D$ so

$$(S,I) \in D \text{ and } S(t) \geq \delta, \ \forall t. \tag{10}$$

By integrating the second equation in system (8) as $\int_0^T \frac{I'}{I} dt = 0$ (we assume that I is T-periodic), it follows that

$$\gamma + \mu + r_1 + r_2 = \frac{1}{T} \int_0^T \beta_\lambda(t) f_1(I) S(t) dt. \tag{11}$$

By hypothesis iii) on f, $f_1(1) \leq f_1(I) \leq f_1(0)$, and using (10) one gets

$$\gamma + \mu + r_1 + r_2 = \frac{1}{T} \int_0^T \beta_\lambda(t) f_1(1) S(t) dt \geq \delta \overline{\beta} f_1(1). \tag{12}$$

Now, from the hypothesis of the lemma 3.2, it follows that

$$\frac{1}{T}\int_0^T \beta_\lambda(t) S f_1(1) dt \geq \delta \frac{f_1(1)}{f_1(0)} f_1(0)\overline{\beta} > \frac{f_1(0)}{\mathcal{R}_0}\overline{\beta} = \gamma + \mu + r_1 + r_2 \tag{13}$$

that is a contradiction.

Therefore, lemma 3.2 is proved. ☑

We now determine the Leray-Schauder degree, $deg(F_0, U)$ and prove the existence of periodic orbit.

Using analogous arguments to [8], the existence of a unique endemic equilibrium (S_1, I_1) in D can be proved. For $\lambda = 0$, system (8) has exactly two periodic orbits, namely, the following equilibrium points: $(1, 0)$ and (S_1, I_1). Where S_1 satisfies the following expression

$$S_1 = \frac{(\gamma + \mu + r_1 + r_2) I_1}{\bar{\beta} f(I_1)}. \tag{14}$$

Next, we will prove the following lemma.

Lemma 3.3. *Let U be an open set as the one defined above. Then $deg(F_0, U) \neq 0$.*

Proof. Since (S_1, I_1) is the unique solution of $F_0(S, I) = 0$ in U, then it suffices to prove that $DF_0(S_1, I_1)$ is invertible. Because F_0 is a compact perturbation of the identity, by the Fredholm alternative, it suffices to prove that

$$Ker(DF_0(S_1, I_1)) = \{0\}. \tag{15}$$

Consider $(V, W) \in \mathcal{C}_T^0$ such that $(V, W) \in Ker(DF_0(S_1, I_1))$. By the definition of F_0, $L(V, W) = DN_0(S_1, I_1)(V, W)$, where $N_0(S_1, I_1) = (-\eta I_1 + (\mu + \eta) - \bar{\beta} S_1 f(I_1), \bar{\beta} S_1 f(I_1))$. Then

$$DN_0(S_1, I_1)(V, W) = (-\eta V - \bar{\beta}(V f(I_1) + S_1 f'(I_1) W), \bar{\beta}(V f(I_1) + S_1 f'(I_1) W)).$$

By rewriting in matrix form the above equation, we obtain the following system.

$$\begin{pmatrix} V' \\ W' \end{pmatrix} = \begin{pmatrix} -(\mu + \eta + \bar{\beta} f(I_1)) & -\eta - \bar{\beta} S_1 f'(I_1) \\ \bar{\beta} f(I_1) & \bar{\beta} S_1 f'(I_1) - (\gamma + \mu + r_1 + r_2) \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}. \tag{16}$$

Equation (15) is fulfilled if and only if the unique periodic solution of the linear system given by (16) is the trivial one. Therefore, it would be enough to prove that its characteristic polynomial is Hurwitz. For this, notice that, the characteristic polynomial associated to (16) is given by

$$p(\lambda) = \lambda^2 - tr(B)\lambda + det(B), \tag{17}$$

where

$$B = \begin{pmatrix} -(\mu+\eta+\bar{\beta}f(I_1)) & -\eta-\bar{\beta}S_1f'(I_1) \\ \bar{\beta}f(I_1) & \bar{\beta}S_1f'(I_1)-(\gamma+\mu+r_1+r_2) \end{pmatrix}. \tag{18}$$

Keeping that in mind, the following result is proved. ☑

Lemma 3.4. *The characteristic polynomial given by (17) is Hurwitz.*

Proof. It is known that if $tr(B) < 0$ and $det(B) > 0$, then the characteristic polynomial is Hurwitz. From conditions i) and ii) we can establish

$$f(I_1) \geq I_1 f'(I_1). \tag{19}$$

Using (19) a direct calculation yields

$$\begin{aligned} tr(B) &= -(\mu+\eta)-\bar{\beta}f(I_1)-(\gamma+\mu+r_1+r_2)+\bar{\beta}S_1f'(I_1) \\ &= \frac{\bar{\beta}(\gamma+\mu+r_1+r_2)I_1f'(I_1)}{\bar{\beta}f(I_1)}-\bar{\beta}f(I_1)-(\mu+\eta)-(\gamma+\mu+r_1+r_2) \\ &\leq -\bar{\beta}f(I_1)-(\mu+\eta), \\ &< 0. \end{aligned}$$

On the other hand, the following result is obtained.

$$\begin{aligned} det(B) &= ((\mu+\eta)+\bar{\beta}f(I_1))(\gamma+\mu+r_1+r_2+\eta)-(\eta+\mu)(\eta+\bar{\beta}S_1f'(I_1)) \\ &\geq ((\mu+\eta)+\bar{\beta}f(I_1))(\gamma+\mu+r_1+r_2+\eta)-(\eta+\mu)(\eta+\gamma+\mu+r_1+r_2) \\ &> 0. \end{aligned}$$

Therefore, the inequalities $tr(B) < 0$ and $det(B) > 0$ are satisfied. So, the characteristic polynomial associated to the system (16) is Hurwitz. In particular, it does not have imaginary or null roots. Therefore, the linear system (16) has no periodic orbits different from the trivial solution.

From Lemma 3.4, we get that (15) is valid, hence $deg(F_0, U) \neq 0$. ☑

Proof of Theorem 3.1. Using the invariance of the Leray-Schauder degree under homotopy, by Lemma 3.2 and Lemma 3.3, we get $deg(F_1, U) \neq 0$. So the system (8) admits a non-trivial periodic solution, which proves Theorem 3.1. ☑

So far, we have analyzed a particular case of the general seasonal epidemiological model (6). Now, the will prove the result for the general case.

For $\tau \in [0, 1]$, we define the homotopy

$$\begin{aligned} S' &= -\eta I+(\mu+\eta)(1-S)-\beta(t)Sf(I), \\ I' &= \beta_\lambda Sf(I)-(\gamma+\mu)I-\tfrac{(r_1+r_2e^{-r_3\tau I})I}{1+k\tau I^n}. \end{aligned} \tag{20}$$

The operator $M_\tau : \mathcal{C}_T^0 \to \mathcal{C}_T^0$, which is needed for applying the Leray-Schauder theory, is given by

$$M_\tau(S,I) := \left(-\eta I + (\mu+\eta) - \beta(t)Sf(I), \beta(t)Sf(I) - \frac{(r_1 + r_2 e^{-r_3\tau I})I}{1+k\tau I^n}\right).$$

Therefore system (20) becomes $L(S,I) = M_\tau(S,I)$ where L is as defined earlier. Consider

$$H_\tau(S,I) := (S,I) - L^{-1} \circ M_\tau(S,I). \tag{21}$$

Thus, (21) is a functional reformulation of the problem (20). In particular, periodic solutions of (20) correspond to zeros of H_τ. Notice that, $H_0 = F_1$, therefore $deg(H_0, U) \neq 0$. Recall that the existence of a solution for H_1 in U is guaranteed via Leray-Schauder degree if both $deg(H_0, U) \neq 0$ and H_τ is an admissible homotopy i.e. $0 \notin H_\tau(\partial U), \forall \tau \in [0,1]$. So, it is necessary to establish that H_τ is an admissible homotopy.

Lemma 3.5. *If $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$, then for any $\tau \in [0,1]$ there are no solutions (S,I) of (20) on the boundary of the set U.*

Proof. Observe that if $(S,I) \in \partial U$, then $(S,I) \notin \partial D$. Therefore,

$$(S,I) \in D \text{ and } S(t) \geq \delta, \ \forall t. \tag{22}$$

Dividing by I and simplifying the second equation in system (20), we obtain the expression

$$\frac{I'}{I} = \frac{\beta(t)Sf(I)}{I} - (\gamma+\mu) - \frac{(r_1 + r_2 e^{-r_3\tau I})}{1+k\tau I^n} \geq \beta(t)Sf_1(I) - (\gamma+\mu+r_1+r_2). \tag{23}$$

Since I is T-periodic then $\int_0^T \frac{I'}{I}dt = 0$, it follows that

$$\gamma + \mu + r_1 + r_2 = \frac{1}{T}\int_0^T \geq \beta(t)Sf_1(I). \tag{24}$$

Now, from the condition $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$ and the fact that f_1 is decreasing, we get

$$\gamma+\mu+r_1+r_2 \geq \frac{1}{T}\int_0^T \beta(t)Sf_1(1)dt \geq \delta f_1(1)\overline{\beta} > \frac{f_1(0)}{\mathcal{R}_0}\overline{\beta} \geq \gamma+\mu+r_1+r_2, \tag{25}$$

which is a contradiction. Therefore, the lemma is proved. ☑

Recapitulating, from the previous lemma, H_τ is an admissible homotopy since $H_0 = F_1$, then $deg(H_0, U) \neq 0$ (see proof of Theorem 3.1) and thus $deg(H_1, U) \neq 0$. Therefore, the Leray-Schauder degree system (6) admits a non-trivial periodic solution. Finally we obtain

Theorem 3.6. *If $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$, then there is at least one T-periodic orbit of* (6) *whose components are positive.*

Notice that, Theorem 3.6 is associated to the two dimensional model given by (6). Therefore, there is at least one positive T-periodic solution $(S(t), I(t), R(t))$ for system (3) such that $N = 1$.

4. Numerical simulations

In this section, we illustrate that model (3) admits a limit cycle for different functions that model the infection force and the treatment rate. To do this, we use the software Mathematica 11. The values of the parameters of the model are chosen so that the conditions in Theorem 3.6 are satisfied. To exemplify seasonal effects in the spreading of the infectious disease, we use the periodic function

$$\beta(t) \ = \ 10 \ (1 \ + \ 0.5\cos(2\pi t)) \tag{26}$$

in all examples shown in this section.

Example 4.1. In this scenario, we use the factor $f_1(I) = \frac{1}{1+\alpha I^2}$ in the incidence of the disease. Therefore, the number of new infections is given by $\beta(t)Sf(I) = \beta(t)SIf_1(I) = \frac{\beta(t)}{1+\alpha I^2}SI$. In this scenario, we can interpret that the infection rate, $\beta(t)$, strongly decreases as a function of the number of infectious individuals. Therefore, the model is given by

$$\begin{aligned} S' &= \eta R + \mu(1-S) - \beta(t)S\tfrac{I}{1+\alpha I^2}, \\ I' &= \beta(t)S\tfrac{I}{1+\alpha I^2} - (\gamma+\mu)I - \tfrac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}, \\ R' &= \gamma I - (\mu+\eta)R + \tfrac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}. \end{aligned} \tag{27}$$

For the numerical simulation, the values of the parameters are $\eta = 0.2$, $\mu = 0.0001$, $\gamma = 0.9$, $k = 0.001$, $r_1 = 0.2$, $r_2 = 0.3$, $r_3 = 0.4$, $\alpha = 0.01$ and $n = 2$. With these values of the parameters of the model, $R_0 = 7.1423$ and $\frac{f_1(0)}{f_1(1)} = 0.99$. Then, the condition $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$ is satisfied. Therefore, the model (27) admits at least one T-periodic orbit with positive components; see Figure 1.

To exemplify that model (27) admits periodic orbits for $f_1(I) = \frac{1}{1+\alpha I^2}$ and other treatment functions, we show an scenario in which $T(I) = \frac{0.2I}{1+0.001I^2}$. In this scenario, $r_2 = 0$, in $\frac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}$, and all other parameters of the model are the same as the values used in the first case. Then, $R_0 = 9.0901$. Notice that the right hand side of the condition $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$ does not depend on the parameter r_2. Particularly, $\frac{f_1(0)}{f_1(1)} = 0.99$. Therefore, the conditions of Theorem 3.6 are satisfied in this scenario. Then, model (27) admits at least one T-periodic solutions with positive components. Figure 2 shows the dynamics of the infectious class when $r_2 > 0$ and $r_2 = 0$.

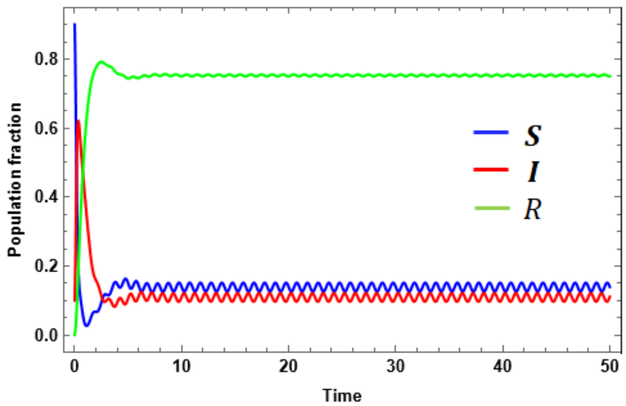

FIGURE 1. Figure shows the existence of a periodic solution of the model (27) when $f_1(I) = \frac{1}{1+\alpha I^2}$ and $T(I) = \frac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}$. For the numerical simulations the initial conditions are given by $S_0 = 0.9$, $I_0 = 0.1$ and $R = 0$.

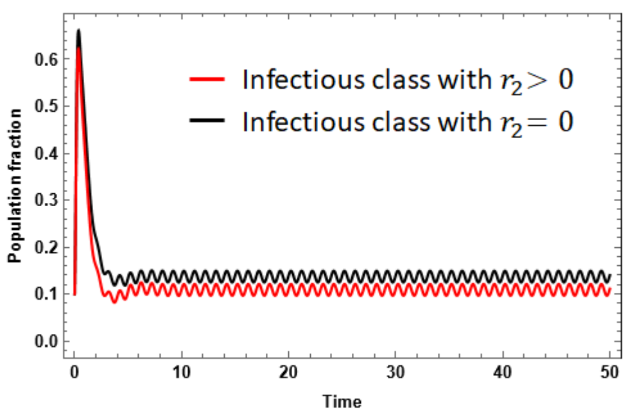

FIGURE 2. Figure shows the dynamics of the infectious class when $f_1(I) = \frac{1}{1+\alpha I^2}$ and $T(I) = \frac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}$ or $T(I) = \frac{r_1 I}{1+kI^n}$. The initial conditions are given by $S_0 = 0.9$, $I_0 = 0.1$ and $R = 0$.

Example 4.2. In this example, we use the function $f_1(I) = e^{-\alpha I}$ as a factor in the incidence of the infectious disease. In this scenario, the new infections are modeled by $\beta(t)Sf(I) = \beta(t)SIf_1(I) = \beta(t)e^{-\alpha I}SI$. Observe that the infection rate $\beta(t)e^{-\alpha I}$ decreases when the number of infectious individuals increases. In this scenario, the seasonal model is

$$\begin{aligned} S' &= \eta R + \mu(1-S) - \beta(t)SIe^{-\alpha I}, \\ I' &= \beta(t)SIe^{-\alpha I} - (\gamma+\mu)I - \tfrac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}, \\ R' &= \gamma I - (\mu+\eta)R + \tfrac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}. \end{aligned}$$

For the numerical simulations, all values of the parameters of the model are the same as the values used in the Example 1. In this example, $R_0 = 7.1423$ and $\frac{f_1(0)}{f_1(1)} = 1.01$. So, the conditions of the Theorem 3.6 are satisfied. Therefore, model (27) with the proposed function $f_1(I) = e^{-\alpha I}$ admits at least a T-periodic orbit. Figure 3 shows the dynamics of the epidemiological classes in this scenario.

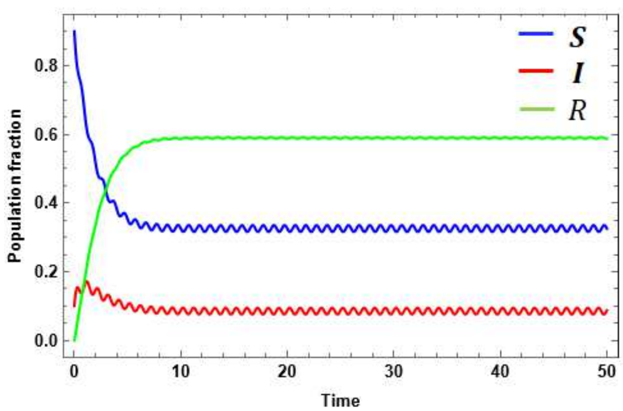

FIGURE 3. Figure shows the existence of sustained oscillations for the epidemiological model given by (27) when $f_1(I) = e^{-\alpha I}$ and $T(I) = \frac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}$. For the numerical simulations, the initial conditions are given by $S_0 = 0.9$, $I_0 = 0.1$ and $R = 0$.

Now, we consider the case where the number of treated individuals is modeled by the treatment rate $T(I) = \frac{0.2I}{1+0.001I^2}$. To do this, we take $r_2 = 0$ in $\frac{(r_1+r_2e^{-r_3 I})I}{1+kI^n}$. In this case, $R_0 = 9.0901$ and $\frac{f_1(0)}{f_1(1)} = 1.01$. Then, model (27) with $f_1(I) = e^{-\alpha I}$ and $T(I) = \frac{0.2I}{1+0.001I^2}$ admits at least a feasible T-periodic solution. Figure 4 shows the behavior of the number of infectious individuals when $r_2 > 0$ and $r_2 = 0$.

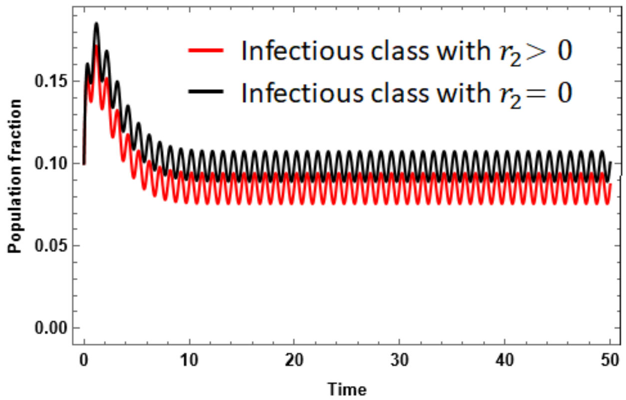

FIGURE 4. Figure shows the behavior of the infectious class when $f_1(I) = e^{-\alpha I}$ and $T(I) = \frac{(r_1 + r_2 e^{-r_3 I})I}{1+kI^n}$ or $T(I) = \frac{r_1 I}{1+kI^n}$. In this scenario, $S_0 = 0.9$, $I_0 = 0.1$ and $R = 0$.

Numerical simulations show that sustained oscillations can appear as solutions of the proposed model for a wide variety of incidence and treatment rates. However, periodic solutions can be avoided if $R_0 < 1$.

5. Conclusions

Modeling the dynamics of infectious diseases can help to make decisions in designing public health strategies. In this direction, it is of paramount importance to know the evolution of infectious diseases when there are environmental or social seasonal drivers. To do this, in the modeling process, it must be considered how the number of new infectious individuals are affected by seasonally. This allows decision makers to propose strategies to control the spreading of the disease.

In epidemiological mathematical literature there are models that describe the incidence of an infectious disease using different functions. In these models, the infection rate can be either a constant or a periodic function; see [6, 16, 21, 20, 9, 15, 2, 19]. Also, there are epidemic models that analyze how the dynamics of an infectious disease can be controlled when infectious individuals are treated against the infection. To do this, the per-capita treatment rate is modeled by different functions that describe a wide variety of social or epidemiological mechanisms; see [7, 10, 11, 23, 30, 27, 31, 26].

In this work, we analyze a general epidemic model that describes a seasonal infectious disease when infectious individuals recover from the disease in a natural way or due to the application of a treatment. For this purpose, we modeled the number of new infectious individuals assuming a general force of infection and a treatment function that generalize some recovery functions that are commonly used.

We proved that periodic solutions exist for a wide variety of $SIRS$ models when a seasonal general force of infection is assumed even though a treatment is applied to infectious individuals. Figures 1-4 show numerical solutions of model (27) when the incidence of the disease is modeled by two different functions and a general treatment function is considered. Particularly, Figures 2 and 4 compare the effects in the number of infectious individuals, that are modeled by the same incidence function, when different treatment rates are used. That is, the application of a treatment to infectious individuals does not avoid undesirable epidemic scenarios such as periodicity. If periodical scenarios want to be excluded, the basic reproduction number must be small enough such that the condition $\mathcal{R}_0 > \frac{f_1(0)}{f_1(1)}$ is not satisfied.

Sustained oscillations in the incidence of the disease can be catastrophic for the population because if the amplitude of the oscillation is large enough, then the number of infectious individuals can increase suddenly until it reaches a critical population threshold. In this work, we proved that, even though a treatment is applied, periodicity persists although the infection rate decreases due to behavioral changes in the susceptible class as a function of the infectious disease. That is, the model admits periodic orbits despite the infection rate decreasing. This decreasing might occur because the susceptible population is now a more cautious population since there are more infectious individuals and there are recovered individuals. In contrast, periodic solutions can be avoided if some conditions on the parameters of the model are satisfied. Therefore, the results obtained in this work can help the health decision makers in the design of public health strategies to control an infectious disease. However, it is of paramount importance to continue with the modeling of seasonal effects in the evolution of infectious diseases, particularly, when changes in environmental or social drivers lead to an asynchronicity that affects the seasonal infection rate making it more difficult to know the evolution of an infectious disease and, consequently, control it.

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