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The Exotic World of Milnor's Spheres

El exótico mundo de las esferas de Milnor

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Abstract. In his celebrated article of 1956, John Milnor established the existence of smooth structures on the 7-dimensional sphere that differs from the usual one. These so-called "exotic" structures have been of great interest ever since. The purpose of this article is to give a clear exposition of the different tools that Milnor used in order to provide an almost self-contained construction of exotic structures on the 7-dimensional sphere and then to show that they are not diffeomorphic to the standard sphere.

Key words and phrases. sphere, signature, characteristic classes.

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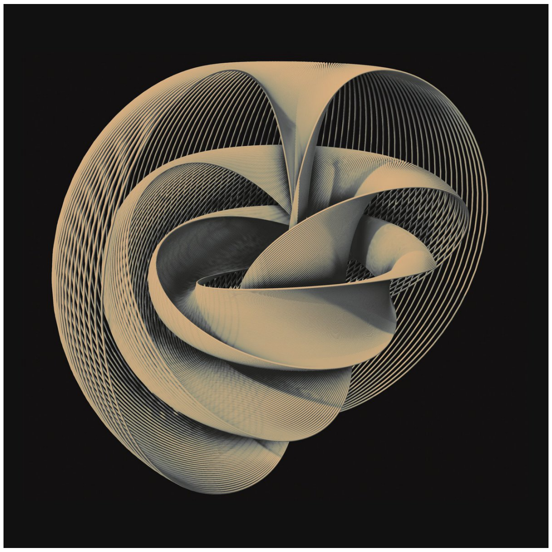
Resumen. En su célebre artículo de 1956, John Milnor estableció la existencia de estructuras suaves en la esfera de dimensión 7 que difieren de la estructura usual. Éstas son llamadas estructuras "exóticas" las cuáles desde entonces fueron de gran interés. El propósito del presente artículo es dar una clara exposición de las diferentes herramientas que Milnor usó para ofrecer una construcción de las esferas exóticas en dimensión 7 y verificar que no son difeomorfas a la esfera usual.

Palabras y frases clave. esfera, signatura, clases características.



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I found I could actually prove that it was homeomorphic to the standard 7−sphere, which made the situation seem even worse!
John Milnor

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[1] Image by Peder Norrby 'ALGOMYSTIC', 'Hopf Fibrations'

1. Introduction

1.1. Spheres and their topological invariants

Among the most classic objects in mathematics are the spheres. As far back as Ancient Greece, the unit circle can be described, although perhaps not in this language, as the set of pairs of real numbers (x, y) that satisfy the equation

$$x^2 + y^2 = 1\,.$$

As we increase the dimension, the sphere consists of all triples $(x, y, z) \in \mathbb{R}^3$ satisfying $x^2+y^2+z^2 = 1$. Continuing with this process, we define the n-sphere as the set of all $(n + 1)$-tuples $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$ that satisfy

$$\sum_{i=1}^{n+1} x_i^2 = 1\,.$$

After the development of concepts such as topology and differential topology, it became clear that spheres had the structure of a topological manifold and, even more, of a smooth manifold. In a certain way, this structure comes from the ambient space.

At the turn of the nineteenth century, Henri Poincaré appeared as one of the protagonists, or more precisely, as the founder of algebraic topology.

FIGURE 1. Henri Poincaré.

He constructed two different invariants associated with manifolds. The first is known as the *fundamental group*, which would later be generalized to the so-called *homotopy groups*. These groups are based on the idea of measuring holes through the obstruction of contracting spheres to a point. The second is the *homology groups* first defined as formal sums of submanifolds up to bounding a higher-dimensional manifold (nowadays, these groups are called *bordism* groups). Nevertheless, an adequate description of homology groups is in terms of triangulations of manifolds. These groups measure the obstruction of a triangle to be the boundary of a higher-dimensional triangle.

Although both invariants look similar at first glance, they are different in calculation complexity, among other properties. Poincaré formulated his first conjecture: if a closed, connected manifold has the same homology groups as the sphere, then it is, in fact, a topological sphere. He gave a counterexample for this conjecture, which nowadays is known as "Poincaré's sphere". Then he formulated a second version of this conjecture, which states that if a closed connected manifold has the same homotopy groups as the sphere, it must be a topological sphere. This was known as *Poincaré's conjecture* until Perelmán came up with the proof.

This question motivated an essential part of the development of mathematics during the twentieth century. At least three Fields medals were attributed to the progress of the Poincaré conjecture (Smale, Freedman, and Perelman). Within the most indirect consequences of this fructiferous program is the heart of the present article, the exotic spheres.

The state of topology during the 1950's

As we mentioned, during the development of point-set theory and differential topology, it became apparent that spheres with their standard structure were not only manifolds but also inherited a smooth structure from the ambient space.

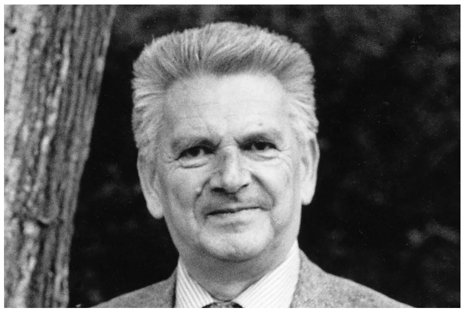

FIGURE 2. René Thom.

The 1950s were quickly marked by the influence of René Thom. His famous *isomorphism theorem* allowed the coherent formulation and proof of various important results and the construction of various new objects. Foremost among them are the topological construction of Chern classes and a description of the bordism ring. This gave mathematicians powerful tools, sometimes becoming the missing piece in their projects.

This was the case of the German mathematician Friedrich Hirzebruch. The story tells that when a new note of Thom came to the institute's library where Hirzebruch was working, it took him a few seconds to complete the proof of the *signature theorem*. This theorem relates two invariants that seemed quite different. On the one hand, the signature of a manifold, a topological index linked to the cohomology of the underlying space, and on the other hand, the Pontryagin classes, which capture the differentiable structure of the space. The equation that would pass to history because of its relevance in Milnor's work takes the form

$$\sigma(M) = \frac{1}{45}\left(7p_2(M) - p_1^2(M)\right).$$

1.2. The road of John Milnor

During the year 1956, at an early age, John Milnor worked on studying the topological invariants of some well-known manifolds. In his own words: "The generalized Poincaré problem of understanding such manifolds seemed too difficult: I had no idea how to get started". He restricted his attention to simpler manifolds: closed $2n$-dimensional manifolds which were $(n-1)$-connected.

Thanks to a paper by Smale and Wall, there was a relatively simple description for $n > 2$.

FIGURE 3. John Milnor.

Indeed, since these spaces have a simple cohomological structure. Milnor further reduced their description to some particular spaces constructed as sphere bundles over the fourth-dimensional sphere. Thanks to Steenrod's work, it is possible to classify all such bundles, and in some cases using a Morse-theoretic argument, namely Reeb's theorem, it is possible to show that their total space is homeomorphic to the 7-sphere. On the other hand, assuming they were diffeomorphic to the sphere, Milnor reached a contradiction with Hirzebruch's formula: he found rational values for an integer value! In conclusion: these spaces were topologically spheres, but their smooth structure did not match the standard one. This was unexpected since the belief was that spheres had a single smooth structure, which was misleading.

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2. Preliminaries

Throughout this paper, we assume that all manifolds are smooth, meaning that the transition maps are $\mathcal{C}^\infty$.

2.1. The signature of a manifold

In this section, we work with rational coefficients. Let M be a connected, oriented, closed $2n$-dimensional manifold. Choose the generator of $H^{2n}(M;\mathbb{Q})$ as the *fundamental class* of M, denoted by $[M]$. The cup product in cohomology induces a bilinear map

$$\omega : H^n(M;\mathbb{Q}) \otimes H^n(M;\mathbb{Q}) \longrightarrow \mathbb{Q}$$

defined by

$$\omega : (\alpha, \beta) \mapsto \langle \alpha \smile \beta, [M] \rangle$$

where $\langle \ , \ \rangle$ denotes the pairing between homology and cohomology.

***Remark* 2.1.** Recall that the cup product is graded-commutative, that is

$$\alpha \smile \beta = (-1)^{pq} \beta \smile \alpha$$

where $\alpha \in H^p(M;\mathbb{Q})$ and $\beta \in H^q(M;\mathbb{Q})$. In particular, for n even ω is symmetric, and for n odd ω is anti-symmetric.

Since $H^n(M;\mathbb{Q})$ is finitely generated, we can represent ω by a square matrix which will be symmetric or anti-symmetric depending on the parity of n.

If n is even, that is M is $4k$-dimensional, then the spectral theorem guarantees the existence of real eigenvalues. We define the *signature* of ω as

$$\operatorname{sign}(\omega) = \#\text{positive eigenvalues} - \#\text{negative eigenvalues}$$

Then we define the *signature* of a manifold M, denoted $\sigma(M)$, as the signature of the associated ω. Note that the signature is always, by definition, an integer.

2.2. Basic properties of the signature

Let us study the behavior of the signature under different operations on manifolds.

First, if we change the orientation of M by $[-M] = -[M]$, then the signature of $-M$ is given by the bilinear form

$$\tilde{\omega}(\alpha, \beta) = \langle \alpha \smile \beta, [-M] \rangle = -\langle \alpha \smile \beta, [M] \rangle = -\omega(\alpha, \beta)\,.$$

Thus the eigenvalues of $\tilde{\omega}$ are those of ω with opposite signs and it follows that $\sigma(-M) = -\sigma(M)$.

Now if we consider the disjoint union of two manifolds $M \sqcup N$, the fundamental class corresponds to the sum of the fundamental classes $[M \sqcup N] = [M] + [N]$. Then the bilinear form associated with the disjoint union is the direct sum of the bilinear forms and $\sigma(M \sqcup N) = \sigma(M) + \sigma(N)$.

Furthermore, we have the following result.

Proposition 2.2. *The signature is a bordism invariant.*

To prove this statement, we will need the next lemma.

Lemma 2.3. *If $\omega : V \times V \longrightarrow \mathbb{Q}$ is a non-degenerate symmetric bilinear form with a subspace W of dimension $\frac{1}{2}\dim V$ such that the restriction of ω to W is identically zero, then the signature of ω is zero. This subspace is called* isotropic *or* Lagrangian.

Proof. The idea is to find a basis of V to 'cancel out' the eigenvalues.

Let $e_1 \in W$ be a non-zero element. Since ω is non-degenerate, there exists $f_1 \in V$ such that $\omega(e_1, f_1) = 1$. If $\omega(f_1, f_1) \neq 0$ we may replace f_1 by $\tilde{f_1} := f_1 - \frac{1}{2}\omega(f_1, f_1)e_1$. Note that $\omega(e_1, \tilde{f_1}) = 1$ and

$$\begin{aligned}\omega(\tilde{f_1}, \tilde{f_1}) &= \omega(f_1, f_1) - \omega(f_1, f_1)\omega(e_1, f_1)\\ &= \omega(f_1, f_1) - \omega(f_1, f_1)\\ &= 0\,.\end{aligned}$$

Thus we assume without loss of generality that $\omega(f_1, f_1) = 0$. Set $S = \mathrm{Span}(e_1, f_1)$. Restricted to S, ω is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has zero signature. Consider the subspace $V_1 = S^{\perp}$. Since ω is non-degenerate, we have that $V = S \oplus V_1$. Defining $W_1 = W \cap V_1$ we have that W_1 has dimension $\frac{1}{2}\dim V_1$ and the restriction of ω to W_1 is identically zero. By induction, we apply the hypothesis to V_1, which has dimension $\dim(V) - 2$; hence the signature of ω is zero. ☑

Now we show the bordism invariance of the signature.

Proof of proposition 2.2. Assume that the $4k$-dimensional manifold M is the boundary of a $(4k+1)$-dimensional manifold W. We denote by $\iota : M \hookrightarrow W$ the inclusion. Using the long exact sequence of the pair and the Poincaré duality, we have the following commutative diagram

$$\begin{array}{ccccc} H^{2k}(W;\mathbb{Q}) & \xrightarrow{\iota^*} & H^{2k}(M;\mathbb{Q}) & \longrightarrow & H^{2k+1}(W,M;\mathbb{Q}) \\ \downarrow D & & \downarrow D & & \downarrow D \\ H_{2k+1}(W,M;\mathbb{Q}) & \longrightarrow & H_{2k}(M;\mathbb{Q}) & \xrightarrow{\iota_*} & H_{2k}(W;\mathbb{Q}) \end{array} \tag{1}$$

where D is the Poincaré isomorphism. The image of ι^* in $H^{2k}(M;\mathbb{Q})$, is a subspace and we claim it is isotropic.

First, the restriction of ω to this subspace is zero,

$$\begin{aligned}\omega(\iota^*(\alpha),\iota^*(\beta)) &= \langle \iota^*(\alpha)\smile\iota^*(\beta),[M]\rangle\\ &= \langle \iota^*(\alpha\smile\beta),\partial[W]\rangle\\ &= \langle \alpha\smile\beta,\iota_*\partial[W]\rangle\\ &= 0\,.\end{aligned}$$

Where we used that the composition $\iota^*\partial$ is zero in the long exact sequence of a pair. This subspace has the half dimension of $H^{2k}(M;\mathbb{Q})$ since

$$\begin{aligned}x\in(\operatorname{im}\iota^*)^\perp &\Leftrightarrow \langle x\smile\iota^*(y),[M]\rangle = 0\ \ \forall y\in H^{2k}(W;\mathbb{Q})\\ &\Leftrightarrow \langle \iota^*(y),[M]\frown x\rangle = \langle\iota^*(y),D(x)\rangle = 0\ \ \forall y\in H^{2k}(W;\mathbb{Q})\\ &\Leftrightarrow \langle y,\iota_*(D(x))\rangle = 0\ \ \forall y\in H^{2k}(W;\mathbb{Q})\\ &\Leftrightarrow \iota_*D(x)=0\\ &\Leftrightarrow D(x)\in\ker\iota_*\,.\end{aligned}$$

On the other hand, we know that

$$\dim(\operatorname{im}\iota^*) + \dim\,(\operatorname{im}\iota^*)^\perp = \dim H^{2k}(M;\mathbb{Q})$$

and because D maps $(\operatorname{im}\iota^*)^\perp$ isomorphically onto $\ker\iota_*$ we can replace the previous equation by

$$\dim\operatorname{im}\iota^* + \dim\ker\iota_* = \dim H^{2k}(M;\mathbb{Q})\,.$$

However, the commutativity of the diagram (1) together with the exactness of the rows imply that D maps $\operatorname{im}\iota^*$ isomorphically onto $\ker\iota_*$. We conclude that

$$\dim\operatorname{im}\iota^* + \dim\operatorname{im}\iota^* = \dim H^{2k}(M;\mathbb{Q})\,.$$

Thus $\operatorname{im}\iota^*$ is an isotropic subspace of $H^{2k}(M;\mathbb{Q})$ of half dimension and by Lemma 2.3, we conclude that $\sigma(M)=0$. ☑

***Remark* 2.4.** Consequently, if two manifolds M and N are equivalent in oriented bordism, they have the same signature. More precisely, denote by W the oriented bordism with $\partial W = M\sqcup -N$, by the previous statements we obtain

$$\sigma(\partial W) = \sigma(M\sqcup -N) = \sigma(M)-\sigma(N) = 0\,.$$

For the product of two manifolds $M\times N$, the signature $\sigma(M\times N)$ uses Künneth's formula

$$H^*(M\times N;\mathbb{Q})\cong H^*(M;\mathbb{Q})\otimes H^*(N;\mathbb{Q})\,.$$

If M is $4k$-dimensional and N is $4l$-dimensional, then ω is a bilinear form on the space

$$\bigoplus_{i+j=2(k+l)} H^i(M;\mathbb{Q}) \otimes H^j(N;\mathbb{Q})\,,$$

which decomposes as the direct sum

$$\left(H^{2k}(M;\mathbb{Q}) \otimes H^{2l}(N;\mathbb{Q})\right) \oplus \bigoplus_{\substack{i+j=2(k+l)\\ i\neq 2k}} H^i(M;\mathbb{Q}) \otimes H^j(N;\mathbb{Q})\,.$$

Notice that the cup product of an element in the first summand with an element in the second summand is trivial [2]. Thus the bilinear form ω is the direct sum of its restriction to each summand. Furthermore, the second summand has an isotropic subspace, and hence the only contribution to the signature is given by the restriction of ω to $H^{2k}(M;\mathbb{Q}) \otimes H^{2l}(N;\mathbb{Q})$. However, the bilinear form restricted to this subspace is the tensor product of the bilinear forms of the factors. Thus the eigenvalues of the original bilinear form correspond to the product of the eigenvalues of the bilinear forms on each of the factors. Therefore, the signature is multiplicative in the sense that $\sigma(M\times N) = \sigma(M)\sigma(N)$.

We illustrate this fact with an example. For the sake of simplicity, in our notation, we will omit the coefficients $\mathbb{Q}$ (only for this example).

Example 2.5. For $M = N = \mathbb{CP}^4$, we apply the Künneth formula, and we get

$$H^*(\mathbb{CP}^4 \times \mathbb{CP}^4) = H^*(\mathbb{CP}^4) \otimes H^*(\mathbb{CP}^4)\,.$$

Since we know $H^*(\mathbb{CP}^4) = \mathbb{Q}[x]/(x^5)$, we have

$$H^*(\mathbb{CP}^4) \otimes H^*(\mathbb{CP}^4) \cong \mathbb{Q}[x,y]/(x^5,y^5)\,.$$

In particular, $H^8(\mathbb{CP}^4 \times \mathbb{CP}^4)$ is generated by $x^2y^2, x^3y, y^3x, x^4, y^4$. Moreover, the subspace generated by x^2y^2 corresponds to $H^4(\mathbb{CP}^4) \otimes H^4(\mathbb{CP}^4)$. In other words, we have a decomposition

$$H^8(\mathbb{CP}^4 \times \mathbb{CP}^4) = \left(H^4(\mathbb{CP}^4) \otimes H^4(\mathbb{CP}^4)\right) \oplus \left(\bigoplus_{\substack{i+j=8\\ i,j\neq 4}} H^i(\mathbb{CP}^4) \otimes H^j(\mathbb{CP}^4) \right).$$

The second summand, say V, is equal to the subspace generated by x^3y, y^3x, x^4, y^4. and consider $W = \mathrm{Span}(y^3x, y^4)$ inside this subspace. We observe that $\dim W = \frac{1}{2}\dim V$ and that the restriction of ω to this subset is zero. Indeed, the product of any two generators in W has a power of y exceeding 5, thus

[2] This follows from a dimension argument where the product exceeds the manifold's dimension.

is trivial. Since W is an isotropic subspace of V of the half dimension, the signature of ω restricted to V is zero. Hence the signature depends only on the factor

$$\omega' : \big(H^4(\mathbb{CP}^4) \otimes H^4(\mathbb{CP}^4)\big) \otimes \big(H^4(\mathbb{CP}^4) \otimes H^4(\mathbb{CP}^4)\big) \longrightarrow \mathbb{Q}\,.$$

But the properties of the cup product imply that ω' is given by

$$\begin{aligned}\omega'(\alpha \otimes \beta, \alpha' \otimes \beta') &= \langle \alpha \smile \alpha' \otimes \beta \smile \beta', [M] \otimes [N]\rangle \\ &= (-1)^{4kl} \langle \alpha \smile \alpha', [M]\rangle \langle \beta \smile \beta', [N]\rangle \\ &= \omega_1(\alpha, \alpha')\omega_2(\beta, \beta')\,.\end{aligned}$$

where ω_i is the bilinear form on M and N respectively. Therefore, ω' is the *tensor product* or *Kronecker product* of the bilinear forms. The eigenvalues correspond to the product of the eigenvalues of each factor, and the product of the signatures gives the signature of ω'.

We summarize our discussion so far in the following theorem.

Theorem 2.6. *he signature σ satisfies the following properties:*

(1) $\sigma(-M) = -\sigma(M)$,

(2) $\sigma(M \sqcup N) = \sigma(M) + \sigma(N)$,

(3) If M and N are the same in bordism, then $\sigma(M) = \sigma(N)$,

(4) $\sigma(M \times N) = \sigma(M)\sigma(N)$.

We remind the reader that the signature is only defined for $4k$-dimensional manifolds.

We conclude this section with a couple of examples, the first of which is of great importance in this work.

Example 2.7. The signature of $\mathbb{CP}^{2l}$.

The computation can be carried out algebraically. We remind the reader that the cohomology ring of $\mathbb{CP}^{2l}$ can be described by polynomials in one variable of degree at most $2l$, i.e., $H^*(\mathbb{CP}^{2l}; \mathbb{Q}) = \mathbb{Q}[x]/(x^{2l+1})$, and under this identification the cup product corresponds to polynomial multiplication. Hence the bilinear form ω is given in terms of the generators by $\omega(x^l, x^l) = \langle x^{2l}, [\mathbb{CP}^{2l}]\rangle = 1$.

We also give a geometric argument. We recall the cell structure of the complex projective space, where the generator of $H^l(\mathbb{CP}^{2l}; \mathbb{Q})$ is dual to the projective subspace $\mathbb{CP}^l \subset \mathbb{CP}^{2l}$. Because of the duality of the cup product and the cap product, the signature is calculated by the self-intersection number of this cell (for more information on intersection theory, we refer the reader to [1]). Consider the usual embedding of $\mathbb{CP}^l$ in $\mathbb{CP}^{2l}$, given by

$$E = [z_1 : z_2 : \cdots : z_l : z_{l+1} : 0 : \cdots : 0] .$$

We can deform this subspace into the following one, where we are going to calculate the intersection,

$$L = \{[z_1 : \cdots : z_{2l+1}] \mid z_{2l+1} = z_1 + \cdots + z_l - z_{l+1}, z_{2l} = z_1 + \cdots + z_{l-1} - z_l \\ \ldots, z_{l+2} = z_1 - z_2\} .$$

Notice that L is the zero-set of a set of homogeneous polynomials, so L is well-defined. Even more, L is the intersection of codimension-increasing planes, and therefore, L is the set of lines through the $(l+1)$-space, i.e. $L \cong \mathbb{CP}^l$. The intersection $E \cap L$ is described by elements of the form $[z_1 : \cdots : z_{l+1} : 0 : \cdots : 0]$ subject to the conditions

$$\begin{cases} z_1 = z_2 \\ z_1 + z_2 = z_3 \\ \vdots \\ z_1 + \cdots + z_l = z_{l+1} \end{cases}$$

hence $z_1 = z_2$, $z_3 = 2z_1$, $z_4 = 6z_1$ and so on. This set consists of a single line; in other words, $E \cap L$ consists of a single element. Moreover, L is homotopic to E by multiplication of each defining polynomial of L by a parameter t. As a consequence, we conclude that the self-intersection number of E is precisely 1, and it follows that $\sigma(\mathbb{CP}^{2l}) = 1$.

Example 2.8. The signature of S^4: the group $H^2(S^4;\mathbb{Q})$ is zero, hence the bilinear form ω is null and $\sigma(S^4) = 0$.

2.3. Characteristic classes

In this section, we review some basic properties of characteristic classes; for a profound and complete exposition, the reader can consult the book of Milnor [9].

We start with a motivation: a vector bundle over a space X consists of a topological space E and a continuous projection $\pi : E \longrightarrow X$ such that each fiber $\pi^{-1}(x)$, for each $x \in X$, has the structure of a vector space. Moreover, they are locally trivial, meaning that for each $x \in X$, we can find an open neighborhood where the restriction is a trivial bundle (a trivial bundle is one of the form $E = M \times \mathbb{R}^k$ and $\pi = p_1$) and the change of coordinates are linear isomorphisms. A vector bundle can be understood as a continuous way of attaching to each point x an n-dimensional vector space. We are interested in a way to classify all vector bundles, but to achieve this, we need to introduce an essential space which we specify in the following paragraph.

The space of n-dimensional planes in $\mathbb{R}^{n+k}$, denoted by $G_n(\mathbb{R}^{n+k})$, is known as the *Grassmannian.* This space has a topological structure induced by the Gaussian elimination on $n\times(n+k)$ matrices with rank n. Thus the dimension of the Grassmannian $G_n(\mathbb{R}^{n+k})$ is kn. Moreover, there is an n-dimensional vector bundle γ_n^k over $G_n(\mathbb{R}^{n+k})$, with total space

$$E(\gamma_n^k) := \{(v, P) \mid P \in G_n(\mathbb{R}^{n+k}) \text{ and } v \in P\}.$$

This vector bundle is known as the *canonical bundle.* The Grassmannian is of great importance since every smooth manifold X with an embedding into $\mathbb{R}^{n+k}$ admits a Gauss map $f : X \to G_n(\mathbb{R}^{n+k})$ which maps each point to its tangent space. This is as follows:

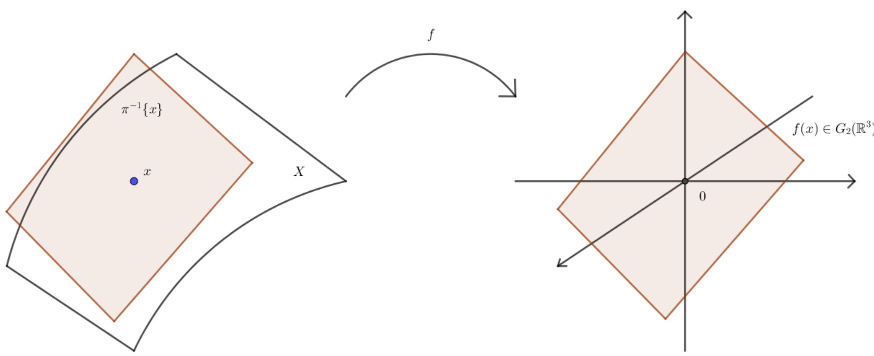


FIGURE 4. The map f associates to each point its corresponding tangent space. In this picture, X is a surface in $\mathbb{R}^3$ and the tangent bundle is of dimension 2.

Notice that the map f is smooth. However, the definition of the Gauss map for an arbitrary vector bundle needs further work using the local trivializations; such construction is explained in full detail in [9]. Now, we can increase k in $G_n(\mathbb{R}^{n+k})$ and take the limit to infinity, and we obtain the infinite Grassmannian

$$G_n := \lim_{k\to\infty} G_n(\mathbb{R}^{n+k}),$$

where G_n has the topology induced by the direct limit of the finite-dimensional Grassmannians. The infinite Grassmannian also inherits a canonical bundle built in a similar way as in the finite-dimensional case.

A remarkable result states that for any n-dimensional vector bundle $\pi : E \longrightarrow X$, any two maps of bundles with domain E and codomain the total space of the canonical bundle are always homotopic through maps of bundles, see [9]. As a consequence, their projections onto the base space are homotopic. More precisely, this is the following theorem.

Theorem 2.9. *Every real vector bundle of dimension n over X determines a* *smooth* classifying map $f : X \longrightarrow G_n$*. Even more, the vector bundle is uniquely determined by the homotopy type of f up to isomorphism.*

In other words, there is a bijection

$$\left\{ \begin{array}{c} \text{Isomorphism classes of n-dimensional} \\ \text{vector bundles over X} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Homotopy classes} \\ f : X \longrightarrow G_n \end{array} \right\}.$$

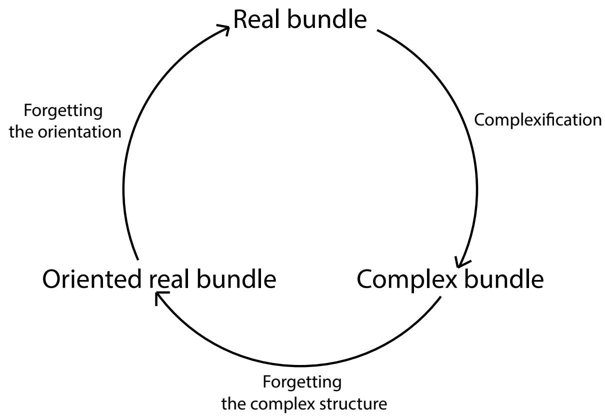
Therefore, the problem of understanding vector bundles over X is equivalent to studying homotopy invariants between X and G_n.

Cohomology is contravariant because the induced map of a continuous map goes in the opposite direction. For this reason, cohomology classes in the Grassmannian produce invariants on the cohomology of the base space. More precisely, given a vector bundle $\pi : E \longrightarrow X$ with classifying map $f : X \longrightarrow G_n$ we have an induced map in cohomology $f^* : H^*(G_n) \longrightarrow H^*(X)$. For c an element of $H^*(G_n)$, we get the invariant $f^*(c) \in H^*(X)$ which we call *characteristic class*. The first step is to take $\mathbb{Z}_2$ coefficients and we obtain the *Stiefel-Whitney classes*.

If we consider complex vector bundles instead of working with real vector bundles, then we get the *complex Grassmanian*. The associated characteristic classes are the *Chern classes*. If we consider real vector bundles but those that are oriented, we get the *oriented Grassmannian*, and the characteristic classes are the *Pontryagin classes*.

These three types of characteristic classes are related via the following constructions: start with an n-dimensional vector bundle ξ, and then we get a complex vector bundle via the complexification $\xi \otimes \mathbb{C}$. Then we forget the complex structure and get a real $2n$-dimensional vector bundle with a canonical orientation. Finally, we forget the orientation and obtain a $2n$-dimensional real vector bundle isomorphic to $\xi \oplus \xi$.

The following diagram schematically represents this situation:



2.3.1. The Thom isomorphism and the Euler class

A fundamental construction in algebraic topology is the *Pontryagin-Thom construction*, which associates with an n-dimensional bundle, the space in which we collapse the complement of the disc bundle to a single point. Despite the simple definition, the implications are remarkable. In a certain sense, this construction has the behavior of an n-suspension of the base space. More precisely, this is the famous theorem of Thom:

Theorem 2.10 (Thom isomorphism theorem)**.** *For an orientable vector bundle, there exists a unique cohomology class $u \in H^n(E, E_0; \mathbb{Z})$ whose restriction to (F, F_0) coincides with $u|_F$ for any fiber F and F_0 its nonzero elements. Furthermore, the map*

$$\smile u : H^i(E; \mathbb{Z}) \longrightarrow H^{i+n}(E, E_0; \mathbb{Z})$$

is an isomorphism.

For a complete proof of this theorem, the reader can see [9]. For $p : E \longrightarrow B$, the projection of an n-dimensional vector bundle, we have an isomorphism defined by the composition:

$$\phi : H^k(B; \mathbb{Z}) \xrightarrow{p^*} H^k(E; \mathbb{Z}) \xrightarrow{\smile u} H^{k+n}(E, E_0; \mathbb{Z}) .$$

Definition 2.11 (Euler class)**.** For ξ an oriented n-dimensional real vector bundle and $j : (E, \varnothing) \hookrightarrow (E, E_0)$ the inclusion, we define the Euler class of ξ, denoted by $e(\xi) \in H^n(B; \mathbb{Z})$, as the only cohomology class that satisfies the following equation

$$p^*(e(\xi)) = j^*(u) .$$

Proposition 2.12. *In case we reverse the orientation of the bundle ξ, the Euler class changes sign.*

For $\mathbb{Z}_2$-coefficients, the Euler class coincides with the top Stiefel-Whitney class $\omega_n(\xi)$.

2.3.2. Stiefel-Whitney classes

The following properties completely determine the Stiefel-Whitney classes:

Theorem 2.13 (Stiefel-Whitney classes)**.** *There exists one and only one sequence of characteristic classes $\omega_0, \omega_1, \ldots$ which assigns to each real n-dimensional vector bundle ξ of the form $E \longrightarrow B$, the class $w_i(\xi) \in H^i(B; \mathbb{Z}_2)$, such that:*

(1) $\omega_0(\xi) = 1$ *and* $w_i(\xi) = 0$ *for* $i > n$*,*

(2) $\omega_i(\xi) = f^*(\omega_i(\eta))$*, for all bundle map* $f : \xi \to \eta$ *,*

(3) $\omega_k(\xi \oplus \eta) = \sum_{i=0}^{k} \omega_i(\xi) \smile \omega_{k-i}(\eta)$*,*

(4) for the canonical bundle γ_1^1 *over* S^1*, we have* $\omega_1(\gamma_1^1) \neq 0$*.*

An intelligent way to show the existence and uniqueness of the Stiefel-Whitney classes and gain an understanding is by means of the Steenrod squares and Thom's isomorphism. In what follows, we present a rough idea of these themes.

Category theory arises to create a common language for the mathematical community. The reader can consult the Founder's book [5] for a historical and mathematical overview. For a modern approach, we refer the reader to [10].

Algebraic topology works with invariants which are functors from the category of topological spaces to some algebraic category: for example, the category of groups for the homotopy groups and the category of rings for cohomology. We wonder what are the natural transformations in cohomology. These are given by linear maps $\theta : H^n(\_; H) \longrightarrow H^m(\_; G)$ which satisfy certain commutative diagrams. Such maps are known as *cohomological operations*. In case we are working with CW-complexes, the functor of cohomology is a representable functor in the sense that it is equivalent to having the homotopy classes of maps from the space in question to the so-called Eilenberg-MacLane spaces, i.e., $H^n(X; H) \cong [X, K(H, n)]$.

For representable functors, a significant result is the Yoneda lemma, which states that the natural transformations from a representable functor to any other functor (forgetting the structure) are in correspondence with the image of the second functor applied to the element which represents the first functor. In symbols

$$\mathbf{Nat}(\mathrm{Hom}(-, X), F) \cong F(X)\,.$$

In our case, we conclude the following bijection:

$$\mathbf{Nat}([-, K(H, n)], [-, K(G, m)]) \cong H^m(K(H, n); G)\,.$$

Consequently, to understand cohomological operations, it is enough to understand the cohomology of the Eilenberg-MacLane spaces.

A basis for the cohomological operations are the *Steenrod squares* Sq^j : $H^i(B; \mathbb{Z}_2) \longrightarrow H^{i+j}(B; \mathbb{Z}_2)$.

For a real n-dimensional vector bundle ξ, of the form $p : E \to B$, we have the Thom isomorphism $\phi : H^k(B) \longrightarrow H^{k+n}(E, E_0)$. The Stiefel-Whitney classes are defined as $\omega_i(\xi) = \phi^{-1} Sq^i \phi(1)$. In other words,

$$\begin{array}{ccc} H^n(E, E_0) & \xrightarrow{Sq^i} & H^{n+i}(E, E_0) \\ \phi \uparrow & & \downarrow \phi^{-1} \\ H^0(B; \mathbb{Z}_2) & \longrightarrow & H^i(B; \mathbb{Z}_2) \end{array}$$

$$1 \longmapsto \omega_i(\xi)$$

As a consequence, we have shown the existence of the Stiefel-Whitney classes. It is relatively easy to show their uniqueness [9].

We finish the section with some bordism invariants known as the Stiefel-Whitney numbers. Take an n-dimensional closed smooth manifold (possibly disconnected). Using $\mathbb{Z}_2$-coefficients there is only one fundamental class in homology $[B] \in H_n(B; \mathbb{Z}_2)$. Consider non-negative integers $r_1, \cdots, r_n$ such that $r_1 + 2r_2 + \cdots + nr_n = n$. For ξ a real vector bundle over B, we can associate the monomial $\omega_1(\xi)^{r_1} \cdots \omega_n(\xi)^{r_n}$ in $H^n(B; \mathbb{Z}_2)$. The Stiefel-Whitney number is defined as the evaluation of this monomial in the fundamental class, i.e.,

$$\omega_1(\xi)^{r_1} \cdots \omega_n(\xi)^{r_n}[B] := \langle \omega_1(\xi)^{r_1} \cdots \omega_n(\xi)^{r_n}, [B] \rangle,$$

which is an element in $\mathbb{Z}_2$. Now, we use the formula $\omega(\mathbb{RP}^n) = (1 + a)^{n+1}$ with a the generator of the cohomology of $\mathbb{RP}^n$, hence we have for n even that $\omega_n(\mathbb{RP}^n) = (n + 1)a^n$ and $\omega_1(\mathbb{RP}^n) = (n + 1)a$ both different form zero. As a consequence, the Stiefel-Whitney numbers $\omega_n[\mathbb{RP}^n]$ and $\omega_1^n[\mathbb{RP}^n]$ are different from zero. In the case $n = 2^k$, these are the only non-trivial Stiefel-Whitney since $\omega(\mathbb{RP}^n) = 1 + a + a^n$. For n odd, it is relatively easy to show that all the Stiefel-Whitney numbers are zero. In bordism theory (non-necessarily oriented), we have that a manifold M of dimension n is the boundary of a manifold of dimension $n+1$, if and only if all the Stiefel-Whitney numbers are zero. The necessity of this fact is straightforward, where we use the duality between the connections maps of the long exact sequence (cohomology/homology) of the pair given by the bordism and M, see Milnor [9]. However, the sufficiency uses the Pontryagin-Thom construction. Let ξ be an n-dimensional vector bundle; the Thom space is defined as the quotient of the total space by the vectors with a norm bigger or equal to 1. In the case we have the canonical bundle γ^k, this space is denoted by $MO(k)$ and an outstanding result of Thom [14] says that the bordism group Ω_n is isomorphic to the following homotopy group,

$$\Omega_n \cong \pi_{n+k}(MO(k))\,,$$

for $k > n+1$. This isomorphism is determined by the Whitney embedding theorem, which embeds any manifold M inside a $\mathbb{R}^{n+k}$ for $k > n+1$. Therefore, the normal bundle of such embedding induces a Thom space with a map to $MO(k)$, using the one-point compactification of $\mathbb{R}^{n+k}$ we obtain a map from the sphere S^{n+k} to $MO(k)$. If M represents a non-trivial element in Ω_n, we have that the associated map $S^{n+k} \to MO(k)$ is not trivial in the homotopy group. Because $H^{n+k}(MO(k);\mathbb{Z}_2)$ is generated by some polynomial, which sent in $H^{n+k}(S^{n+k};\mathbb{Z}_2) = \mathbb{Z}_2$ at least a non-trivial Stiefel-Whitney number.

2.3.3. Chern and Pontryagin classes

Chern classes are associated with complex vector bundles, which are completely determined by the following properties:

Theorem 2.14 (Chern classes)**.** *There exists one and only one sequence of characteristic classes $c_1, c_2, \ldots$ which assigns to each complex n-dimensional vector bundle ξ of the form $E \longrightarrow B$, the class $c_i(E) \in H^{2i}(B;\mathbb{Z})$, such that:*

(1) $c_0(\xi) = 1$ *and* $c_i(\xi) = 0$ *for* $i > n$*,*

(2) $c_i(\xi) = f^*(c_i(\eta))$*, for all bundle maps* $f : \xi \to \eta$*,*

(3) $c_k(\xi \oplus \eta) = \sum_{i=0}^{k} c_i(\xi) \smile c_{k-i}(\eta)$*,*

(4) for the canonical bundle γ_1^1 *over* S^2*, we have* $c_1(\gamma_1^1)$ *which is the generator of* $H^2(\mathbb{CP}^1, \mathbb{Z})$*.*

In this case, the existence of the characteristic classes is explained relatively easily as follows: given a complex n-dimensional vector bundle ξ of the form $E \longrightarrow B$ equipped with a Hermitian metric, we form a new bundle ξ_0 over E_0 whose fiber over each point is the orthogonal complement of the given vector.

As a consequence, ξ_0 is an $(n-1)$-dimensional complex vector bundle. We then use the Gysin sequence with integer coefficients,

$$\cdots \longrightarrow H^{i-2n}(B) \xrightarrow{\cup e} H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \to H^{i-2n+1}(B) \to \cdots$$

for $i < 2n-1$ the groups $H^{i-2n}(B)$ and $H^{i-2n+1}(B)$ are zero, hence we have the isomorphism $\pi_0^* : H^i(B) \to H^i(E_0)$. Take $c_n(\xi)$ as the Euler class of the induced $2n$-dimensional real vector bundle $e(\xi_{\mathbb{R}})$. We define for $i < n$ the Chern class c_i as

$$c_i(\xi) = \pi_0^{*-1} c_i(\xi_0)\,,$$

and for $i > n$ the class $c_i(\xi)$ is defined to be zero.

These classes satisfy the axioms of Theorem 2.14.

An important property of Chern classes is their behavior under the conjugation $x + iy \longmapsto x - iy$ of a complex vector bundle ξ, where we have the following identity

$$c_k(\xi) = (-1)^k c_k(\overline{\xi})\,,$$

hence the total class of the conjugated bundle $\overline{\xi}$ is given as

$$c(\overline{\xi}) = 1 - c_1(\xi) + c_2(\xi) - + \cdots \pm c_n(\xi)\,.$$

The Pontryagin classes are defined using the Chern classes for an n-dimensional real vector bundle. More precisely, we consider the *complexification* $\xi \otimes \mathbb{C}$ given by the tensor product over the reals of each fiber with the complex numbers. The bundle $\xi \otimes \mathbb{C}$ has an induced structure of real vector bundle given by the Whitney sum $\xi \oplus \xi$ with complex structure $J(x, y) = (-y, x)$.

Now, we consider the conjugate $\overline{\xi \otimes \mathbb{C}}$ which is isomorphic to the complexification $\xi \otimes \mathbb{C}$, hence the odd Chern classes $c_1(\xi \otimes \mathbb{C})$, $c_3(\xi \otimes \mathbb{C})$, $\cdots$ are zero. We define the i-th Pontryagin class for an n-dimensional real vector bundle as

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}),$$

which is an element in $H^{4i}(B; \mathbb{Z})$.

There are similar properties as in the case of Stiefel-Whitney classes and Chern classes. We have $p_0(\xi) = 1$ and $p_i(\xi) = 0$ for $i > n/2$. For a trivial bundle ϵ^k, we obtain $p(\xi \oplus \epsilon^k) = p(\xi)$. In this case, the total class is defined as

$$p(\xi) = 1 + p_1(\xi) + \cdots + p_{\lceil n/2 \rceil}(\xi)\,,$$

where $\lceil n/2 \rceil$ denotes the smallest integer that is not smaller than $n/2$. In this case, we have the Whitney sum satisfies the formula

$$p(\xi \oplus \eta) = p(\xi)p(\eta) \bmod 2\,.$$

We end with two properties that determine the Pontryagin classes:

i) For ξ an n-dimensional complex vector bundle, we have the underlying $2n$-dimensional real vector bundle satisfies the following identity

$$1-p_1+p_2-+\cdots\pm p_n = (1-c_1+c_2-+\cdots\pm c_n)(1+c_1+c_2+\cdots+c_n) \quad (2)$$

where $c_i = c_i(\xi)$ and $p_k = p_k(\xi_{\mathbb{R}})$. As a consequence, the class $p_k(\xi_{\mathbb{R}})$ is equal to

$$c_k(\xi)^2 - 2c_{k-1}(\xi)c_{k+1}(\xi) + - \cdots \pm 2c_1(\xi)c_{2k-1}(\xi) \mp 2c_{2k}(\xi)\,.$$

ii) For ξ a $2n$-dimensional oriented real vector bundle, we have the Pontryagin class $p_n(\xi)$ is equal to the square of the Euler class $e(\xi)$.

Finally, we define the Pontryagin numbers associated with a smooth, compact, oriented manifold of dimension $4n$, which we denote by M. To this end, recall that a partition of a positive integer n is an ordered collection of positive numbers $I = \{i_1, \ldots, i_r\}$ with a sum equal to n (notice that in this collection some numbers can be repeated). For a partition I of n, the I-th Pontryagin number is defined as the evaluation of the polynomial $p_{i_1}(\tau_M)\ldots p_{i_r}(\tau_M)$ in the fundamental class, i.e.,

$$p_{i_1}\cdots p_{i_r}[M] = \langle p_{i_1}(\tau_M)\ldots p_{i_r}(\tau_M), [M]\rangle\,,$$

where τ_M represents the tangent bundle and $[M] \in H_{4n}(M;\mathbb{Z})$ is the fundamental class. For the complex projective spaces $\mathbb{CP}^{2n}$ such numbers have the value

$$p_{i_1}\cdots p_{i_r}[\mathbb{CP}^{2n}] = \binom{2n+1}{i_1}\cdots\binom{2n+1}{i_r}.$$

Just as for the Stiefel-Whitney numbers, if an oriented smooth manifold of dimension n is the boundary of an oriented smooth manifold of dimension $n+1$, all the Pontryagin numbers are zero. The converse is satisfied when we tensor with the rational numbers.

3. The Hirzebruch signature theorem

A central element in the proof of the exotic spheres is the famous *Hirzebruch signature theorem*. This theorem determines the signature of a manifold in terms of a polynomial in the Pontryagin classes with rational coefficients. In this section, we give the proof of this theorem after introducing some algebraic background.

3.1. Multiplicative sequences

We start with a commutative graded algebra over a commutative and unitary ring Λ:

$$A = \bigoplus_{i=0}^{\infty} A^i.$$

By A^{Π} we understand the ring of formal series $a_0 + a_1 + a_2 + \dots$ with $a_i \in A^i$. Of particular interest is the subset $(A^{\Pi})^{\times}$ consisting of formal series of the form $1 + a_1 + a_2 + \dots$.

***Remark* 3.1.** It is a classical exercise to show that $(A^{\Pi})^{\times}$ is a group. Set $a = 1 + a_1 + a_2 + \dots$. To construct the inverse, we proceed inductively: consider $b = 1 + b_1 + b_2 + \cdots$ such that $ab = 1$ and expand the product

$$\begin{aligned} ab &= (1 + a_1 + a_2 + a_3 + \dots)(1 + b_1 + b_2 + b_3 + \dots) \\ &= 1 + (a_1 + b_1) + (a_2 + a_1 b_1 + b_2) + (b_3 + b_2 a_1 + a_2 b_1 + a_3) + \dots \\ &= 1 + 0 + 0 + \dots . \end{aligned}$$

Therefore, we define $b_1 = -a_1$ for the first coefficient. Then we have $a_2 + a_1 b_1 + b_2 = 0$ and hence $b_2 = -a_2 - a_1 b_1 = -a_2 + a_1^2$, and so on.

We consider a sequence of polynomials $K_1(x_1)$, $K_2(x_1, x_2), \dots$ subject to the following two properties:

- the polynomial K_i has degree i;
- the polynomial K_i is homogeneous where x_j has weight j.

For example, these properties are satisfied for the sequence of polynomials:

$$\begin{cases} K_1(x_1) = x_1 \\ K_2(x_1, x_2) = x_1^2 + x_2 \\ K_3(x_1, x_2, x_3) = x_1^3 + x_1 x_2 + x_3 \\ \vdots \end{cases}$$

For an element $a = 1 + a_1 + a_2 + \cdots \in (A^{\Pi})^{\times}$, we can evaluate the sequence of polynomials in a as follows

$$K(a) := 1 + K_1(a_1) + K_2(a_1, a_2) + K_3(a_1, a_2, a_3) + \dots .$$

A sequence of polynomials K_i subject to the two aforementioned properties, is called *multiplicative* if for any $a, b \in (A^{\Pi})^{\times}$ we have the equation

$$K(ab) = K(a)K(b) .$$

We give below some examples.

Example 3.2.

(1) Take $\lambda \in \Lambda$ and define

$$K_i(x_1, \dots, x_i) = \lambda^i x_i .$$

For $a, b \in \left(A^{\Pi}\right)^{\times}$, we compute

$$\begin{aligned} K(ab) &= K(1 + (a_1 + b_1) + (a_2 + a_1 b_1 + b_2) + \dots) \\ &= 1 + \lambda(a_1 + b_1) + \lambda^2(a_2 + a_1 b_1 + b_2) + \dots \end{aligned}$$

and

$$\begin{aligned} K(a)K(b) &= (1 + \lambda a_1 + \lambda^2 a_2 + \dots)(1 + \lambda b_1 + \lambda^2 b_2 + \dots) \\ &= 1 + (\lambda a_1 + \lambda b_1) + (\lambda^2 a_2 + \lambda^2 a_1 b_1 + \lambda^2 b_2) + \dots \\ &= 1 + \lambda(a_1 + b_1) + \lambda^2(a_2 + a_1 b_1 + b_2) + \dots \\ &= K(ab)\,. \end{aligned}$$

This shows that the sequence is multiplicative.

(2) Define $K_i(x_1, \dots, x_i)$ to be the i-th coefficient of $(1 + x_1 + x_2 + \dots)^{-1}$. It is easy to see that this sequence is homogeneous of degree i, and moreover, this sequence is multiplicative because, by definition, $K(a) = a^{-1}$. Therefore,

$$K(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = K(a)K(b)\,.$$

Now, we see that multiplicative sequences are closely related to power series. Given a multiplicative sequence $\{K_n\}_{n\in\mathbb{N}}$, we can associate a power series by setting

$$f(t) = K(1 + t) = 1 + K_1(t) + K_2(t, 0) + K_3(t, 0, 0) + \dots.$$

The important point here is the reverse process; that is, given a power series f we can associate a multiplicative sequence such that $f(t) = K(1 + t)$. This is the purpose of the next section.

3.2. Digression: symmetric polynomials and the Hirzebruch's lemma

Among all polynomials, some are distinguished for being invariant under the action of the *symmetric group*, i.e., under permutations of their variables.

Example 3.3.

- The polynomial $p(x, y, z) = x + y + z$ is invariant under the action of the symmetric group. Indeed, any permutation, for instance $\tau : x \longrightarrow y \longrightarrow z$ gives

$$p(\tau(x), \tau(y), \tau(z)) = p(y, z, x) = y + z + x = p(x, y, z)\,.$$

- The polynomial $q(x, y) = x^2 + xy + y^2$ is also invariant.

Such polynomials are called *symmetric polynomials.*

Lemma 3.4. *Consider the polynomials $\sigma_1, \dots, \sigma_n$ where σ_i is the component of degree i of the product $(1+t_1)\cdot(1+t_2)\cdot\dots\cdot(1+t_n)$. Then each σ_i is a symmetric polynomial in n variables.*

Proof. This follows from the equation

$$1 + \sigma_1 + \sigma_2 + \dots + \sigma_n = \prod_{i=1}^{n}(1+t_i)$$

the right-hand side is invariant under permutations, so the i-th degree component is also invariant. ☑

The polynomials σ_i in the previous lemma are called *elementary symmetric polynomials.*

Example 3.5. In two variables, there are two elementary symmetric polynomials. Indeed, they are the components of the product

$$(1+x)(1+y) = 1 + (x+y) + xy\,.$$

Therefore, $\sigma_1(x,y) = x+y$ and $\sigma_2(x,y) = xy$.

In three variables, there are three elementary symmetric polynomials. Namely, the components of the product

$$(1+x)(1+y)(1+z) = 1 + (x+y+z) + (xy+yz+xz) + (xyz)\,.$$

Elementary symmetric polynomials are fundamental in mathematics due to the following theorem; we refer the reader to [4] for proof.

Theorem 3.6 (Fundamental theorem of elementary symmetric polynomials)**.** *Elementary symmetric polynomials form a basis for the set of symmetric polynomials, in the sense that each symmetric polynomial of degree n can be uniquely written as a polynomial in the variables $\sigma_1, \dots, \sigma_n$.*

For example, the symmetric polynomial

$$q(x,y) = x^2 + xy + y^2$$

can be written as

$$q(x,y) = (x+y)^2 - xy = \sigma_1^2 - \sigma_2\,.$$

***Remark* 3.7.** Any monomial yields a symmetric polynomial by summing over all equivalent monomials [3] For example, the monomial $m(x, y, z) = x^2yz$ is not symmetric, however, the mentioned sum is the symmetric polynomial

$$x^2yz + y^2xz + z^2xy\,.$$

It is easy to see that this polynomial is, in fact, symmetric. In general, we will denote the polynomial obtained by this process by Σm called the "symmetrization" of m. With this notation, the elementary symmetric polynomials in n variables are elegantly given by

$$\sigma_i = \Sigma t_1 \cdots t_i\,.$$

We return to the power series. Start with

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \dots$$

and consider the partition $I = \{i_1, \dots, i_r\}$ of n (that is, they are all positive integers with $i_1 + \cdots + i_r = n$). We define λ_I as the product $\lambda_{i_1} \cdot \dots \cdot \lambda_{i_r}$ and s_I as the unique polynomial such that

$$s_I(\sigma_1, \dots, \sigma_n) = \Sigma t_1^{i_1} \cdot \dots \cdot t_r^{i_r}\,.$$

The existence of s_I is a direct consequence of the fundamental theorem of elementary symmetric polynomials. Thus we define

$$K_n(x_1, \dots, x_n) := \sum_{I \text{ partition of } n} \lambda_I s_I(x_1, \dots, x_n)\,.$$

Example 3.8. Suppose we have a power series

$$f(t) = 1 + \frac{t}{3} - \frac{t^2}{45} + \dots$$

and then we calculate the first two terms of the sequence above. For the first term, there is only one partition of the number 1, namely the number 1, which we call I (this may seem unnecessary, but it is meant to show the general procedure). To calculate s_I we observe that $\Sigma t^1 = t$, in particular $s_I(\sigma_1) = \sigma_1$ and since λ_I is just the first coefficient λ_1, so

$$K_1(x) = \frac{1}{3}x\,.$$

For the second term, we have two partitions of the number 2, given by $1 + 1$ and $2 + 0$, denoted by J and H. Finding s_J amounts to finding a polynomial

[3] Two monomials are equivalent if a permutation relates them.

such that $s_J(\sigma_1, \sigma_2) = \Sigma t_1^1 t_2^1 = t_1 t_2$, i.e., $s_J(\sigma_1, \sigma_2) = \sigma_2$ (recall that $\sigma_1 = x+y$ and $\sigma_2 = xy$). The coefficient λ_J is given by $\lambda_1 \cdot \lambda_1$, so the first summand is

$$\lambda_J s_J(x, y) = \frac{1}{9} y \, .$$

For the partition H, we see the polynomial s_H satisfies $s_H(\sigma_1, \sigma_2) = \Sigma t_1^2 = t_1^2 + t_2^2$ and hence $s_H(\sigma_1, \sigma_2) = \sigma_1^2 - 2\sigma_2$ and we conclude

$$s_H(x, y) = (x+y)^2 - 2xy = x^2 + y^2 \, .$$

The coefficient is just $\lambda_H = \lambda_2$, hence the second summand is

$$\lambda_H s_H(x, y) = -\frac{1}{45}(x^2 - 2y) \, .$$

Combining our computations yields

$$K_2(x, y) = \frac{1}{9} y - \frac{1}{45}(x^2 - 2y) = \frac{1}{45}(7y - x^2) \, . \tag{3}$$

We return with the multiplicative property of the sequence $K_n(x_1, \cdots, x_n)$ associated with the power series. Denote by σ_i the i-th elementary symmetric polynomial in the variables $x_1, \ldots, x_n$ and by σ'_j the j-th elementary symmetric polynomial in the variables $y_1, \ldots, y_n$. Then

$$\sigma''_k = \sum_{i=0}^{k} \sigma_i \sigma'_{k-i}$$

is the k-th elementary symmetric polynomial in the variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. This is because we can compare the product $\prod_{i=1}^n (1+x_i) \prod_{j=1}^n (1+y_j)$ with the definition of the k-th elementary symmetric polynomial.

Given two disjoint partitions, say J and K, their juxtaposition is also a partition. More precisely, if $J = \{j_1, \ldots, j_r\}$ is a partition of l and $K = \{k_1, \ldots, k_p\}$ is a partition of m, then

$$JK = \{j_1, \ldots, j_r, k_1, \ldots, k_p\}$$

is a partition of $l+m$.

Going back to our polynomial sequence, we claim that

$$s_I(\sigma''_1, \ldots, \sigma''_k) = \sum_{JK=I} s_J(\sigma_1, \sigma_2, \ldots) \cdot s_K(\sigma'_1, \sigma'_2, \ldots) \, ,$$

where the sum is taken over all partitions J, K such that their juxtaposition is I. For this purpose, we use that

$$s_I(\sigma''_1, \ldots, \sigma''_k) = \Sigma t_1^{i_1} \cdots t_r^{i_r} \, ,$$

where in the right hand side appear all possible monomials $t_{\alpha_1}^{i_1} \cdots t_{\alpha_r}^{i_r}$ with $0 \leq \alpha_i \leq 2n$. For each monomial, let J be the partition formed by all exponents i_q such that $1 \leq \alpha_q \leq n$ and let K be the partition formed by all exponents i_q such that $n+1 \leq \alpha_q \leq 2n$. By construction, the product $s_J(\sigma_1, \sigma_2, \dots) s_K(\sigma_1', \sigma_2', \dots)$ has all the possible combinations of this distribution of exponents in both variables. The sum of all such decompositions implies the claim.

From the previous discussion, we can conclude the multiplicativity of the sequence $K_n(x_1, \dots, x_n)$. Indeed, for $a, b \in \left(A^\Pi\right)^\times$, we obtain

$$\begin{aligned} K(ab) &= \sum_I \lambda_I s_I(ab) \\ &= \sum_I \lambda_I \sum_{HJ=I} s_H(a) s_J(b) \\ &= \sum_{HJ=I} \lambda_H s_H(a) \lambda_J s_J(b) \\ &= K(a)K(b)\,. \end{aligned}$$

Furthermore, $K_n(t, 0, \dots, 0) = \lambda_n t^n$ since the only partition involving this term is the trivial one (see example 3.8). Consequently, $K(1+t) = f(t)$ which is the half of the following lemma:

Lemma 3.9 (Hirzebruch). *Let*

$$f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \cdots \in \Lambda[[t]]$$

be a formal power series. Then there exists a unique multiplicative sequence $\{K_n\}_{n\in\mathbb{N}}$ *satisfying* $K(1+t) = f(t)$.

To show uniqueness, if

$$\sigma = (1+t_1)\cdots(1+t_n) \in \left(A^\Pi\right)^\times,$$

then

$$K(\sigma) = K((1+t_1)\cdots(1+t_n)) = K(1+t_1)\cdots K(1+t_n) = f(t_1)\cdots f(t_n).$$

We compare the homogeneous component of each side, and we observe that $K_n(\sigma_1, \dots, \sigma_n)$ is determined only by the values of f. We use the fundamental theorem of elementary symmetric polynomials to conclude that the variables $\sigma_1, \dots, \sigma_n$ completely determine the polynomial; hence the K_n must be unique.

3.3. K-genus and the Hirzebruch theorem

For a multiplicative sequence K_n, we define the K-genus of a smooth, closed, oriented manifold M, denoted by $K[M] \in \mathbb{Q}$, as follows

$$K_n[M] = \begin{cases} 0 & 4 \not| \dim M \\ \langle K_n(p_1, \dots, p_n), [M] \rangle & \dim M = 4n, \end{cases}$$

where p_i denotes the i-th Pontryagin class of M. Notice that the K-genus is a *rational combination* of the Pontryagin numbers of M. In particular, if M is the boundary of a compact, oriented manifold, then the Pontryagin numbers are zero, and $K[M] = 0$.

The K-genus satisfies the following essential properties: for M, N two manifolds, we have $K[M \sqcup N] = K[M] + K[N]$ which, combined with the previous observation implies that the K-genus is a *bordism invariant*. Furthermore, we have the multiplicative property $K[M \times N] = K[M]K[N]$. This property is deduced as follows: for p, p' the total Pontryagin classes of M and N respectively, the total class of $M \times N$ is congruent to $p \times p'$ modulo torsion, and in addition, the codomain of the K-genus is the field of rational numbers, so the torsion elements automatically vanish. More precisely, we have shown the following lemma.

Lemma 3.10. *The K-genus gives rise to a ring homomorphism*

$$K : \Omega_*^{SO} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}.$$

Now we are ready to state the main theorem of this section.

Theorem 3.11 (Hirzebruch signature theorem)**.** *Let L_n be the multiplicative sequence associated with the power series*

$$\frac{\sqrt{x}}{\tanh \sqrt{x}} = 1 + \frac{x}{3} - \frac{x^2}{45} + \cdots + \frac{(-1)^{k-1} 2^{2k} B_k x^k}{(2k)!} + \dots ,$$

where B_k is the k-th Bernoulli number*. Then for any compact, oriented smooth manifold M, $\sigma(M) = L[M]$.*

The proof of this theorem depends on the following fact. Since both L and σ define ring homomorphisms $\Omega_*^{SO} \otimes \mathbb{Q} \longrightarrow \mathbb{Q}$ (where here we implicitly extend the signature by 0 on dimensions not divisible by 4), it is enough to verify that they agree on the generators, which we know thanks to the following result of Thom [14]:

Theorem 3.12 (Thom)**.** *The oriented cobordism ring Ω_*^{SO} is finitely generated in dimensions divisible by* 4 *and finite otherwise. In particular*

$$\Omega_*^{SO} \otimes \mathbb{Q} = \bigoplus_{k=1}^{\infty} \Omega_{4k}^{SO} \otimes \mathbb{Q}\,.$$

Furthermore, the generators are given by combinations of the form

$$\mathbb{CP}^{2i_1} \times \mathbb{CP}^{2i_2} \times \cdots \times \mathbb{CP}^{2i_r}\,.$$

We have already computed the signature of these complex projective spaces, which was precisely 1; see Example 2.7. Therefore, it is enough to show the identity $L(\mathbb{CP}^{2l}) = 1$, which is as follows: First, recall that the total Pontryagin class of $\mathbb{CP}^{2l}$ is given by $p(\mathbb{CP}^{2l}) = (1+a^2)^{2l+1}$. Since $L(1+t) = \frac{\sqrt{t}}{\tanh\sqrt{t}}$, it follows that

$$L(1+a^2+0+0+\dots) = \frac{\sqrt{a^2}}{\tanh\sqrt{a}^2} = \frac{a}{\tanh a}\,.$$

Now, we use the multiplicative property of L and we see that

$$L((1+a^2)^{2l+1}) = \Big(\frac{a}{\tanh a}\Big)^{2l+1}\,.$$

Thus the L-genus will be determined by the coefficient of a^{2l} in the power series of $(a/\tanh a)^{2l+1}$. For this, we recall that in complex analysis, we can recover a coefficient of a power series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m + \dots\,,$$

say c_m, by first dividing by z^{m+1}

$$\frac{f(z)}{z^{m+1}} = \frac{c_0}{z^{m+1}} + \cdots + \frac{c_m}{z} + c_{m+1} + \dots$$

and then integrating around the origin

$$\oint \frac{f(z)}{z^{m+1}}dz = \oint \frac{c_m}{z}dz = 2\pi i c_m\,.$$

As a consequence, replacing a by z in the power series of $(a/\tanh a)^{2l+1}$, we obtain

$$L[\mathbb{C}P^{2l}] = \frac{1}{2\pi i}\int \frac{dz}{z^{2l+1}}\Big(\frac{z}{\tanh z}\Big)^{2l+1} = \frac{1}{2\pi i}\oint \frac{dz}{\tanh z^{2l+1}}.$$

The change of coordinates $u = \tanh z$ implies that $dz = \frac{du}{1-u^2} = (1+u^2+u^4+\dots)du$ and we get the result

$$L[\mathbb{CP}^{2l}] = \frac{1}{2\pi i}\oint \frac{1+u^2+u^4+\cdots+}{u^{2l+1}}du = \frac{1}{2\pi i}\oint \frac{u^{2k}}{u^{2k+1}}du = 1\,.$$

This proves the Hirzebruch signature theorem.

We use the formula (3) in Example 3.8 and deduce the following.

Corollary 3.13. If M is an 8-dimensional compact oriented manifold then

$$\sigma(M) = \frac{1}{45}\left(7p_2(M) - p_1^2(M)\right)\,.$$

4. Milnor's explicit construction

4.1. Construction in terms of the canonical fibration

This section aims to construct a family of manifolds, some of which are exotic spheres. They are the total space of fiber bundles over S^4 with fiber S^3 and structural group $SO(4)$ (the transition maps are given by matrices in the group $SO(4)$). We have to classify all such fiber bundles to identify which are exotic spheres. This section follows some parts of [6], [15].

These fiber bundles have fibers identified with the ring of quaternions. The quaternions $\mathbb{H}$ are the set of numbers of the form $a+bi+cj+dk$ where a, b, c and d are real numbers and the symbols i, j, k are subject to the following rules:

$$\begin{aligned} i^2 = j^2 = k^2 &= -1\,, \\ ij = -ji &= k\,, \\ jk = -kj &= i\,, \\ ki = -ik &= j\,. \end{aligned}$$

Observe that quaternion multiplication is *not* commutative. The ring of quaternions $\mathbb{H}$ is a 4-dimensional real vector space with the component-wise sum and scalar real multiplication. Similarly, there is a conjugacy operator, as with the complex numbers, also a norm and an inverse for a quaternion $h = a+bi+cj+dk$:

$$\overline{h} = a - bi - cj - dk\,, \quad \|h\| = \sqrt{\overline{h}h} = \sqrt{a^2+b^2+c^2+d^2} \text{ and } h^{-1} = \frac{\overline{h}}{\|h\|^2}\,.$$

We can define the quaternionic projective line $\mathbb{HP}^1$, which consists of all quaternionic lines in $\mathbb{H}^2$. The elements are denoted by classes $[h_1 : h_2] \in \mathbb{HP}^1$, where $[h_1 : h_2] = [\lambda h_1 : \lambda h_2]$ for each $\lambda \in \mathbb{H}$ non-zero. The canonical bundle over $\mathbb{HP}^1$, denoted by γ^1, has total space

$$E(\gamma^1) = \{((x,y),[z:w]) \in \mathbb{H}^2 \times \mathbb{HP}^1 \mid (x,y) \in [z:w]\}\,.$$

The projection map $\pi : E(\gamma^1) \longrightarrow \mathbb{HP}^1$ is given by the projection onto the second coordinate. Notice that the canonical bundle is a 4-dimensional real vector bundle. To construct the exotic spheres, we first consider a family of fiber bundles constructed from the canonical bundle. These bundles only depend on the usual construction of $\mathbb{HP}^1$ by two charts, and we calculate the local trivializations. We consider the open sets $U_1 = \{[z:w] \in \mathbb{HP}^1 \mid w \neq 0\}$ and $U_2 = \{[z:w] \in \mathbb{HP}^1 \mid z \neq 0\}$ where the first chart is

$$\begin{aligned} \phi_1 : U_1 &\longrightarrow \mathbb{H} \cong \mathbb{R}^4 \\ [z:w] &\longmapsto w^{-1}z \end{aligned}$$

and the second chart is given by

$$\begin{array}{rcl}\phi_2 : U_2 & \longrightarrow & \mathbb{H} \cong \mathbb{R}^4 \\ [z:w] & \longmapsto & z^{-1}w\,.\end{array}$$

Thus for the projection map $\pi : E(\gamma^1) \longrightarrow \mathbb{HP}^1$, we obtain

$\pi^{-1}(U_1) = \{((x,y),[z:1]) \mid yz = x\}$ and $\pi^{-1}(U_2) = \{((x,y),[1:w]) \mid xw = y\}$.

Therefore, the local trivializations are

$$\begin{array}{rcl}\rho_1 : \pi^{-1}(U_1) & \longrightarrow & \phi_1(U_1) \times \mathbb{H} \\ ((x,y),[z:1]) & \longmapsto & (z,y)\end{array}$$

and

$$\begin{array}{rcl}\rho_2 \colon \pi^{-1}(U_2) & \longrightarrow & \phi_2(U_2) \times \mathbb{H} \\ ((x,y),[1:w]) & \longmapsto & (w,x).\end{array}$$

Finally, the transition map $\rho_2 \circ \rho_1^{-1} : \phi_1(U_1 \cap U_2) \times \mathbb{H} \longrightarrow \phi_1(U_1 \cap U_2) \times \mathbb{H}$ is given by

$$\begin{aligned}\rho_2 \circ \rho_1^{-1}((z,y)) &= \rho_2\left((yz,y), \left[\frac{1}{z}:1\right]\right) \\ &= \left(\frac{1}{z}, yz\right).\end{aligned}$$

Consequently, excluding the poles, we are gluing at the point $y \in \pi^{-1}([z:1])$ with the point $yz \in \pi^{-1}([1/z:1])$ two fibers which can be identified with $\mathbb{H}$. Since multiplication in $\mathbb{H}$ is not commutative, we can have a different bundle if we glued y with zy. These provide a family of gluing maps $f_{h,l} : \phi_1(U_1 \cap U_2) \times \mathbb{H} \longrightarrow \phi_1(U_1 \cap U_2) \times \mathbb{H}$ defined as follows:

$$f_{h,l}((z,y)) = \left(\frac{1}{z}, z^h y z^l\right).$$

Thus, each gluing map has associated a vector bundle, denoted by $\xi_{h,l}$. For example, the bundle $\xi_{0,1}$ is precisely the canonical bundle γ^1.

However, our initial purpose was to build bundles over S^4 with fiber S^3. Thus in the previous vector bundles, we identify $\mathbb{HP}^1$ with S^4 by means of the diffeomorphism $\mathbb{HP}^1 \longrightarrow S^4 \subset \mathbb{R}^5$, which is given by

$$[z:w] \mapsto \left(\frac{2\overline{w}z}{\|z\|^2 + \|w\|^2}, \frac{\|z\|^2 - \|w\|^2}{\|z\|^2 + \|w\|^2}\right),$$

and we restrict the fibers to S^3 since $S^3 = \{h \in \mathbb{H} \mid \|h\| = 1)\}$. Therefore, the gluing maps are now of the form $f_{h,l} : \phi_1(U_1 \cap U_2) \times S^3 \longrightarrow \phi_1(U_1 \cap U_2) \times S^3$. These maps have to be normalized in the second coordinate in order to be coherent with the restriction, so we set

$$f_{h,l}((z,y)) = \left(\frac{1}{z}, \frac{z^h y z^l}{\|z\|^{h+l}} \right) .$$

Thus we have constructed for each vector bundle $\xi_{h,l}$, via the restriction, an induced sphere bundle. We denote these sphere bundles by $\sigma_{h,l}$ and their total space by $M_{h,l}$. These spaces are manifolds of dimension seven, which can be exotic spheres. In what follows, we show that for some particular h and l, the space $M_{h,l}$ is homeomorphic to the sphere S^7 (see section 4.2) but not diffeomorphic to it (see section 4.3). For this purpose, we show that these sphere bundles $\sigma_{h,l}$ are, in essence, all possible bundles with the property that the transition map is orientation-preserving. This is stated in the following theorem:

Theorem 4.1. *There is a bijection between the isomorphism classes of fiber bundles over S^4 with fiber S^3 and structural group $SO(4)$ and the homotopy classes of maps from S^3 to $SO(4)$.*

As a consequence, each sphere bundles $\sigma_{h,l}$ is classified up to isomorphism by an element in $\pi_3(SO(4))$. This group is relatively easy to understand since $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$ (see section 4.1.1) and thus the sphere bundles $\sigma_{h,l}$ and consequently the manifolds $M_{h,l}$ are completely determined by a pair of integers which are precisely (h, l).

4.1.1. Calculating $\pi_3(SO(4))$

The orthogonal group $O(n)$ consists of all matrices $n \times n$, representing all the distance-preserving transformations of the Euclidean space $\mathbb{R}^n$. They are given by matrices $A \in \mathrm{Gl}(n, \mathbb{R})$ such that $A^{tr}A = AA^{tr} = I$. If we consider orientation-preserving transformations, we obtain matrices in $O(n)$, with a determinant equal to 1. This subgroup is denoted by $SO(n)$ and is called the special orthogonal group of dimension n.

Consider S^3 as the unit quaternions. There is a well-defined homomorphism

$$P : S^3 \times S^3 \longrightarrow SO(4) ,$$

which for each pair $(u, v) \in S^3 \times S^3$, assigns the linear transformation $f_{(u,v)} : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ defined for $x \in \mathbb{R}^4$, be the product uxv^{-1}. The homomorphism P is a continuous map with the following properties:

- The image of P is contained in $SO(4)$ as a connected subset of $O(4)$, since $S^3 \times S^3$ is connected. Moreover, the image of P is in the same connected component of the identity because $P(1,1) = \mathrm{Id}$.

- P is a group homomorphism (as claimed). Indeed we have the equality

$$P(uu', vv') = P(u,v) \circ P(u', v'),$$

since both sides are equal to the map $x \mapsto uu'xv'^{-1}v^{-1}$.

- We have the identity $P(u,v) = P(-u,-v)$.

- Assume $P(u,v) = \mathrm{Id}$ and hence $P(u,v)(1) = 1$. Thus $u1v^{-1} = 1$ and therefore $uv^{-1} = 1$, which is equivalent to $u = v$. In addition, we have the equations

$$P(u,u)(i) = uiu^{-1} = i, \qquad P(u,u)(j) = uju^{-1} = j$$
$$P(u,u)(k) = uku^{-1} = k.$$

For the first equation, set $u = a + bi + cj + dk$, and we get

$$\begin{aligned} uiu^{-1} &= (a + bi + cj + dk)i(a - bi - cj - dk) \\ &= (a + bi + cj + dk)(ai + b - ck + dj) \\ &= a^2 i + ab - ack + adj \\ &- ab + b^2 i + bcj + bdk \\ &+ ack + bcj - c^2 i - cd \\ &- adj + bdk + cd - d^2 i \\ &= (a^2 + b^2 - c^2 - d^2)i + 2bcj + 2bdk = i, \end{aligned}$$

from which we deduce the equations $a^2 + b^2 - c^2 - d^2 = 1$ and $bc = bd = 0$. Recalling that $a^2 + b^2 + c^2 + d^2 = 1$ we get that $c = d = 0$. Proceeding similarly with the other equations, we conclude that $b = c = d = 0$. Consequently, we obtain $u = \pm 1$ and the kernel of P is the group with only two elements $\mathbb{Z}_2 \cong \{(1,1), (-1,-1)\}$.

- The kernel of P acts properly and discontinuously on $S^3 \times S^3$ from which it follows that the image of P is a 6-dimensional open submanifold of $SO(4)$ (since $\dim S^3 \times S^3 = 6$). Because P is continuous and $S^3 \times S^3$ is compact, we have that $P(S^3 \times S^3)$ is compact, and since $SO(4)$ is Hausdorff, the image of P is also closed. Thus since $SO(4)$ is connected, we have that P is surjective.

In conclusion, since the kernel of P is discrete and the homomorphism $P : S^3 \times S^3 \longrightarrow SO(4)$ is a 2-fold covering, see [11]. In other words, every point in $SO(4)$ has a neighborhood covered by two copies of itself, as shown in the picture

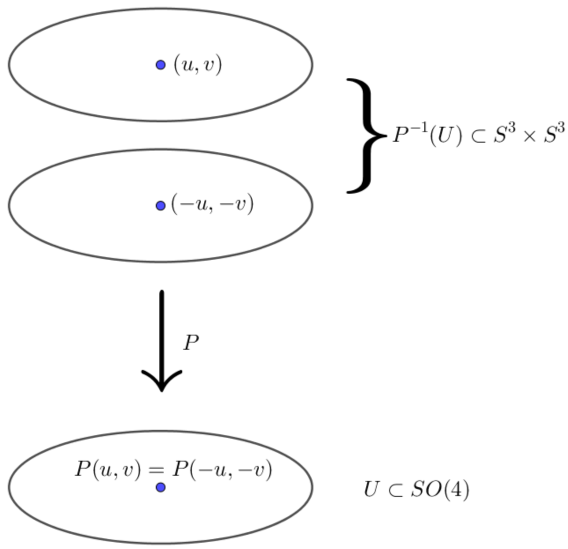


FIGURE 5. The homomorphism P is a 2-fold covering.

As a consequence of the homotopy lifting property for covering spaces, see [1], we have the following theorem.

Theorem 4.2. *If $P : Y \longrightarrow X$ is a covering map between connected spaces, then P induces an isomorphism between the higher homotopy groups $P_* : \pi_n(Y) \longrightarrow \pi_n(X)$ (i.e. with $n > 1$).*

Finally, we obtain the following result.

Corollary 4.3.

$$\pi_3(SO(4)) \cong \pi_3(S^3 \times S^3) \cong \mathbb{Z} \oplus \mathbb{Z}\,.$$

4.2. They are homeomorphic to the sphere S^7

This section aims to show that if $h+l = \pm 1$, then $M_{h,l}$ is homeomorphic to the standard sphere. We need the concept of *Morse function*, a smooth function $f : M \to \mathbb{R}$ such that all critical points are non-degenerate (the Hessian matrix is non-degenerate). A significant result in Morse theory is the following, see [8].

Theorem 4.4 (Reeb)**.** *If M is a compact manifold with a Morse function F such that F has exactly two critical points, then M is homeomorphic to the sphere in the corresponding dimension.*

We apply this result to our manifolds $M_{h,l}$. As we have seen previously, $M_{h,l}$ has a cover by two charts, $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$. We start with the first chart $\rho_1 : \pi^{-1}(U_1) \longrightarrow \phi_1(U_1) \times S^3$ defined by $\rho_1([x,y],[z,1]) = (z,y)$. Take the smooth function $F_1 : \pi^{-1}(U_1) \longrightarrow \mathbb{R}$ by the composition

$$F_1 \circ \rho_1^{-1} : \phi_1(U_1) \times S^3 \longrightarrow \mathbb{R}$$

which has the form

$$F_1 \circ \rho_1^{-1} : (z,v) \longmapsto \frac{\operatorname{Re}(v)}{\sqrt{1+\|z\|^2}} .$$

What are the critical points of $F_1 \circ \rho_1^{-1}$? Since the domain of $F_1 \circ \rho_1^{-1}$ is a product, the derivative must vanish in each component. So we ask ourselves: fixing z, what are the critical points of $F_1 \circ \rho_1^{-1}$?

Observe that, restricted to the second component, the map is just given by $v \longmapsto \operatorname{Re}(v)$ with a re-scaling. However, this map is just $a + ib + cj + dk \mapsto a$ (the projection onto the first coordinate). The critical points of this map in the sphere are just the poles ± 1. Thus we have established that $v = \pm 1$, we have to find the critical points for the restriction of $F_1 \circ \rho_1^{-1}$ to $\phi_1(U_1)$, which has the form

$$(z, \pm 1) \longmapsto \frac{\pm 1}{\sqrt{1+\|z\|^2}} .$$

Since $\phi_1(U_1)$ is isomorphic to $\mathbb{R}^4$, hence we have a problem in multivariable calculus for $x := (x_1, x_2, x_3, x_4)$:

$$\begin{aligned} \nabla F_1 \circ \rho_1^{-1}|_{(z,\pm 1)}(x) &= \left(\frac{\partial F_1 \circ \rho_1^{-1}}{\partial x_1}, \frac{\partial F_1 \circ \rho_1^{-1}}{\partial x_2}, \frac{\partial F_1 \circ \rho_1^{-1}}{\partial x_3}, \frac{\partial F_1 \circ \rho_1^{-1}}{\partial x_4} \right) \\ &= \frac{\pm 1}{(1+\|z\|^2)^{\frac{3}{2}}} (x_1, x_2, x_3, x_4) \\ &= \frac{\pm z}{(1+\|z\|^2)^{\frac{3}{2}}}. \end{aligned}$$

This gradient is null only if $z = 0$. Therefore, we show that in $\pi^{-1}(U_1)$ there are only two critical points given by $(0, \pm 1)$. It is a straightforward computation to see that the Hessian is $\mp \operatorname{Id}$. Thus the critical points we have found so far are non-degenerate.

Now, we consider the second chart $\rho_2 : \pi^{-1}(U_2) \longrightarrow \phi_2(U_2) \times S^3$ defined by $\rho_2 : ((x,y),[1:w]) = (w,x)$. Take the smooth function $F_2 : \pi^{-1}(U_2) \longrightarrow \mathbb{R}$ by the composition

$$F_2 \circ \rho_2^{-1} : \phi_2(U_2) \times S^3 \longrightarrow \mathbb{R}$$

which has the form

$$F_2 \circ \rho_2^{-1} : (w,u) \mapsto \frac{\operatorname{Re}(wu^{-1})}{\sqrt{1+\|wu^{-1}\|^2}} = \frac{\operatorname{Re}(wu^{-1})}{\sqrt{1+\|w\|^2}} .$$

Here we used the multiplicativity of the norm and the fact that $\|u\| = 1$, since $u \in S^3$.

Now, if $u^{-1} = a + ib + jc + kd$ and $w = x_1 + ix_2 + jx_3 + kx_4$, then we differentiate with respect to the first variable and we obtain

$$\begin{aligned} \nabla \operatorname{Re}(wu^{-1})|_w &= \nabla(ax_1 - bx_2 - cx_3 - dx_4)|_{w=(x_1,x_2,x_3,x_4)} \\ &= (a, -b, -c, -d) \\ &= u\,. \end{aligned}$$

Since the conjugate of u^{-1} is its own inverse for the unit quaternions, we get

$$\begin{aligned} \nabla F_2 \circ \rho_2^{-1}|_w &= \frac{\nabla \operatorname{Re}(wu^{-1})|_w \cdot \sqrt{1+\|w\|^2} - \frac{1}{2\sqrt{1+\|w\|^2}} 2w \operatorname{Re}(wu^{-1})}{(1+\|w\|^2)} \\ &= \frac{u(1+\|w\|^2) - w \operatorname{Re}(wu^{-1})}{(1+\|w\|^2)^{\frac{3}{2}}}. \end{aligned}$$

Notice the numerator is never zero; if not, we have $u(1+\|w\|^2) = w \operatorname{Re}(wu^{-1})$ and considering the norm on both sides, we have the inequality

$$1 + \|w\|^2 = \|w\| \, |\operatorname{Re}(wu^{-1})| \leq \|w\| \, \|wu^{-1}\| = \|w\|^2 \,,$$

which is impossible. As a consequence, there are no critical points in the second chart.

It remains to show the compatibility in $\pi^{-1}(U_1) \cap \pi^{-1}(U_2)$ whenever $h + l = -1$. It is enough to show the following commutative diagram

$$\begin{array}{ccc} (z, v) & \xrightarrow{F_1 \circ \rho_1^{-1}} & \frac{\operatorname{Re}(v)}{\sqrt{1+\|z\|^2}} \\ \Big\downarrow{\scriptstyle \rho_2 \circ \rho_1^{-1}} & & \Big\downarrow{\scriptstyle =} \\ \left(\frac{1}{z}, \frac{z^h v z^l}{\|z\|^{h+l}}\right) & \xrightarrow{F_2 \circ \rho_2^{-1}} & \frac{\operatorname{Re}(wu^{-1})}{\sqrt{1+\|wu^{-1}\|^2}}. \end{array}$$

For this purpose, we express wu^{-1} in terms of z and v. Recall that $u^{-1} = \frac{\overline{u}}{\|u\|^2}$ and we know $u = \frac{z^h v z^l}{\|z\|^{h+l}}$. Thus we use the properties of the norm and conjugate in order to obtain the following

$$\begin{aligned} u^{-1} &= \frac{\overline{z^h v z^l}}{\|z\|^{h+l}} \frac{1}{\|v\|^2} \\ &= \frac{\overline{z}^l \overline{v} \overline{z}^h}{\|z\|^{h+l}}\,, \end{aligned}$$

and we multiply by $w = \frac{1}{z}$:

$$
\begin{aligned}
wu^{-1} &= \frac{1}{z} \frac{\overline{z}^{l}\overline{vz}^{h}}{\|z\|^{h+l}} \\
&= \frac{\overline{z}^{l+1}\overline{vz}^{h}}{\|z\|^{h+l+2}}
\end{aligned}
$$

For the numerator, we recall that the real part of a quaternion is unchanged by conjugation with an element x, i.e., $\mathrm{Re}(xyx^{-1}) = \mathrm{Re}(y)$. For $h+l=-1$, we have $h = -1-l$ and $h+l+2 = 1$. Consequently, we get

$$
\begin{aligned}
\mathrm{Re}\left(\frac{\overline{z}^{l+1}\overline{vz}^{h}}{\|z\|^{h+l+2}}\right) &= \frac{\mathrm{Re}\left(\overline{z}^{l+1}\overline{vz}^{-1-l}\right)}{\|z\|} \\
&= \frac{\mathrm{Re}(\overline{v})}{\|z\|} \\
&= \frac{\mathrm{Re}(v)}{\|z\|}\,.
\end{aligned}
$$

For the denominator, we first calculate $\left\|wu^{-1}\right\|^2$:

$$
\begin{aligned}
\left\|wu^{-1}\right\|^2 &= \left\|\frac{\overline{z}^{l+1}\overline{vz}^{h}}{\|z\|^{h+l+2}}\right\|^2 \\
&= \frac{\|v\|^2}{\|z\|^2} \\
&= \frac{1}{\|z\|^2}\,.
\end{aligned}
$$

Hence, we substitute into the denominator:

$$
\begin{aligned}
\frac{\mathrm{Re}(wu^{-1})}{\sqrt{1+\|wu^{-1}\|^2}} &= \frac{\mathrm{Re}(v)}{\|z\|}\frac{1}{\sqrt{1+\frac{1}{\|z\|^2}}} \\
&= \frac{\mathrm{Re}(v)}{\sqrt{1+\|z\|^2}}
\end{aligned}
$$

Therefore, we have constructed maps that agree on the overlap, and they are glued together to form a smooth map F defined on $M_{h,l}$.

Lastly, we have shown that if $h+l=-1$, then $M_{h,l}$ is homeomorphic to S^7 using Reeb's Theorem 4.4. We will see in Section 4.3.2 that there exists an orientation-reversing isomorphism between $\xi_{h,l}$ and $\xi_{-l,-h}$. We conclude that if $h+l=\pm 1$, then $M_{h,l}$ is homeomorphic to S^7.

4.3. They are non-diffeomorphic to the sphere S^7

Let us stand back for a moment and consider the different spaces involved. From Section 4.1, we have a family of vector bundles $\xi_{h,l}$, and take the associated fibration given by all vectors of norm less or equal to 1. Denote by $N_{h,l}$ the total space of the fibration associated to $\xi_{h,l}$. Moreover, the boundary of $N_{h,l}$ consists of all vectors of norm equal to 1. Notice this space is precisely the manifold $M_{h,l}$.

Now we show that $M_{h,l}$ is not always diffeomorphic to S^7 through a contradiction. Assume that $M_{h,l}$ is diffeomorphic to S^7, then we can attach an 8-disc smoothly along the boundary using a collar to get a closed manifold $K_{h,l}$, as shown in the following picture:

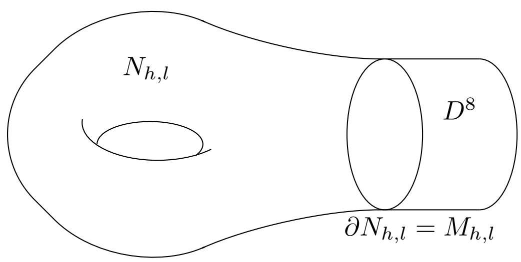


FIGURE 6. The manifold $K_{h,l}$ obtained by gluing an 8-disc along $M_{h,l}$.

In Section 4.3.3, we find the first Pontryagin class of $K_{h,l}$ using the first Pontryagin class of the total space $\xi_{h,l}$.

4.3.1. The characteristic classes of $\xi_{h,l}$

We recall two theorems from [13]:

Theorem 4.5. *For any topological group, there exists a group isomorphism*

$$\pi_n(BG) \cong \pi_{n-1}(G)\,.$$

Theorem 4.6 (Steenrod)**.** *A bijection exists between isomorphism classes of orientable n-dimensional vector bundles and homotopy classes of maps from the base space to $BSO(n)$.*

Thus $\pi_4(BSO(4)) \cong \pi_3(SO(4))$ which is $\mathbb{Z} \oplus \mathbb{Z}$ by Section 4.1.1. By Steenrod's Theorem 4.6, every 4-dimensional oriented vector bundle over S^4, is defined by a continuous map $f : S^4 \longrightarrow BSO(4)$. Then f as an element of $\pi_4(BSO(4))$, coincides with an element in $\pi_3(SO(4))$. This is precisely the pair of integers (h, l) defining the vector bundle $\xi_{h,l}$.

In addition, there is a group homomorphism for every $\alpha \in H^4(BSO(4))$

$$\begin{aligned} \Psi : \pi_4(BSO(4)) &\longrightarrow H^4(S^4) \\ [f] &\longmapsto f^*(\alpha) \end{aligned}$$

where $[f]$ denotes the homotopy class of f. We show that Ψ is a group homomorphism: recall the group structure of $\pi_4(BSO(4))$ where for two maps $f, g : S^4 \longrightarrow BSO(4)$ we have a composition with the "pinching" map along the equator $\mu : S^4 \longrightarrow S^4 \vee S^4$ as in the picture

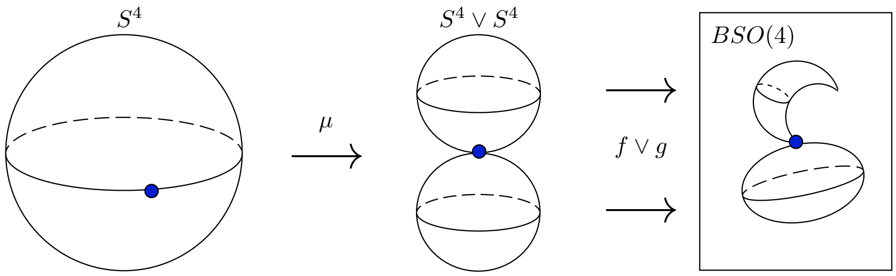


FIGURE 7. The group structure in $\pi_4(BSO(4))$ is given by this composition.

Thus

$$f + g := (f \vee g) \circ \mu : S^4 \longrightarrow S^4 \vee S^4 \longrightarrow BSO(4).$$

Besides that we have two maps $c_i : S^4 \vee S^4 \longrightarrow S^4$ where c_i collapses the i-th sphere, for $i = 1, 2$. Considering the cell structure of S^4 with 4-dimensional cells, one for each hemisphere, it is not hard to verify that

$$\mu^* : H^4(S^4 \vee S^4) \longrightarrow H^4(S^4)$$

maps the sum of both 4-dimensional cells to a generator (the sum of both hemispheres). As a consequence, the composition

$$H^4(S^4) \times H^4(S^4) \longrightarrow H^4(S^4 \vee S^4) \longrightarrow H^4(S^4)$$

is given by

$$(\alpha, \beta) \mapsto c_1^*(\alpha) + c_2^*(\beta) \mapsto \alpha + \beta$$

where $\eta : H^4(S^4) \times H^4(S^4) \longrightarrow H^4(S^4 \vee S^4)$ is an isomorphism obtained by the Mayer-Vietoris sequence.

Therefore, for two maps $f, g : S^4 \longrightarrow BSO(4)$ we obtain

$$\begin{aligned} (f+g)^*(\alpha) &= \mu^*(f \vee g)^*(\alpha) \\ &= (\mu^* \circ \eta^*) \circ ((\eta^{-1})^* \circ (f \vee g)^*)(\alpha) \\ &= (\mu^* \circ \eta^*)(f^*(\alpha), g^*(\alpha)) \\ &= f^*(\alpha) + g^*(\alpha)\,, \end{aligned}$$

which shows that Ψ is a group homomorphism.

Denote by φ the isomorphism between $\pi_3(SO(4))$ and $\pi_4(BSO(4))$. We have the following commutative triangle

$$\begin{array}{ccc} \pi_3(SO(4)) \cong \mathbb{Z}\oplus\mathbb{Z} & & \\ \downarrow \varphi & \searrow & \\ \pi_4(BSO(4)) & \xrightarrow[\Psi]{} & H^4(S^4)\cong\mathbb{Z}\,. \end{array}$$

Now we know $\xi_{h,l}$ is represented by the element $(h,l)\in\mathbb{Z}\oplus\mathbb{Z}\cong\pi_3(SO(4))$. In particular, if $g=\varphi(h,l)\in\pi_4(BSO(4))$ is represented by a classifying map with the same name $g : S^4 \longrightarrow BSO(4))$, then we use the naturality of the Pontryagin classes to deduce that

$$g^*(p_1) = p_1(\xi_{h,l})\,.$$

In other words, if we choose $p_1 \in H^4(BSO(4))$, the canonical Pontryagin class over $BSO(4)$ as our cohomology class, it follows that

$$p_1(\xi_{h,l}) = \Psi(g) = \Psi(\varphi(h,l))\,.$$

Since $\Psi\circ\varphi : \mathbb{Z}\oplus\mathbb{Z}\longrightarrow\mathbb{Z}$ is a group homomorphism, there exist integers m, k such that

$$\Psi\circ\varphi(h,l) = m\cdot h + k\cdot l\,.$$

In the next section, we calculate the coefficients m and k.

4.3.2. Determining the coefficients

Recall that if $x = a + bi + cj + dk$ is a quaternion, its conjugate is given by $\overline{x} = a - bi - cj - dk$. Furthermore, the transformation

$$\begin{array}{rcl} T : \mathbb{H} & \longrightarrow & \mathbb{H} \\ x & \longmapsto & \overline{x} \end{array}$$

is $\mathbb{R}$-linear and reverses the orientation of $\mathbb{H}$ since its determinant is -1 (where we identify $\mathbb{H}$ with $\mathbb{R}^4$). For a 4-dimensional oriented vector bundle with quaternion fiber, say ξ, we can consider its conjugate $\overline{\xi}$. This consists of taking the same underlying 4-dimensional real bundle but changing the multiplication structure to conjugate multiplication in $\mathbb{H}$. In other words, the identity transformation (in a set-theoretic sense) between the total spaces

$$\mathrm{id} : E(\xi) \longrightarrow E(\overline{\xi})$$

is turned into a conjugate-linear transformation in such a way that $\mathrm{id}(\lambda v) = \overline{\lambda} v$.

Thus if a transition map is given by $f : U_i \cap U_j \longrightarrow SO(k)$, then our new transition map is subject to the condition

$$f(x)(\overline{v}) = \overline{f(x)(v)}\,.$$

Consequently, by construction, there exists a bundle isomorphism between ξ and $\overline{\xi}$ that reverses the orientation (conjugating each fiber).

Going back to our particular case, if we conjugate $\xi_{h,l}$ where the transition map is given by $f_{h,l}(u)(v) = u^h v u^l$, then the transition map of $\overline{\xi}_{(h,l)}$ is given by

$$\tilde{f}(u)(\overline{v}) = \overline{f_{h,l}(u)(v)} = \overline{u^h v u^l} = u^{-l}\overline{v}u^{-h}.$$

Here we used that u is an element of S^3, and so its conjugate coincides with its inverse. From this it follows (switching $\overline{v}$ by v) that the transition map is

$$\tilde{f}(u)(v) = u^{-l} v u^{-h} = f_{-l,-h}(u)(v)\,.$$

This proves the following lemma:

Lemma 4.7. *There exists an orientation-reversing isomorphism which is given by the conjugate transformation*

$$\xi_{h,l} \cong \overline{\xi_{h,l}} \cong \xi_{-l,-h}\,.$$

For 4-dimensional bundles, the top Pontryagin class (in this case p_1) is independent of the orientation. Thus the first class of $\xi_{h,l}$ and of $\xi_{-l,-h}$ coincide and we obtain

$$m \cdot h + k \cdot l = m \cdot (-l) + k \cdot (-h)\,.$$

In particular, setting $(h, l) = (1, 0)$ we have that

$$m = -k\,,$$

and so

$$p_1(\xi_{h,l}) = m(h - l)\,.$$

To determine the constant m, it would suffice to evaluate in $(1, 0)$ or $(0, 1)$ and calculate the Pontryagin class of the resulting space. Luckily, $\xi_{0,1}$ is the canonical bundle over $\mathbb{HP}^1$, and the characteristic classes are already calculated. For this purpose, we need the following lemma.

Lemma 4.8. *The cohomology ring of* $\mathbb{HP}^n$ *is described as*

$$H^*(\mathbb{HP}^n) \cong \mathbb{Z}[e]/(e^{n+1})$$

where e *is the Euler class of the canonical bundle.*

Proof. Since $\mathbb{HP}^n$ has a cell structure that involves only 4-dimensional cells (the reader may compare this to the construction of $\mathbb{CP}^n$, which has only cells of even dimension) the only non-zero cohomology groups are those whose dimension is divisible by 4.

Let E be the total space of the canonical bundle γ^n. Denote by Σ the zero section and take

$$E \setminus \Sigma = \{([x], v) \mid v \in [x], v \neq 0\}.$$

However, this space is homotopy equivalent (as a bundle) to a sphere bundle with total space S^{4n+3} via the maps $([x], v) \longmapsto \frac{v}{\|v\|}$ and $v \longmapsto ([v], v)$.

Using the Gysin sequence:

$$H^i(\mathbb{HP}^n) \xrightarrow{\smile e} H^{i+4}(\mathbb{HP}^n) \xrightarrow{\pi_0^*} H^{i+4}(S^{4n+3}) \longrightarrow H^{i+1}(\mathbb{HP}^n)$$

since most $H^i(S^{4n+3})$ are zero, we have that multiplication by e gives an isomorphism that jumps 4 dimensions each time. Starting with $H^0(\mathbb{HP}) \cong \mathbb{Z}$ (since $\mathbb{HP}$ is connected), it follows that $H^4(\mathbb{HP}^n) = e\mathbb{Z}$ and so on, while the other groups are zero. This proves the lemma. ☑

Observe that the cohomological description given above has an interesting consequence: the first Chern class of the canonical bundle is zero because $c_1(\gamma^n) \in H^2(\mathbb{HP}^n) = 0$. Also, the second Chern class agrees with the Euler class $c_2(\gamma^n) = e$. Thus

$$c(\gamma^n) = 1 + c_1(\gamma^n) + c_2(\gamma^n) = 1 + e.$$

On the other hand, by equation (2), we obtain

$$1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \dots)(1 + c_1 + c_2 + \dots)$$

and so

$$1 - p_1(\gamma^n) + p_2(\gamma^n) = (1 + c_2(\gamma^n))(1 + c_2(\gamma^n)) = (1 + e)^2.$$

We conclude

$$p(\gamma^n) = 1 - 2e + e^2.$$

Now we are ready to determine the coefficients of the first Pontryagin class. Recall

$$p_1(\xi_{h,l}) = m(h - l)\eta$$

where η is a generator in cohomology. But then

$$p_1(\xi_{0,1}) = m(0 - 1)\eta = -2e.$$

Thus, depending on our choice of the generator, we have that $m = \pm 2$, and we have shown the following.

Proposition 4.9.

$$p_1(\xi_{h,l}) = \pm 2(h - l)\eta.$$

4.3.3. Calculating $p_1(K_{h,l})$

Now we use the characteristic classes of the bundles $\xi_{h,l}$ in order to calculate the characteristic classes of $K_{h,l}$ from Section 4.3.

To a vector bundle $\pi : E \longrightarrow M$ we can associate the commutative diagram

$$\begin{array}{ccc} \pi^*TM & \longrightarrow & TM \\ \downarrow & & \downarrow \pi' \\ E & \xrightarrow{\pi} & M \end{array}$$

where $\pi' : TM \longrightarrow M$ is the standard projection for the tangent bundle. In particular, we have an exact sequence

$$0 \longrightarrow \pi^*E \longrightarrow TE \longrightarrow \pi^*TM \longrightarrow 0 \ .$$

In this sequence, we write π^*E as the set of pairs that commute with both projections, that is

$$\pi^*E = \{(x, f) \in E \times E \mid \pi(x) = \pi(f)\}$$

and

$$\pi^*TM = \{(x, v) \in E \times TM \mid \pi(x) = \pi'(v)\} \, .$$

The first map in the sequence can be defined by identifying $f \in E$ as an element of T_xE (since the fiber over x is just a copy of Euclidean space, take $f - x$). The second map simply projects (tangentially) the second coordinate, $(x, v) \mapsto (x, \pi_*(v))$. It is clear that the image of the first map is contained in the kernel of the second map. By a dimension argument, this sequence is exact. Moreover, choosing a Riemannian metric on E, this sequence splits, i.e.,

$$TE \cong \pi^*E \oplus \pi^*TM \, .$$

Now we restrict both tangent bundles and the projections to vectors of norm less or equal to 1. We obtain a similar splitting and for the space $N_{h,l}$ we get

$$TN_{h,l} \cong \pi^*\xi_{h,l} \oplus \pi^*TS^4 \, .$$

It is known that by adding a trivial one-dimensional bundle to the tangent bundle of the sphere, one gets a trivial bundle, i.e.,

$$TS^4 \oplus \varepsilon^1 \cong \varepsilon^5 \, .$$

As a consequence,

$$\begin{aligned} TN_{h,l} \oplus \varepsilon^1 &\cong \pi^*\xi_{h,l} \oplus \pi^*TS^4 \oplus \varepsilon^1 \\ &\cong \pi^*\xi_{h,l} \oplus \pi^*\left(TS^4 \oplus \varepsilon^1\right) \\ &\cong \pi^*\xi_{h,l} \oplus \varepsilon^5 \, , \end{aligned}$$

hence

$$\begin{aligned} p_1(N_{h,l}) &= p_1(\pi^*\xi_{h,l} \oplus \varepsilon^5) \\ &= p_1(\pi^*\xi_{h,l}) \\ &= \pi^* p_1(\xi_{h,l}). \end{aligned}$$

Since $\pi : N_{h,l} \longrightarrow S^4$ is a homotopy equivalence, hence the map $\pi^* : H^4(S^4) \longrightarrow H^4(N_{h,l})$ is an isomorphism. Then

$$\pi^*(p_1(\xi_{h,l})) = \pi^*(2(h-l)\eta) = 2(h-l)\pi^*(\eta)$$

where $\pi^*(\eta)$ is a generator.

Now the inclusion

$$\iota : N_{h,l} \hookrightarrow K_{h,l}$$

induces an isomorphism ι^* in degree four cohomology because $K_{h,l}$ differs from $N_{h,l}$ by the addition of an 8-cell (this does not affect the lower-degree cohomology). Therefore, we have a natural identification

$$p_1(K_{h,l}) = 2(h-l)\beta$$

where β is a generator in degree four cohomology.

Finally, by Hirzebruch's signature theorem and corollary 3.13, we have the equation

$$\sigma(K_{h,l}) = \frac{1}{45}\left(7p_2(K_{h,l}) - (\pm 2(h-l))^2\right).$$

On the left-hand side, since $H^4(K_{h,l})$ is of dimension one, we have that $\sigma(K_{h,l}) = \pm 1$. We choose the fundamental class in such a way that $\sigma(K_{h,l}) = 1$, i.e., such that $\langle \beta^2, [K_{h,l}] \rangle = 1$ (we can always do this by reversing the orientation). Thus we get an equation of the form

$$\begin{aligned} 45 &= 7p_2(K_{h,l}) - \langle (\pm 2(h-l)\beta)^2, [K_{h,l}] \rangle \\ &= 7p_2(K_{h,l}) - 4(h-l)^2 \langle \beta^2, [K_{h,l}] \rangle \\ &= 7p_2(K_{h,l}) - 4(h-l)^2 . \end{aligned}$$

Reducing modulo 7 we have

$$\begin{aligned} 3 &= -4(h-l)^2 \mod 7 \\ &= 3(h-l)^2 \mod 7 , \end{aligned}$$

which simplifies to

$$(h-l)^2 = 1 \mod 7 .$$

This does not always hold! Just take h, l such that $(h-l)^2 \neq 1 \mod 7$. By way of contradiction, we have shown that the differentiable structure cannot coincide with the standard one.

5. A comparison with Milnor's original work

In his famous paper of 1956, see [7], Milnor defines an invariant associated with 7-manifolds.

We begin with a 7-dimensional, compact, oriented manifold M subject to the following condition

$$H^3(M) = 0 = H^4(M).$$

An important result is the following:

Lemma 5.1 (Thom)**.** *The oriented bordism group in degree 7 is trivial.*

As a consequence, M is the boundary of an 8-dimensional manifold, which we denoted by B. The Poincaré duality relates the long exact sequence of the pair (B, M) in cohomology and homology. This is the following commutative diagram

$$\begin{array}{ccccccccc} \longrightarrow H^3(M) & \longrightarrow & H^4(B,M) & \xrightarrow{j} & H^4(B) & \longrightarrow & H^4(M) & \longrightarrow \\ \downarrow D & & \downarrow D & & \downarrow D & & \downarrow D & \\ \longrightarrow H_3(M) & \longrightarrow & H_4(M) & \xrightarrow{j} & H_4(B,M) & \longrightarrow & H_3(M) & \longrightarrow \end{array} .$$

Since $H^3(M) = H_3(M) = H_4(M) = H^4(M) = 0$ we get that the morphisms j's are isomorphisms. For the fundamental classes $[B] \in H_8(B, M)$ and $[M] \in H_7(M)$, we set $V = H^4(B, M)/\text{Torsion}$ and we get a quadratic form $Q : V \times V \longrightarrow \mathbb{Z}$ given by

$$Q(\alpha) = \langle \alpha \smile \alpha, [B] \rangle.$$

The Poincaré duality implies that Q is non-degenerate.

Denote by $\tau(B)$ the index of Q and since j is an isomorphism we define

$$q(B) := \langle \left(j^{-1} p_1\right)^2, [B] \rangle.$$

The invariant $\lambda(M)$ is the residue modulo 7 of $2q(B) - \tau(B)$. Now we show that $\lambda(M)$ is well-defined (it is independent of B and only depends on M).

Take two disjoint manifolds B_1,B_2 such that $\partial B_1 = \partial B_2 = M$. We construct a new manifold C obtained by smoothly gluing B_1 and B_2 along M, where we keep the original orientation of B_1 and reverse the orientation of B_2. We illustrate C in the following picture:

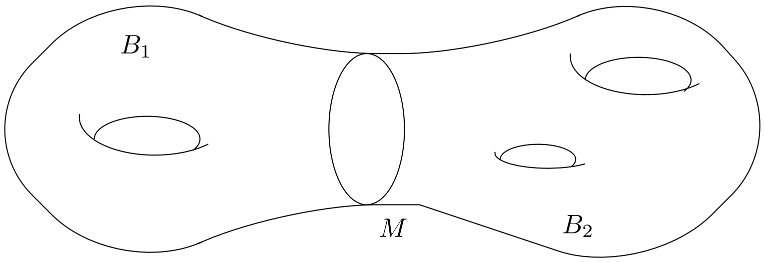


FIGURE 8. The manifold $C := B_1 \cup -M_2$.

By our choice of orientation, the fundamental class $[C]$ restricts to $[B_1]$ and $-[B_2]$.

Lemma 5.2. *The following equalities hold:*

$$\sigma(C) = \tau(B_1) - \tau(B_2)$$
$$\langle p_1^2(C), [C]\rangle = q(B_1) - q(B_2).$$

Proof. Using the Mayer-Vietories exact sequence, we have a commutative square

$$\begin{array}{ccc} H^n(B_1, M) \oplus H^n(B_2, M) & \xleftarrow[h]{} & H^n(C, M) \\ \Big\downarrow{\scriptstyle j_1 \oplus j_2} & & \Big\downarrow{\scriptstyle j'} \\ H^n(B_1) \oplus H^n(B_2) & \xleftarrow[k]{} & H^n(C)\,. \end{array} \tag{4}$$

Since $H^3(M) = H^4(M) = 0$, for $n = 4$, the square consists of isomorphisms. In particular, if $\alpha \in H^4(C)$ is any cohomology class, then there exist α_1, α_2 such that $\alpha = j'h^{-1}(\alpha_1 \oplus \alpha_2)$. Thus

$$\begin{aligned} \langle \alpha^2, [C]\rangle &= \langle (j'h^{-1}(\alpha_1 \oplus \alpha_2))^2, [C]\rangle \\ &= \langle \alpha_1^2 \oplus \alpha_2^2, [B_1] \oplus (-[B_2])\rangle \\ &= \langle \alpha_1^2, [B_1]\rangle - \langle \alpha_2^2, [B_2]\rangle\,. \end{aligned}$$

The index of the left-hand side is simply the signature of C (compare with section 2.1). This implies that $\sigma(C) = \tau(B_1) - \tau(B_2)$.

Moreover, let α_1, α_2 be defined by $\alpha_1 = j_1^{-1}p_1(B_1)$ and $\alpha_2 = j_2^{-1}p_1(B_2)$. If $\iota_i : B_i \hookrightarrow M$ denote the inclusions, then we have $\iota_i^* p_1(C) = p_1(B_i)$ by naturality of characteristic classes. As a consequence,

$$k(p_1(C)) = p_1(B_1) \oplus p_1(B_2)$$

where k is the isomorphism in (4). This implies

$$j'h^{-1}(\alpha_1 \oplus \alpha_2) = p_1(C)\,.$$

Similarly, as in the computation for the signature, we get

$$\langle p_1^2(C), [C]\rangle = \langle \alpha_1^2, [B_1]\rangle - \langle \alpha_2^2, [B_2]\rangle = q(B_1) - q(B_2)\,.$$

☑

Recall Hirzebruch's signature theorem (Corollary 3.13):

$$\sigma(C) = \langle \frac{1}{45}\left(7p_2(C) - p_1^2(C)\right), [C]\rangle\,.$$

After some manipulation we obtain

$$\langle p_1^2(C), [C]\rangle + 45\sigma(C) = 7\langle p_2, [C]\rangle\,.$$

Reducing modulo 7 it follows that

$$\langle p_1^2(C), [C]\rangle + 3\sigma(C) = 0 \mod 7$$

and multiplying by 2 and reducing the coefficients gives

$$2\langle p_1^2(C), [C]\rangle - \sigma(C) = 0 \mod 7\,.$$

Lemma 5.2 implies the following

$$2q(B_1) - \tau(B_1) = 2q(B_2) - \tau(B_2) \mod 7\,.$$

This implies that $\lambda(M)$ is well-defined.

In particular, if $h + l = -1$ we know that $M_{h,l}$ is homeomorphic to S^7, which obviously satisfies the condition $H^4(S^7) = H^3(S^7) = 0$. Furthermore, we can explicitly calculate $\lambda(M_{h,l})$ using that $\partial N_{h,l} = M_{h,l}$. In Section 4.3.3 we have computed

$$p_1(N_{h,l}) = \pm 2(h - l)\zeta$$

with $\zeta = \pi^*(\eta)$. We chose an orientation of $N_{h,l}$ such that we have the identity $\langle (j^{-1}\zeta)^2, [N_{h,l}]\rangle = 1$ (we can always do this, up to reversing the orientation) and from this we see

$$q(N_{h,l}) = \langle (j^{-1}(\pm 2(h - l)\zeta))^2, [N_{h,l}]\rangle = 4(h - l)^2\,.$$

Besides that the index τ is given by $\langle (j^{-1}\zeta)^2, [N_{h,l}]\rangle$ which is exactly 1 due to our choice of orientation. Therefore,

$$\lambda(M_{h,l}) = 2q - \tau = 8(h - l)^2 - 1 \equiv (h - l)^2 - 1 \mod 7\,.$$

If $M_{h,l}$ is diffeomorphic to the standard sphere, we take the standard 8-ball as a bounding manifold. For this case, both q and τ are zero since the fourth cohomology group is trivial. Thus $\lambda(M_{h,l}) = 0$ which means that for all values of h, l with $h + l = -1$, we must have $(h - l)^2 - 1 \neq 0 \mod 7$. This is false.

6. Closing remarks

6.1. In summary

The proof of the existence of exotic spheres resides in a plethora of topological results that were freshly developed in the fifties.

On one hand, the simple classification of oriented vector bundles over the sphere due to Steenrod's theorem and the relatively easy computation of the group $\pi_3(SO(4))$ allow us to understand all sphere bundles with structure group $SO(4)$ over the 4-sphere. Then we use Reeb's theorem to conclude that among all those bundles, some are homeomorphic to the sphere.

The work of Thom and Hirzebruch provides powerful invariants associated with manifolds.

The cohomological description of these spaces implies prescribed values for the signature and the first Pontryagin class. If they were diffeomorphic to the standard sphere, it would be possible to construct new spaces that also admit a simple but rigid description of these invariants. Eliminating the second Pontryagin class by working modulo 7 we get a contradiction by a specific choice of indices. From this, we conclude that said manifolds are homeomorphic to the 7-sphere but not diffeomorphic.

6.2. A glimpse ahead

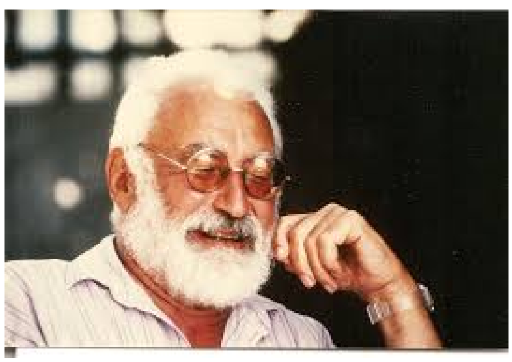


FIGURE 9. Michel Kervaire.

This was only the beginning of discovering the so-called 'Exotic structures'. The most immediate progress came from Milnor and Kervaire [3], who enumerated all exotic spheres in 1963, summing up to 28 different exotic spheres in dimension 7. The monoid of exotic structures in dimensions different from four has been extensively studied and turns out to be a group.

It is worth mentioning that even if many important results have been obtained in this direction, we still need to understand more about exotic structures. A combination of the work by Moise and Stallings [12] shows that $\mathbb{R}^n$ has no exotic structure for n different from 4, while Freedman was the first to exhibit the existence of an "exotic $\mathbb{R}^4$" [2]. A continuum of exotic structures has been found for $\mathbb{R}^4$. Finally, the question about exotic structures in the 4-sphere remains open.

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