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Numerical representation of topological real algebras

Representación numérica de álgebras reales topológicas

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Abstract. We show that the isomorphism provided by the Gelfand-Mazur theorem for a commutative pre-ordered real Banach algebra $\mathcal{A}$ with unit defines a numerical representation which is compatible with the order structure.

Key words and phrases. Gelfand-Mazur's theorem, Real algebra, Continuous numerical representations of total pre-orders.

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Resumen. Utilizamos que el isomorfismo dado por el teorema de Gelfand-Mazur para álgebras reales de Banach reales preordenadas conmutativas con unidad a su vez define una representación numérica compatible con la estructura de orden.

Palabras y frases clave. Teorema de Gelfand-Mazur, álgebra real, representaciones reales continuas numéricas de preóordenes.

1. Introduction

Let $(\mathcal{X}, \preceq)$ be a pre-ordered set and let $(\mathbb{R}, \leq)$ denote the ordered field of the real numbers. A real-valued function $f : \mathcal{X} \to \mathbb{R}$ is called a *numerical representation* of $(\mathcal{X}, \preceq)$ if $f(x) \leq f(y)$, whenever $x \preceq y$.

In economic theory, if $(\mathcal{X}, \preceq)$ is a totally pre-ordered topological space and $\preceq$ verifies some topological compatibility, f is called a utility function, usually denoted by u. This interplay between topology and order is an important tool to characterize topological properties of the underlying topological space $\mathcal{X}$ and has many applications ranging from economic theory to functional analysis, see [3, 4, 2] and the bibliography therein.

On the other hand, the study of orders in Banach and C^*-algebras has attracted the attention of operator algebraists for a long time, see [6, 7, 9]. In [3] and [4] the authors studied numerical representations on semi-topological real algebras and in [5] numerical representations of pre-ordered semigroups were considered. In both articles, the order and algebraic structures are compatible.

In this note we consider a similar algebraic setting. Let $(\mathcal{A}, \preceq)$ be a pre-ordered topological real algebra. Our aim is to show that the isomorphism provided by the Gelfand-Mazur theorem can be used to provide a numerical representation of $\preceq$ which is compatible with the algebraic structure on $\mathcal{A}$. In particular, the pre-order $\preceq$ induced by a suitable cone on $\mathcal{A}$ extends to a total pre-order. The proofs use an argument due to Oudadess [8] on a sharp version of the Gelfand-Mazur theorem for real topological algebras.

2. Preliminaries

For convenience of the reader, we begin by recalling some background on ordered structures, continuous numerical representability and ordered Banach algebras.

2.1. Ordered structures and numerical representation functions

A pre-order $\preceq$ on a nonempty set $\mathcal{X}$ is a binary relation which is *reflexive* and *transitive*. An *antisymmetric* pre-order is called an order. A *total pre-order* (resp., *total order*) on $\mathcal{X}$ is a pre-order (resp., order) such that if $x, y \in \mathcal{X}$ then $x \preceq y$ or $y \preceq x$.

The *asymmetric relation* $\prec$ associated with a total pre-order $\preceq$ on $\mathcal{X}$ is defined to be

$$x \prec y \Leftrightarrow x \preceq y \wedge \neg(y \preceq x).$$

A *linear* (or total) *extension* of a partial order $\preceq$ on $\mathcal{X}$ is a total order $\preceq^*$ on $\mathcal{X}$ that is compatible with $\preceq$, i.e.

$$(x \preceq y \Rightarrow x \preceq^* y) \text{ and } (x \prec y \Rightarrow x \prec^* y).$$

According to the order extension principle, every partial order has a linear extension.

Given a totally pre-ordered set $(\mathcal{X}, \preceq)$, a real-valued function $f : \mathcal{X} \to \mathbb{R}$ is said to be order preserving, also known as a *utility function* or a *numerical representation* for $\preceq$, if

$$x \preceq y \Leftrightarrow f(x) \leq f(y), \forall x, y \in \mathcal{X}.$$

We shall adopt the latter designation throughout the paper. Now, suppose $\mathcal{X}$ is a topological space. A total pre-order $\preceq$ on $\mathcal{X}$ is said to be *continuously representable* if there is a continuous numerical representation $f : \mathcal{X} \to \mathbb{R}$ representing $\preceq$, where $\mathbb{R}$ is endowed with the usual topology.

Define the lower and upper contour sets on $(\mathcal{X}, \preceq)$ by $L(x) = \{y \in \mathcal{X} : y \preceq x\}$ and $G(x) = \{y \in \mathcal{X} : x \preceq y\}$, respectively. Then, the family of sets $\{L(x), G(x) : x \in \mathcal{X}\}$ form a sub-basis for a topology on $\mathcal{X}$, called the *order topology*. If $(\mathcal{X}, \preceq)$ is a topological totally pre-ordered space, then $\preceq$ is said to be *continuous* if $L(x)$ and $G(x)$ are closed sets, for all $x \in \mathcal{X}$. On the other hand, a topology on $\mathcal{X}$ is said to be totally pre-orderable if it coincides with the order topology, for some total pre-order $\preceq$ on $\mathcal{X}$.

Next result provides topological conditions for a continuous total pre-order defined on a topological space to be continuously representable by a numerical representation.

Theorem 2.1. (Eilenberg) *Let $(\mathcal{X}, \tau)$ be a connected and separable topological space. Let $\preceq$ be a continuous total pre-order on $\mathcal{X}$. Then, there is a numerical representation function which represents $\preceq$.*

2.2. Ordered Banach Algebras

By $\mathcal{A}$ we denote an associative algebra over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). $\mathcal{A}$ is a *topological algebra* provided that it has a topology and addition, multiplication and scalar multiplication are continuous functions with respect to that topology. If $\mathcal{A}$ has a unit $e = e_{\mathcal{A}}$ we call it a *unital* algebra. It is always possible to adjoin a unit to an algebra by a well known procedure. A special case of topological algebras are *normed algebras*, i.e., those whose topology is induced by a norm $\|.\|$. A *Banach algebra* is a complete normed algebra. It is called a *real Banach algebra* or *complex Banach algebra* depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, respectively.

A (two-sided) *ideal* of $\mathcal{A}$ is a subspace $\mathcal{I}$ of $\mathcal{A}$, denoted by $\mathcal{I} \lhd \mathcal{A}$, such that $\mathcal{A}\mathcal{I} \subset \mathcal{I}$ and $\mathcal{I}\mathcal{A} \subset \mathcal{I}$. It is called *proper* if $\mathcal{I} \neq \{0\}$ and $\mathcal{I} \neq \mathcal{A}$. An ideal $\mathcal{M} \lhd \mathcal{A}$ is called *maximal* if, for every ideal $\mathcal{I}$ such that $\mathcal{I} \supset \mathcal{M}$ we have that $\mathcal{I} = \mathcal{M}$. We denote by $\mathcal{A}_+ = \{x^2 : x \in \mathcal{A}\}$ the set of squares of $\mathcal{A}$. Ideals of $\mathcal{A}$ containing $\mathcal{A}_+$ will play a role in the sequel.

A *homomorphism* of Banach $\mathbb{K}$-algebras is a $\mathbb{K}$-linear map $\phi : \mathcal{A} \to \mathcal{B}$ such that $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in \mathcal{A}$. If ϕ is a real or complex valued homomorphism of $\mathcal{A}$, that is, $\mathcal{B}$ is either $\mathbb{R}$ or $\mathbb{C}$, then Kerϕ is a maximal ideal of $\mathcal{A}$. In particular, if $\mathcal{A}$ is commutative then $\mathcal{A}/\text{Ker}\phi$ is a commutative real division algebra.

Every real banach algebra $\mathcal{A}$ may be complexified as follows. Write the direct sum $\mathcal{A}_{\mathbb{C}} = \mathcal{A} \oplus i\mathcal{A}$ in the category of real vector spaces and define in $\mathcal{A}_{\mathbb{C}}$ addition in the usual way, scalar multiplication by $\alpha + i\beta \in \mathbb{C}$ as

$$(\alpha, \beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y),$$

and multiplication by

$$(x, y)(x', y') = (xx' - yy', yx' + xy').$$

Then, $\mathcal{A}_{\mathbb{C}}$ is a complex Banach algebra and $\mathcal{A}$ is identified with its copy $\mathcal{A} \oplus i0$ in $\mathcal{A}_{\mathbb{C}}$. Now we recall the notion of spectrum of an element of $\mathcal{A}$. As pointed out in [1, p.294], contrary to the complex case, an element in a real Banach algebra may have empty spectrum. Denote by $G(\mathcal{A})$ the set of invertible elements of $\mathcal{A}$. The *spectrum* of an element $x \in \mathcal{A}$ is the set

$$\sigma(x) = \{\alpha \in \mathbb{R} : x - \alpha e \notin G(\mathcal{A})\}$$

and the *complex spectrum* of x is the set

$$\sigma_{\mathbb{C}}(x) = \{\alpha + i\beta \in \mathbb{C} : (x - \alpha e)^2 + \beta^2 e \notin G(\mathcal{A})\}.$$

To define a partial pre-order on a real Banach algebra $\mathcal{A}$ we now recall the notion of an algebra cone.

Definition 2.2. A nonempty subset $\mathcal{C} \subset \mathcal{A}$ is called a cone if it satisfies the following conditions:

(i) $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$,

(ii) $\lambda\mathcal{C} \subset \mathcal{C}$, for all $\lambda \geq 0$.

If, in addition, the following properties hold

(iii) $\mathcal{C}.\mathcal{C} \subset \mathcal{C}$,

(iv) $1 \in \mathcal{C}$

then $\mathcal{C}$ is called an algebra cone. Moreover, if

(v) $\mathcal{C} \cap -\mathcal{C} = \{0\}$

the cone $\mathcal{C}$ is said to be proper.

A cone $\mathcal{C}$ induces a binary relation $\preceq$ on $\mathcal{A}$ as follows. Given $x, y \in \mathcal{A}$,

$$x \preceq y \text{ if, and only if, } y - x \in \mathcal{C}.$$

This allows us to write

$$\mathcal{C} = \{x \in \mathcal{A} : x \succeq 0\}. \tag{1}$$

It can easily be shown that the binary relation $\preceq$ induced by a cone $\mathcal{C}$ is a pre-order, i.e it is reflexive and transitive. However, it does not have to be antisymmetric. In view of the above characterization, $\mathcal{C}$ is also called the *positive cone* of the partially ordered algebra $(\mathcal{A}, \preceq)$.

We have the following characterization of properness.

Proposition 2.3. *The cone $\mathcal{C}$ is proper if and only if the ordering is antisymmetric.*

Proof. Suppose $\mathcal{C}$ is a proper cone and let $x \preceq y$ and $y \preceq x$. Then,

$$y - x \in \mathcal{C} \text{ and } x - y = -(y - x) \in \mathcal{C} \Leftrightarrow y - x \in -\mathcal{C}.$$

So, $y - x \in \mathcal{C} \cap -\mathcal{C} = \{0\}$ and $y = x$.

Conversely, suppose $\mathcal{C}$ is not proper. Then, there exist $x \in \mathcal{C} \cap -\mathcal{C}$, with $x \neq 0$. Let $x = -y$. We have

$$x + (-y) = 2x \in \mathcal{C} \Rightarrow y \preceq x$$

$$y + (-x) = 2y \in \mathcal{C} \Rightarrow x \preceq y$$

Therefore, $x \preceq y$, $y \preceq x$ and $x - y = 2x \neq 0$. ☑

When the algebraic structure of $\mathcal{A}$ is compatible with the order structure, we have a (partially) ordered Banach algebra. Taking into account the partial pre-order $\preceq$ induced by a cone, we may rewrite definition 2.2 and call the pair $(\mathcal{A}, \preceq)$ a partially ordered Banach algebra ($\mathcal{A}$ a real or complex unital Banach algebra), if the ordering $\preceq$ verifies the following conditions:

(i) $0 \preceq x, y \Rightarrow 0 \preceq x + y$,

(ii) $0 \preceq x, \lambda \geq 0 \Rightarrow 0 \preceq \lambda x$,

(iii) $0 \preceq x, y \Rightarrow 0 \preceq xy$,

(iv) $0 \preceq 1$.

If $\phi : \mathcal{A} \to \mathcal{B}$ is a homomorphism of Banach algebras (real or complex) and $\mathcal{C}$ is a cone in $\mathcal{A}$ then the set $\phi(\mathcal{C}) = \{\phi(c) : c \in \mathcal{C}\}$ is an algebra cone in $\mathcal{B}$. In particular, if $\mathcal{I} \lhd \mathcal{A}$ is a closed ideal and $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is the canonical surjection then $\pi(\mathcal{C})$ is an algebra cone in $\mathcal{A}/\mathcal{I}$, although we cannot guarantee the closedness of $\pi(\mathcal{C})$ in case $\mathcal{C}$ is closed, see [9, p.492]. Similarly, it is an easy exercise to show that, if $\mathcal{C}'$ is an algebra cone of $\mathcal{B}$, then the pre-image $\phi^{-1}(\mathcal{C}')$ is an algebra cone of $\mathcal{A}$.

3. Main result

A theorem of Mazur states that every topological real division algebra is isomorphic either to the field of real numbers $\mathbb{R}$, the field of complex numbers $\mathbb{C}$ or the skew field of quaternions $\mathbb{H}$. This result became known as the Gelfand-Mazur theorem for real topological algebras. On the other hand, the classical complex Gelfand-Mazur theorem, established by Gelfand, says that every normed division algebra over the complex field $\mathbb{C}$ is isomorphic to $\mathbb{C}$. In [8], Oudadess defines sufficient conditions for the real Gelfand-Mazur theorem to be as sharp as the complex case, i.e to guarantee, under certain conditions, that a topological real division algebra is isomorphic to $\mathbb{R}$.

Definition 3.1. Let $\mathcal{A}$ be a real topological algebra. We say that $\mathcal{A}$ satisfies the *Real Gelfand-Mazur conditions* (*RGM* for short) if $\sigma(x) \neq \varnothing$ for every $x \in \mathcal{A}$, and if there is a cone $\mathcal{C}_{RGM} \subset \mathcal{A}$ containing all the squares, i.e $\mathcal{C} \supset \mathcal{A}_+$.

The above definition is pertinent since, as mentioned before, although the complex spectrum $\sigma_{\mathbb{C}}(x)$ of an element x in the complexification $\mathcal{A}_{\mathbb{C}}$ of $\mathcal{A}$ is never empty, the real spectrum may be empty. Note also that in the definition we do not require the algebra $\mathcal{A}$ to be commutative.

Example 3.2. A commutative example is given by the algebra of real valued continuous functions on a compact set $\mathcal{X}$, $\mathcal{A} = C(\mathcal{X}, \mathbb{R})$, where we have

$$\mathcal{A}_+ = \{f \in C(\mathcal{X}, \mathbb{R}) : f \geq 0\}.$$

3.1. Commutative algebras

Next result is due to Oudadess and establishes that the *RGM* conditions are sufficient conditions for a topological real division algebra $\mathcal{A}$ to be isomorphic to $\mathbb{R}$. For the sake of completeness we include the proof.

Proposition 3.3. [8] *Let $\mathcal{A}$ be a real commutative unital topological algebra verifying the RGM condition. If $\mathcal{A}$ is a division algebra then $\mathcal{A}$ is isomorphic to $\mathbb{R}$.*

Proof. Since $(x+e)^2, (x-e)^2 \in \mathcal{C}_{RGM}$, for all x in $\mathcal{A}$, it follows that

$$x = \frac{1}{4}(x+e)^2 - \frac{1}{4}(x-e)^2$$

and so $\mathcal{A} = \mathcal{C}_{RGM} - \mathcal{C}_{RGM}$.

Define $\mathcal{I} = \mathcal{C}_{RGM} \cap -\mathcal{C}_{RGM}$. The set $\mathcal{I}$ is clearly a subspace. It is also easy to show that $\mathcal{I}\mathcal{A} \subset \mathcal{I}$. In fact, given $xa \in \mathcal{I}\mathcal{A}$, we have $xa = xr - xs$, for some $r, s \in \mathcal{C}_{RGM}$. Since xr and xs are in $\mathcal{I}$, the result follows. A similar argument shows that $\mathcal{A}\mathcal{I} \subset \mathcal{I}$ and we conclude that $\mathcal{I}$ is a two-sided ideal of $\mathcal{A}$.

By hypothesis, $\mathcal{A}$ is a division algebra and so it is simple. It follows that

$$\mathcal{I} = \mathcal{C}_{RGM} \cap -\mathcal{C}_{RGM} = \{0\}.$$

Let $x \in \mathcal{A}$. Since $\sigma(x) \neq \varnothing$ there is $\alpha + i\beta \in \sigma_{\mathbb{C}}(x)$ such that

$$x - (\alpha + i\beta)e \notin G(\mathcal{A}).$$

Since $\mathcal{A}$ is a division algebra, $x - (\alpha + i\beta)e = 0$. But

$$(x - \alpha e)^2 = -\beta^2 e \Rightarrow x - \alpha e \in \mathcal{C} \cap -\mathcal{C} = \{0\}.$$

Therefore, $x = \alpha e$. ☑

We may now prove our main result.

Theorem 3.4. *Let $\mathcal{A}$ be a unital real commutative topological algebra verifying the RGM condition. If the cone $\mathcal{C}_{RGM}$ is closed and proper then $\mathcal{C}_{RGM}$ defines a continuous partial order $\preceq$ on $\mathcal{A}$ which extends to a total pre-order $\preceq^*$ and is compatible with the algebraic structure. In particular, $\mathcal{A}$ admits a character $\chi : \mathcal{A} \to \mathbb{R}$.*

Proof. Define as usual

$$x \preceq y \Leftrightarrow y - x \in \mathcal{C}_{RGM}.$$

As we have seen in Proposition 3.3, the set $\mathcal{I} = \mathcal{C}_{RGM} \cap -\mathcal{C}_{RGM}$ is a two sided ideal of $\mathcal{A}$. Let $\mathcal{M}$ be a maximal ideal containing $\mathcal{I}$. Since $\mathcal{A}$ is commutative, $\mathcal{A}/\mathcal{M}$ is isomorphic to the topological real algebra $\mathbb{R}$. Now, since $\mathcal{C}_{RGM}$ is closed, the upper and lower contour sets

$$G(x) = \{y \in \mathcal{A} : x \preceq y\} = x + \mathcal{C}_{RGM}$$

and

$$L(x) = \{y \in \mathcal{A} : y \preceq x\} = x - \mathcal{C}_{RGM}$$

are also closed. Therefore, $\preceq$ is continuous.

Now, since $\mathcal{A}/\mathcal{M}$ is isomorphic to $\mathbb{R}$ as real algebras, using the canonical surjection $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{M}$, there is a continuous total pre-order $\preceq^*$ such that $(\mathcal{A}, \preceq^*)$ is a totally pre-ordered real algebra. We claim that $\preceq^*$ is a linear extension of the continuous order $\preceq$ on $\mathcal{A}$ induced by $\mathcal{C}_{RGM}$.

In fact, for $x, y \in \mathcal{A}$,

$$x \preceq y \Leftrightarrow y - x \in \mathcal{C}_{RGM} \Rightarrow \pi(y - x) \in \pi(\mathcal{C}_{RGM})$$

and

$$(\rho \circ \pi)(y - x) \in (\rho \circ \pi)(\mathcal{C}_{RGM}) = \mathbb{R}_0^+,$$

that is, $(\rho \circ \pi)(y - x) \geq 0 \Leftrightarrow (\rho \circ \pi)(x) \leq (\rho \circ \pi)(y)$. Hence,

$$x \preceq y \Rightarrow x \preceq^* y.$$

The proof that

$$x \prec y \Rightarrow x \prec^* y$$

is similar. Finally, $\chi = \rho \circ \pi$ is clearly a character of $\mathcal{A}$ and moreover, if the maximal ideal $\mathcal{M}$ is proper, then χ is nontrivial. ☑

In the conditions of Theorem 3.4, we conclude the following result.

Corollary 3.5. *The total pre-order $\preceq^*$ is continuous representable by the character $\chi = \rho \circ \pi$.*

Proof. Clearly, χ is continuous. Since $\preceq^*$ is induced by the cone $\mathcal{C}^* = \{x \in \mathcal{A} : x \succeq^* 0\}$, which in turn is the pre-image of the usual total order of the real algebra $\mathbb{R}$, we have:

$$\begin{aligned} x \preceq^* y &\Leftrightarrow y - x \in \mathcal{C}^* \\ &\Leftrightarrow (\rho \circ \pi)(y - x) = \chi(y - x) \geq 0 \\ &\Leftrightarrow \chi(x) \leq \chi(y). \end{aligned}$$

☑

***Remark* 3.6.** In [4], the authors defined a maximal ideal $\mathcal{I}(0)$ which is crucial in the proof of their Theorem 3.14. In our setting, the ideal $\mathcal{I}(0)$ admits the following interpretation:

$$\begin{aligned} \mathcal{I}(0) &= \{x \in \mathcal{A} : x \sim 0\} \\ &= \{x \in \mathcal{A} : 0 \preceq^* x \preceq^* 0\} \\ &= \{x \in \mathcal{A} : \chi(x) = 0\} \\ &= ker(\chi). \end{aligned}$$

Example 3.7. Let ℓ^∞ denote the real space of bounded real-valued sequences. Define a multiplication on ℓ^∞ componentwise, so that $x.y$ is the sequence

$$(x.y)_n = x_n y_n \, , \, n \in \mathbb{N},$$

with $x, y \in \ell^\infty$. Let $\Sigma(\ell^\infty)$ denote the spectrum of the real Banach algebra ℓ^∞, i.e., the set of multiplicative linear functionals. For each natural number $n \in \mathbb{N}$, the n-th projection $\pi_n : x \in \ell^\infty \mapsto x_n \in \mathbb{R}$ defines a multiplicative linear functional of ℓ^∞ and so $\mathbb{N} \subset \Sigma(\ell^\infty)$. However, the inclusion is proper since $\mathbb{N}$ is not w^*-compact. In fact, the spectrum $\Sigma(\ell^\infty)$, can be identified with the Stone-Čech compatification $\beta(\mathbb{N})$ of $\mathbb{N}$.

Now, we have:

$$\ell^\infty / \ker \pi_n \simeq \mathbb{R},$$

and the evaluation map $\mathrm{ev}_{\pi_n} : \ell^\infty \to \mathbb{R} \, , \, x \mapsto \mathrm{ev}_{\pi_n}(x) = \pi_n(x) = x_n$ defines a nontrivial real character of ℓ^∞. In particular, we have a total pre-order $\preceq_n$ on ℓ^∞ which is continuously representable by the evaluation map, that is,

$$x \preceq_n y \Leftrightarrow \mathrm{ev}_{\pi_n}(x) \leq \mathrm{ev}_{\pi_n}(y) \Leftrightarrow x_n \leq y_n.$$

The above total pre-order is compatible with the algebraic structure of ℓ^∞.

3.2. Noncommutative algebras

We will now extend the results of the previous section to unital noncommutative algebras.

Definition 3.8. A real Banach algebra $\mathcal{A}$ is called strictly real if every element has real spectrum.

A simple criteria for strictly real Banach algebras is the following. An algebra $\mathcal{A}$ is strictly real if and only if $e + x^2$ is invertible for every x in $\mathcal{A}$, see [6, p.250].

The radical of $\mathcal{A}$, denoted $Rad(\mathcal{A})$, is the intersection of all its maximal ideals. Clearly, $Rad(\mathcal{A})$ is an ideal of $\mathcal{A}$. Write $\mathcal{A}' = \mathcal{A}/Rad(\mathcal{A})$.

Next result is elementary but we include a proof for convenience of the reader.

Lemma 3.9. The pre-image of a total pre-order under an algebra homomorphism is a total pre-order.

Proof. Let $\phi : \mathcal{A} \to \mathcal{B}$ be an algebra homomorphism and let $\mathcal{C}$ be a cone in $\mathcal{B}$ inducing a total pre-order $\preceq_{\mathcal{C}}$ on $\mathcal{B}$. Let $x, y \in \mathcal{A}$. Then, either we have

$$\phi(x) \preceq_{\mathcal{C}} \phi(y) \Leftrightarrow \phi(y) - \phi(x) = \phi(y - x) \in \mathcal{C} \tag{2}$$

or we have

$$\phi(y) \preceq_{\mathcal{C}} \phi(x) \Leftrightarrow \phi(x) - \phi(y) = \phi(x - y) \in \mathcal{C}. \tag{3}$$

From (2) and (3), and denoting by $\preceq_{\phi^{-1}(\mathcal{C})}$ the pre-order induced by the pre-image of $\mathcal{C}$ under ϕ we conclude that

$$x \preceq_{\phi^{-1}(\mathcal{C})} y \text{ or } y \preceq_{\phi^{-1}(\mathcal{C})} x,$$

that is, the pre-order induced by the pre-image is total. ☑

Now, we restrict to the case of strictly real Banach algebras. Theorem 3.10 below extends, under certain conditions, Theorem 3.4 and Corollary 3.5 to noncommutative real Banach algebras.

Theorem 3.10. *Let $\mathcal{A}$ noncommutative, strictly real, unital Banach algebra such that $\mathcal{A}' = \mathcal{A}/\mathrm{Rad}(\mathcal{A})$ verifies the RGM conditions. Suppose that the corresponding cone $\mathcal{C}_{RGM}$ is closed and proper. Then there exists a total pre-order $\preceq^*$ on $\mathcal{A}$ which is continuously representable by a numerical representation $f : \mathcal{A} \to \mathbb{R}$. Moreover, f is a real algebra homomorphism.*

Proof. Since $\mathcal{A}$ is strictly real, by Kaplansky's theorem [7, Th.4.8], $\mathcal{A}'$ is commutative. From Theorem 3.4, the cone $\mathcal{C}_{RGM}$ defines a continuous partial order $\preceq'$ which extends to a total pre-order $\preceq'^{*}$ on $\mathcal{A}'$.

Let $\preceq$ denote the partial order on $\mathcal{A}$ induced by the pre-image cone $\pi^{-1}(\mathcal{C}_{RGM})$. Since $\mathcal{C}_{RGM}$ is closed and π is continuous then $\pi^{-1}(\mathcal{C}_{RGM})$ is closed. In particular,

$$G(x) = \{y \in \mathcal{A} : x \preceq y\} = x + \pi^{-1}(\mathcal{C}_{RGM})$$

and

$$L(x) = \{y \in \mathcal{A} : y \preceq x\} = x - \pi^{-1}(\mathcal{C}_{RGM})$$

are also closed and so $\preceq$ is continuous.

Denote by $\mathcal{C}^{'*}$ the cone associated with the total pre-order $\preceq^{'*}$ on $\mathcal{A}'$. By Lemma 3.9, the pre-image $\pi^{-1}(\mathcal{C}^{'*}) := \mathcal{C}^*$ induces a total pre-order $\preceq^*$ on $\mathcal{A}$ which is clearly a linear extension of $\preceq$ because the inclusion $\mathcal{C}_{RGM} \subset \mathcal{C}^{'*}$ implies $\pi^{-1}(\mathcal{C}_{RGM}) \subset \pi^{-1}(\mathcal{C}^{'*})$.

Finally, from Corollary 3.5, the total pre-order $\preceq^{'*}$ is continuously representable by a character $\chi : \mathcal{A}' \to \mathbb{R}$. Therefore,

$$x \preceq^* y \Leftrightarrow \pi(x) \preceq^{'*} \pi(y) \Rightarrow (\chi \circ \pi)(x) \leq (\chi \circ \pi)(y).$$

and

$$x \prec^* y \Leftrightarrow \pi(x) \prec^{'*} \pi(y) \Rightarrow (\chi \circ \pi)(x) < (\chi \circ \pi)(y)$$

and the result follows, with f given by the composition

$$f = \chi \circ \pi : \mathcal{A} \to \mathcal{A}' \to \mathbb{R}.$$

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