Revista Colombiana de Matemáticas
Volumen 57(2023)1, páginas 123-154

On the some equations inequalities in Musielak–Orlicz spaces with measure data

Desigualdades de ecuaciones sobre espacios Musielak–Orlicz con condiciones iniciales medibles

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Abstract. Our aim in this present work is to prove the existence of a solution for the nonlinear unilateral parabolic problems associated to the equation

$$\frac{\partial u}{\partial t} - \operatorname{div}\ a(x,t,u,\nabla u) - \operatorname{div}\Phi(x,t,u) = \mu \quad \text{in } Q_T = \Omega \times (0,T),$$

where the lower order term Φ satisfies a generalized natural growth condition and the datum μ belongs to $L^1(Q) + E_\psi(Q)$.

Key words and phrases. Musielak-Orlicz-Sobolev spaces, Unilateral parabolic problem, entropy solutions, truncations.

2020 Mathematics Subject Classification. 35J25, 35J60, 46E30.

Resumen. El objetivo de este trabajo es probar la existencia de soluciones para problemas parabólicos unilaterales asociados a la ecuación

$$\frac{\partial u}{\partial t} - \operatorname{div}\ a(x,t,u,\nabla u) - \operatorname{div}\Phi(x,t,u) = \mu \quad \text{in } Q_T = \Omega \times (0,T),$$

donde el término Φ satisface una condición natural generalizada de crecimiento y la condición μ pertenece al espacio $L^1(Q) + E_\psi(Q)$.

Palabras y frases clave. Espacios de Musielak-Orlicz-Sobolev, problemas parabólicos unilaterales, soluciones de entropía.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, $Q_T = \Omega \times (0,T)$ where T is a positive real number and ψ is an Orlicz function. Let $\mathrm{A} : D(\mathrm{A}) \subset W_0^{1,x} L_\varphi(Q_T) \to W^{-1,x} L_{\overline{\varphi}}(Q_T)$ be an operator of Leray-Lions type of the form:

$$\mathrm{A}(u) := -\operatorname{div}\ a(x,t,u,\nabla u).$$

In this note, we are interested in the existence of entropy solutions to the following nonlinear unilateral parabolic problems

$$\begin{cases} u \in \mathbf{K}_\psi = \Big\{ u \in W_0^{1,x} L_\varphi(Q_T) : u \geq \psi \text{ a.e. in } Q_T \Big\} & \\ \dfrac{\partial u}{\partial t} + \mathrm{A}(u) - \operatorname{div} \Phi(x,t,u) = f - \operatorname{div}\ F & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{1}$$

where $u_0 \in L^1(\Omega)$, $f \in L^1(Q_T)$, $F \in \left(E_\psi(Q)\right)^N$ and Φ satisfies a natural growth condition (see assumption (H_4)).

The notion of entropy solution was used due to very weak assumptions on the given data, this notion of solution was introduced in [10] in the elliptic case and by Prignet [35] in the parabolic case.

The existence of solutions for a parabolic equation with measure data in Orlicz-Sobolev spaces was shown by Meskine in [31], see also [20]. The existence of solutions to some unilateral parabolic problems in the framework of Orlicz spaces with given data in L^1 is shown by Meskine et al. in [1], see also [8]. Elmahi and Meskine in [18] study the Cauchy-Dirichlet problem and prove the existence of at least one entropy solution to strongly nonlinear parabolic equations with natural growth terms in Orlicz spaces (isotropic, homogeneous) provided that the data is in L^1.

In the framework of variable exponent Sobolev spaces, Azroul, Redwane and Yazough have shown in [9] the existence of solutions for the unilateral elliptic problem where the second member f is in $L^1(\Omega)$.

In the setting of Musielak-Orlicz spaces and in the variational case, Benkirane and Sidi El vally [15] proved the existence of solutions for the obstacle elliptic problem where they generalized the work of Gossez and Mustonen in [19], there are several papers worth mentioning that deal with the existence solutions of elliptic and parabolic problems under various assumptions (see [13, 25, 24, 21, 12, 22, 23, 11, 14, 3, 4, 2, 5, 7, 6] for more details).

The purpose of this paper is to prove, in the setting of Musielak spaces, an existence result for unilateral problem corresponding to problem (1) in the case where the datum μ belongs to $L^1(Q) + E_\psi(Q)$ and no Δ_2-condition will be assumed on the conjugate function of the Musielak Orlicz function φ.

Let us briefly summarize the contents of this article: in Section 2 we compile some well-known preliminaries, results and properties of Musielak-Orlicz-Sobolev spaces and inhomogeneous Musielak-Orlicz-Sobolev spaces. Section 3 is devoted to setting the assumptions and the proof of the main result.

2. Background

Here we give some definitions and properties that concern Musielak-Orlicz spaces (see [33]).

2.1. Musielak-Orlicz functions

Let Ω be an open subset of $\mathbb{R}^n$.

A Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that

a) $\varphi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0, \varphi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0, \qquad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0.$$

b) $\varphi(\cdot,t)$ is a Lebesgue measurable function.

Now, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e., the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2 -condition if for some $k > 0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2}$$

When (2) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2 -condition near infinity. Let φ and γ be two Musielak-orlicz functions, we say that φ dominates γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0)\,.$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0, \quad \left(\text{ resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega} \frac{\gamma(x,ct)}{\varphi(x,t)}\right) = 0\right).$$

***Remark* 2.1.** (see [28]) If $\gamma \prec\prec \varphi$ near infinity, then $\forall\varepsilon > 0$ there exists a nonnegative integrable function h, such that

$$\gamma(x,t) \leq \varphi(x,\varepsilon t) + h(x). \text{ for all } t \geq 0 \text{ and for a. e. } x \in \Omega. \tag{3}$$

2.2. Musielak-Orlicz-Sobolev spaces

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

The set $K_\varphi(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } /\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}.$$

For a Musielak-Orlicz function φ we put: $\psi(x, s) = \sup_{t>0}\{st - \varphi(x, t)\}, \psi$ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s in the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0/ \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\},$$

which is called the Luxemburg norm and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent (see [33]).

We will also use the space $E_\varphi(\Omega)$ defined by

$$E_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } /\rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for all } \lambda > 0\right\}.$$

A Musielak function φ is called locally integrable on Ω if $\rho_\varphi(t\chi_D) < \infty$ for all $t > 0$ and all measurable $D \subset \Omega$ with meas $(D) < \infty$.

Let φ be a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable (see [33], Theorem 7.10).

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall |\alpha| \leq m, D^\alpha u \in L_\varphi(\Omega)\}$$

and

$$W^m E_\varphi(\Omega) = \{u \in E_\varphi(\Omega) : \forall |\alpha| \leq m, D^\alpha u \in E_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \ldots + |\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives.

The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\bar{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha| \leq m} \rho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \bar{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

for $u \in W^m L_\varphi(\Omega)$.

These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), \|\|^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the condition (see[33])

$$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c_0. \tag{4}$$

The space $W^m L_\varphi(\Omega)$ will always be identified with a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$ and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, the following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_{\bar{\varphi}}(\Omega) \right\},$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_{\bar{\varphi}}(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

For φ and her complementary function $\bar{\varphi}$, the following inequality is called the Young inequality (see[33]):

$$ts \leq \varphi(x,t) + \bar{\varphi}(x,s), \quad \forall t, s \geq 0, x \in \Omega, \tag{5}$$

this inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{6}$$

In $L_{\varphi}(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1, \tag{7}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{8}$$

For two complementary Musielak Orlicz functions φ and $\bar{\varphi}$, let $u \in L_{\varphi}(\Omega)$ and $v \in L_{\bar{\varphi}}(\Omega)$, then we have the Holder inequality (see[33]):

$$\left| \int_{\Omega} u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \||v|\|_{\bar{\varphi},\Omega}. \tag{9}$$

2.3. Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega \times]0, T[$ with some given $T > 0$. Let φ and $\bar{\varphi}$ be two complementary Musielak-Orlicz functions. For each $\alpha \in \mathbb{N}^N$ denote by D_x^{α} the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows:

$$W^{1,x} L_{\varphi}(Q) = \{u \in L_{\varphi}(Q) : \forall |\alpha| \leq 1 D_x^{\alpha} u \in L_{\varphi}(Q)\}$$

and

$$W^{1,x} E_{\varphi}(Q) = \{u \in E_{\varphi}(Q) : \forall |\alpha| \leq 1 D_x^{\alpha} u \in E_{\varphi}(Q)\}.$$

This second space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \leq 1} \|D_x^{\alpha} u\|_{\varphi,Q}.$$

These spaces constitute a complementary system since Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_{\varphi}(Q)$ which has $(N+1)$ copies.

We shall also consider the weak topologies $\sigma(\Pi L_{\varphi}, \Pi E_{\bar{\varphi}})$ and $\sigma(\Pi L_{\varphi}, \Pi L_{\bar{\varphi}})$. If $u \in W^{1,x} L_{\varphi}(Q)$ then the function $t \to u(t) = u(\cdot, t)$ is defined on $[0, T]$ with values in $W^1 L_{\varphi}(\Omega)$. If $u \in W^{1,x} E_{\varphi}(Q)$, then $u \in W^1 E_{\varphi}(\Omega)$ and it is strongly measurable. Furthermore, the imbedding $W^{1,x} E_{\varphi}(Q) \subset L^1(0, T, W^1 E_{\varphi}(\Omega))$

holds. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, for $u \in W^{1,x}L_\varphi(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t \to \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{1,x}E_\varphi(Q)$. It is proved that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak $*$ topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is a limit in $W^{1,x}L_\varphi(Q)$ of some subsequence $(v_j) \in \mathcal{D}(Q)$ for the modular convergence, i.e., there exists $\lambda > 0$ such that for all $|\alpha| \le 1$

$$\int_Q \varphi\left(x, \left(\frac{D_x^\alpha v_j - D_x^\alpha u}{\lambda}\right)\right) dxdt \to 0 \text{ as } j \to \infty;$$

this implies that (v_j) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_{\bar\varphi})$. Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_{\bar\varphi})} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_{\bar\varphi})}.$$

The space of functions satisfying such a property will be denoted by $W_0^{1,x}L_{\bar\varphi}(Q)$, furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi(Q)$. Thus, both sides of the last inequality are equivalent norms on $W_0^{1,x}L_\varphi(Q)$. We then have the following complementary system:

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix}$$

where F states for the dual space of $W_0^{1,x}E_\varphi(Q)$ and can be defined, except for an isomorphism, as the quotient of $\Pi L_{\bar\varphi}$ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$. It will be denoted by $F = W^{-1,x}L_\psi(Q)$, where

$$W^{-1,x}L_\psi(Q) = \left\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\bar\varphi}(Q) \right\}$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\bar\varphi,Q}$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_{\bar\varphi}(Q).$$

The space F_0 is then given by

$$F_0 = \left\{ f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\bar\varphi}(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$.

Theorem 2.2. [34] *Let φ be a Musielak-Orlicz function satisfies the log-Hölder continuity condition on Ω (see definition 2.4). If $u \in W^{1,x}L_\varphi(Q) \cap L^2(Q)$ $\Big($respectively $u \in W_0^{1,x}L_\varphi(Q) \cap L^2(Q)\Big)$ and $\frac{\partial u}{\partial t} \in W^{-1,x}L_{\bar\varphi}(Q) + L^2(Q)$, then there exists a sequence $(v_j) \in D(\bar Q)$ (respectively $D(\bar I, D(\Omega))$) such that $v_j \longrightarrow u$ in $W^{1,x}L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial v_j}{\partial t} \longrightarrow \frac{\partial u}{\partial t}$ in $W^{-1,x}L_{\bar\varphi}(Q) + L^2(Q)$ for the modular convergence.*

Lemma 2.3. [34] *Let $a < b \in \mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then*

$$\left\{u \in W_0^{1,x}L_\varphi(\Omega\times]a,b[) : \frac{\partial u}{\partial t} \in W^{-1,x}L_\psi(\Omega\times]a,b[) + L^1(\Omega\times]a,b[)\right\}$$

is a subset of $\mathcal{C}(]a,b\left[, L^1(\Omega)\right)$.

2.4. Some Auxiliary Lemmas

Definition 2.4. A Musielak function φ satisfies the log-Hölder continuity condition on Ω if there exists a constant $A > 0$ such that

$$\frac{\varphi(x,t)}{\varphi(y,t)} \le t^{\left(\frac{A_1}{\log\left(\frac{1}{x-y|}\right)}\right)}$$

for all $t \ge 1$ and for all $x, y \in \Omega$ with $|x-y| \le \frac{1}{2}$.

We will use the following technical lemmas.

Lemma 2.5. (See Theorem 2 in [28]) *Let Ω be a bounded Lipschitz domain in $R^N (N \ge 2)$ and let φ be a Musielak function satisfying the log-Hölder continuity such that*

$$\bar\varphi(x,1) \le c_1 \quad \text{a.e in } \Omega \text{ for some } c_1 > 0. \tag{10}$$

Then, $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1 L_\varphi(\Omega)$ for the modular convergence.

Remark **2.6.** Note that if $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$, then (10) holds.

Example 2.7. Let $p \in P(\Omega)$ be a bounded variable exponent on Ω, such that there exists a constant $A > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, we have the inequality

$$|p(x) - p(y)| \le \frac{A}{\log\left(\frac{1}{|x-y|}\right)}.$$

We can verify that the Musielak function defined by $\varphi(x,t) = t^{p(x)}\log(1+t)$ satisfies the conditions of Lemma 2.5

Lemma 2.8. (See Theorem 3 in [28]) *(Poincare's inequality: Integral form) Let Ω be a bounded Lipschitz domain of $R^N(N \geq 2)$ and let φ be a Musielak function satisfying the conditions of Lemma 2.5. Then there exists positive constants β, η and λ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x,|v|)dx \leq \beta + \eta \int_\Omega \varphi(x, \lambda|\nabla v|)dx \text{ for all } v \in W_0^1 L_\varphi(\Omega). \tag{11}$$

Corollary 2.9. (See Corollary 1 in [28]) *(Poincare's inequality) Let Ω be a bounded Lipchitz domain of $R^N(N \geq 2)$ and let φ be a Musielak function satisfying the same conditions of Lemma 2.8. Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \forall v \in W_0^1 L_\varphi(\Omega).$$

Lemma 2.10. ([36]) *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$.*

Moreover, if the set D of discontinuity points of F' is finite, we have

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e in } \{x \in \Omega : u(x) \in D\}, \\ 0 & \text{a.e in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 2.11. [15] *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 2.12. [26] *Let (f_n), $f \in L^1(\Omega)$ be such that*

i) $f_n \geq 0$ *a.e in* Ω,

ii) $f_n \longrightarrow f$ *a.e in* Ω,

iii) $\displaystyle\int_\Omega f_n(x)\,dx \longrightarrow \int_\Omega f(x)\,dx$.

Then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Lemma 2.13. [16] *If a sequence $g_n \in L_\varphi(\Omega)$ converges in measure to a measurable function g and if g_n remains bounded in $L_\varphi(\Omega)$, then $g \in L_\varphi(\Omega)$ and $g_n \rightharpoonup g$ for $\sigma(\Pi L_\varphi, \Pi E_{\bar{\varphi}})$.*

Lemma 2.14. [16] *Let u_n, $u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_{\bar{\varphi}}(\Omega))$.*

Lemma 2.15. [17] *If $P \prec \varphi$ and $u_n \to u$ for the modular convergence in $L_\varphi(\Omega)$ then $u_n \to u$ strongly in $E_P(\Omega)$.*

Lemma 2.16. [38] *(Jensen inequality). Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $g : \Omega \longrightarrow \mathbb{R}$ is function measurable, then*

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g\, d\mu.$$

Lemma 2.17. [27] *(The Nemytskii Operator). Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x, s)| \leq c(x) + k_1 \psi_x^{-1} \varphi\left(x, k_2 |s|\right)$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$. Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\mathcal{P}\left(E_\varphi(\Omega), \frac{1}{k_2}\right)^p = \prod \left\{u \in L_\varphi(\Omega) : d\left(u, E_\varphi(\Omega)\right) < \frac{1}{k_2}\right\}$$

into $(L_\psi(\Omega))^q$ for the modular convergence. Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\left(E_\varphi(\Omega), \frac{1}{k_2}\right)^p$ to $(E_\gamma(\Omega))^q$.

3. Basic Assumptions and Main Result

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property and let φ be a Musielak-Orlicz function. Consider the following convex set

$$\mathbf{K}_\psi = \left\{u \in W_0^{1,x} L_\varphi(Q_T) : u \geq \psi \text{ a.e. in } Q_T\right\}, \tag{12}$$

where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function. Defining the following set

$$\mathcal{T}_0^{1,\varphi}(Q_T) := \left\{u : Q_T \to \mathbb{R} \text{ measurable} : T_k(u) \in W_0^{1,x} L_\varphi(Q_T)\right\}.$$

On the convex $\mathbf{K}_\psi$ we assume that

(C_1) $\psi^+ \in W_0^{1,x} L_\varphi(Q_T) \cap L^\infty(Q_T)$,

(C_2) For each $v \in \mathbf{K}_\psi \cap L^\infty(Q_T)$, there exists a sequence $\{v_j\} \subset \mathbf{K}_\psi \cap W_0^{1,x} E_\varphi(Q_T) \cap L^\infty(Q_T)$ such that $v_j \to v$ for the modular convergence.

(C_3) $\mathbf{K}_\psi \cap L^\infty(Q_T) \neq \varnothing$.

Let A : $D(\mathrm{A}) \subset W_0^{1,x} L_\varphi(Q_T) \to W^{-1,x} L_{\overline{\varphi}}(Q_T)$ be an operator of Leray-Lions type of the form

$$\mathrm{A}(u) := -\operatorname{div}\, a(x, t, u, \nabla u).$$

The goal of this work is to prove the existence of entropy solutions in the framework of the Museilak-Orlicz-Sobolev spaces for the nonlinear problem.

$$\begin{cases} u \geq \psi \text{ a.e. in } Q_T \\ \dfrac{\partial u}{\partial t} - \operatorname{div}\, a(x,t,u,\nabla u) - \operatorname{div}\Phi(x,t,u) = f - \operatorname{div}\, F & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{13}$$

where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for almost every $(x,t) \in Q_T$ and for all $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N (\xi \neq \eta)$ the following conditions:

(H_1) There exist a function $c(x,t) \in E_{\overline{\varphi}}(Q_T)$ and some positive constants k_1, k_2, k_3 and an Orlicz function $\gamma \prec\prec \varphi$ such that

$$|a(x,t,s,\xi)| \leq \beta\Big[c(x,t) + k_1\overline{\varphi}^{-1}(x,\gamma(k_2|s|)) + \overline{\varphi}^{-1}(\varphi(x,k_3|\xi|))\Big].$$

(H_2) a is strictly monotone

$$\Big(a(x,t,s,\xi) - a(x,t,s,\eta)\Big) \cdot \Big(\xi - \eta\Big) > 0.$$

(H_3) a is coercive, there exists a constant $\alpha > 0$ such that

$$a(x,t,s,\xi) \cdot \xi \geq \alpha\varphi(x,|\xi|).$$

For the lower order term, we assume $\Phi : Q_T \times \mathbb{R} \to \mathbb{R}^N$ be a Caratheodory function satisfying:

(H_4) For all $s \in \mathbb{R}$, $\delta > 0$ and for almost every $x \in \Omega$,

$$|\Phi(x,t,s)| \leq \gamma(x,t) + \overline{\varphi}^{-1}(\varphi(x,|s|)) \text{ where } \gamma \in E_{\overline{\varphi}}(Q_T).$$

(H_5) $f \in L^1(Q_T)$. u_0 is an element of $L^1(\Omega)$,

(H_6) $F \in (E_\psi(Q))^N$.

Lemma 3.1. [30] *Under assumptions (H_1)-(H_3), let (Z_n) be a sequence in $W_0^{1,x}L_\varphi(Q_T)$ such that*

$$Z_n \rightharpoonup Z \quad \text{in } W_0^{1,x}L_\varphi(Q_T) \text{ for } \sigma(\Pi L_\varphi(Q_T), \Pi E_{\overline{\varphi}}(Q_T)), \tag{14}$$

$$\Big(a(x,t,Z_n,\nabla Z_n)\Big)_n \quad \text{is bounded in } \Big(L_{\overline{\varphi}}(Q_T)\Big)^N, \tag{15}$$

$$\lim_{n,s\to\infty} \int_{Q_T} \Big(a(x,t,Z_n,\nabla Z_n) - a(x,t,Z_n,\nabla Z\chi_s)\Big) \cdot \Big(\nabla Z_n - \nabla Z\chi_s\Big) dxdt = 0, \tag{16}$$

where χ_s *denote the characteristic function of the set* $\Omega_s = \Big\{x \in \Omega : |\nabla Z| \le s\Big\}$. *Then,*

$$\nabla Z_n \to \nabla Z \quad \text{a.e. in } Q_T, \tag{17}$$

$$\lim_{n\to\infty} \int_{Q_T} a(x,t,Z_n,\nabla Z_n)\nabla Z_n \, dx = \int_{Q_T} a(x,t,Z,\nabla Z)\nabla Z \, dxdt, \tag{18}$$

$$\varphi(x,|\nabla Z_n|) \longrightarrow \varphi(x,|\nabla Z|) \quad \text{ in } L^1(Q_T). \tag{19}$$

In what follows, we will use the following real function of a real variable, called the truncation at height $k > 0$,

$$T_k(s) = \max\Big(-k, \min(k,s)\Big) = \begin{cases} s & \text{if } |s| \le k \\ k\dfrac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and its primitive is defined by

$$\widetilde{T}_k(s) = \int_0^s T_k(t)\, dt.$$

Note that $\widetilde{T}_k$ have the properties: $\widetilde{T}_k(s) \ge 0$ and $\widetilde{T}_k(s) \le k|s|$.

Definition 3.2. A measurable function u defined on Q_T is said a solution for the obstacle problem associated to (13) if $T_k(u) \in \mathcal{T}_0^{1,\varphi}(Q_T)$, $\forall k > 0$ and $\widetilde{T}_k(u(\cdot,t)) \in L^1(\Omega)$ for every $t \in [0,T]$, we have

$$\begin{aligned} \int_\Omega \widetilde{T}_k(u-v)\, dx &+ \Big\langle \frac{\partial v}{\partial t}, T_k(u-v)\Big\rangle_{Q_\tau} \\ &+ \int_{Q_\tau} a(x,t,u,\nabla u)\nabla T_k(u-v))\, dx\, dt \\ &\quad + \int_{Q_\tau} \Phi(x,t,u)\nabla T_k(u-v))\, dx\, dt + \int_{Q_\tau} F\nabla T_k(u-v))\, dx\, dt \\ &\quad\ \le \int_{Q_\tau} fT_k(u-v))\, dx\, dt + \int_\Omega \widetilde{T}_k(u_0 - v(0))\, dx, \end{aligned} \tag{20}$$

and

$$u(x,0) = u_0(x) \text{ for a.e } x \in \Omega, \tag{21}$$

for every $\tau \in [0,T]$, $k > 0$ and for all $v \in W_0^{1,x}L_\varphi(Q_T) \cap L^\infty(Q_T)$ such that $\dfrac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{\varphi}}(Q_T) + L^1(Q_T)$, $\widetilde{T}_k(u(.,t)) \in L^1(\Omega)$ is the primitive function of the truncation function T_k defined above.

***Remark* 3.3.** Equation (20) is formally obtained by multiplication of the problem (13) by $T_k(u-v)$. Notice that since $T_k(u-v) \in W_0^{1,x}L_\varphi(Q_T) \cap L^\infty(Q_T)$ then each term in (20) is well defined. Moreover by Lemma 2.3, we have $v \in C\Big([0,T]; L^1(\Omega)\Big)$ and then the first and the last terms of (20) are well defined.

The following theorem is our main result.

Theorem 3.4. *Suppose that the assumptions* $(H_1)-(H_6)$ *hold true and* $f \in L^1(Q_T)$, *then there exists at least one solution for problem* (13) *in the sense of definition 3.2.*

The proof of the above theorem is divided into four steps.

3.1. Step 1: Approximate problems.

Let f_n be a sequence of regular function in $\mathcal{C}_0^\infty(Q_T)$ which converges strongly to f in $L^1(Q_T)$ and such that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$ and for each $n \in \mathbb{N}^*$, put

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi) \text{ a.e } (x,t) \in Q_T, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N,$$

and

$$\Phi_n(x,t,s) = \Phi(x,t,T_n(s)) \text{ a.e } (x,t) \in Q_T, \forall s \in \mathbb{R}.$$

And let $u_{0n} \in \mathcal{C}_0^\infty(\Omega)$ be such that

$$\| u_{0n} \|_{L^1} \leq \| u_0 \|_{L^1} \text{ and } u_{0n} \longrightarrow u_0 \text{ in } L^1(\Omega).$$

Considering the following approximate problem

$$\begin{cases} u_n \in \mathbf{K}_\psi \\ \dfrac{\partial u_n}{\partial t} - \operatorname{div} a(x,t,u_n,\nabla u_n) - \operatorname{div} \Phi_n(x,t,u_n) = f_n - \operatorname{div} F & \text{in } Q_T \\ u_n(x,t=0) = u_{0n} & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{22}$$

Let $z_n(x,t,u_n,\nabla u_n) = a_n(x,t,u_n,\nabla u_n) + \Phi_n(x,t,u_n)$, which satisfies (A_1), (A_2), (A_3) and (A_4) of [19], it remains to prove (A_4). In order to prove it, we use Young's inequality technically as follows:

$$\begin{aligned} |\Phi_n(x,t,u_n)\nabla u_n| &\leq |\gamma(x,t)||\nabla u_n| + \overline{\varphi}^{-1}(\varphi(x,|T_n(u_n)|))|\nabla u_n| \\ &= \frac{\alpha^2}{\alpha+2}\frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla u_n| \\ &\quad + \frac{\alpha+1}{\alpha}\overline{\varphi}^{-1}(\varphi(x,|T_n(u_n)|))\frac{\alpha}{\alpha+1}|\nabla u_n| \\ &\leq \frac{\alpha^2}{\alpha+2}\Big(\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + \varphi\Big(x,|\nabla u_n|\Big)\Big) \\ &\quad + \overline{\varphi}\Big(x,\frac{\alpha+1}{\alpha}\overline{\varphi}^{-1}(\varphi(x,|T_n(u_n)|))\Big) + \varphi\Big(x,\frac{\alpha}{\alpha+1}|\nabla u_n|\Big). \end{aligned}$$

While $\dfrac{\alpha}{\alpha+1}<1$, using the convexity of φ and the fact that $\overline{\varphi}$ and $\overline{\varphi}^{-1}\circ\varphi$ are increasing functions, one has

$$\begin{aligned}|\Phi_n(x,t,u_n)\nabla u_n| \leq{}& \frac{\alpha^2}{\alpha+2}\overline{\varphi}\Big(x,\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)+\frac{\alpha^2}{\alpha+2}\varphi\Big(x,|\nabla u_n|\Big)\\ &+\overline{\varphi}\Big(x,\frac{\alpha+1}{\alpha}\overline{\varphi}^{-1}(\varphi(x,n))\Big)+\frac{\alpha}{\alpha+1}\varphi\Big(x,|\nabla u_n|\Big).\end{aligned}$$

Since $\gamma\in E_{\overline{\varphi}}(Q_T)$, $\overline{\varphi}\Big(x,\dfrac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\in L^1(\Omega)$, then we get

$$\Phi_n(x,t,u_n)\nabla u_n \geq -\Big(\frac{\alpha^2}{\alpha+2}+\frac{\alpha}{\alpha+1}\Big)\varphi\Big(x,|\nabla u_n|\Big)-C_n-\text{ fixed } L^1 function.$$

Using this last inequality and (H_3) we obtain

$$\begin{aligned}z_n(x,t,u_n,\nabla u_n)\nabla u_n&\geq \Big(\alpha-\frac{\alpha^2}{\alpha+2}-\frac{\alpha}{\alpha+1}\Big)\varphi\Big(x,|\nabla u_n|\Big)-C_n-\text{fixed } L^1 function\\ &\geq \frac{\alpha^2}{(\alpha+1)(\alpha+2)}\varphi\Big(x,|\nabla u_n|\Big)-\text{fixed } L^1 function.\end{aligned}$$

Thus, from [29], we can deduce that the approximate problem (22) has at least one weak solution $u_n\in W_0^{1,x}L_\varphi(Q_T)$.

3.2. Step 2: A Priori Estimates.

We prove some results which will be used later.

Proposition 3.5. *Suppose that the assumptions $(H_1)-(H_5)$ hold true and let $(u_n)_n$ be a solution of the approximate problem (22). Then, for all $k>0$, there exists a constant C_k (not depending on n), such that:*

$$\| T_k(u_n) \|_{W_0^{1,x}L_\varphi(Q_T)}\leq C_k, \tag{23}$$

and

$$\lim_{k\to\infty}\text{meas}\Big\{(x,t)\in Q_T:|u_n|>k\Big\}=0. \tag{24}$$

Proof. First, by $(C_1)-(C_3)$, there exists $v_0\in \mathbf{K}_\psi\cap L^\infty(Q_T)\cap W_0^{1,x}E_\varphi(Q_T)$. Testing the approximate problem (22) with $v=u_n-T_k(u_n-v_0)$, one has for every $\tau\in(0,T)$

$$\begin{aligned}&\Big\langle\frac{\partial u_n}{\partial t},(u_n-v_0)\Big\rangle_{Q_\tau}+\int_{Q_\tau}a(x,t,u_n,\nabla u_n)\nabla T_k(u_n-v_0)\,dx\,dt\\ &+\int_{Q_\tau}\Phi_n(x,t,u_n)\nabla T_k(u_n-v_0)\,dx\,dt=\int_{Q_\tau}f_nT_k(u_n-v_0)\,dx\,dt\\ &+\int_{Q_\tau}F_n\nabla T_k(u_n-v_0)\,dx\,dt.\end{aligned} \tag{25}$$

It follows that

$$\begin{aligned}
&\int_\Omega \widetilde{T}_k(u_n - v_0)(\tau)\,dx + \Big\langle \frac{\partial v_0}{\partial t}, T_k(u_n - v_0) \Big\rangle_{Q_\tau} \\
&\quad + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n - v_0))\,dx\,dt \\
&\quad\quad + \int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n - v_0))\,dx\,dt \\
&\quad\quad\quad \le \int_{Q_\tau} f T_k(u_n - v_0))\,dx\,dt + \int_{Q_\tau} F_n \nabla T_k(u_n - v_0))\,dx\,dt \\
&\quad\quad\quad\quad + \int_\Omega \widetilde{T}_k(u_{n0} - v_0(0))\,dx.
\end{aligned}$$

We have

$$\widetilde{T}_k(u_n - v_0)(\tau) \ge 0,$$

$$\int_\Omega \widetilde{T}_k(u_{n0} - v_0(0))\,dx \le \int_\Omega k \mid (u_{n0} - v_0(0)) \mid \,dx \le kC_1,$$

$$\Big\langle \frac{\partial v_0}{\partial t}, T_k(u_n - v_0) \Big\rangle_{Q_\tau} \le kC_2$$

and

$$\int_{Q_T} f_n T_k(u_n - v_0)\,dx\,dt \le k\|f\|_{L^1(Q_T)} \le kC_3.$$

On the other hand we have

$$\begin{aligned}
\int_{Q_T} F_n \nabla T_k\left(u_n - v_0\right) dx &\le \frac{\alpha}{2(\alpha+1)} \Big(\int_{Q_T} \overline{\varphi}\left(x, \frac{2(\alpha+1)}{\alpha}|F|\right) dxdt\Big) \\
&+ \frac{\alpha}{2(\alpha+1)} \int_{Q_T} \varphi\left(x, |\nabla T_k\left(u_n - v_0\right)|\right) dxdt.
\end{aligned} \tag{26}$$

Notice that $\Phi_n(x,t,u_n)\nabla T_k(u_n)$ is different from zero only on the set $\{|u_n| \le k\}$ where $T_k(u_n) = u_n$, we have

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v_0)\,dx\,dt \\
&\le \int_{\{|u_n - v_0| \le k\}} |\Phi(x,t,T_{k+\|v_0\|_\infty}(u_n))||\nabla u_n|\,dx\,dt \\
&\quad + \int_{\{|u_n - v_0| \le k\}} |\Phi(x,t,T_{k+\|v_0\|_\infty}(u_n))||\nabla v_0|\,dx\,dt \\
&\quad + \frac{\alpha}{2(\alpha+1)} \Big(\int_{Q_T} \overline{\varphi}\left(x, \frac{2(\alpha+1)}{\alpha}|F|\right)\,dx\,dt\Big) + \frac{\alpha}{2(\alpha+1)} \\
&\qquad\qquad \int_{Q_T} \varphi\left(x, |\nabla T_k\left(u_n - v_0\right)|\right)\,dx\,dt + kC_4.
\end{aligned} \tag{27}$$

From (H_4) and using Young's inequality for an arbitrary $\alpha > 0$ (the constant of coercivity), using the convexity of φ with $\frac{\alpha}{2(\alpha+3)} < 1$ and since $\gamma \in E_{\overline{\varphi}}(Q_T)$, $(\nabla v_0) \in (L_\varphi(\Omega))^N$, we have

$$\begin{aligned}
&\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - v_0)\,dx\,dt\\
&\leq \int_{\{|u_n-v_0|\leq k\}} \frac{4(\alpha+3)}{\alpha}\Big(\gamma(x,t) + \overline{\varphi}^{-1}(\varphi(x,|T_{k+\|v_0\|_\infty}(u_n)|))\Big)\\
&\qquad\qquad \frac{\alpha}{4(\alpha+3)}|\nabla u_n|\,dx\,dt\\
&\quad + \int_{\{|u_n-v_0|\leq k\}} \Big(\gamma(x,t) + \overline{\varphi}^{-1}(\varphi(x,|T_{k+\|v_0\|_\infty}(u_n)|))\Big)|\nabla v_0|\,dx\,dt + kC_4\\
&\leq \frac{\alpha}{4(\alpha+3)}\int_{\{|u_n-v_0|\leq k\}} \varphi(x,|\nabla u_n|)\,dx\,dt\\
&\quad + \frac{\alpha}{2(\alpha+1)}\int_{\{|u_n-v_0|\leq k\}} \varphi\left(x,|\nabla u_n|\right)\,dx\,dt + C_5(k,\alpha).
\end{aligned} \tag{28}$$

On the other hand, we can write,

$$\begin{aligned}
&\int_{\{|u_n-v_0|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&\leq \frac{\alpha}{\alpha+1}\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\frac{\alpha+1}{\alpha}\nabla v_0\,dx\,dt\\
&\quad + \frac{\alpha}{4(\alpha+3)}\int_{\{|u_n-v_0|\leq k\}} \varphi(x,|\nabla u_n|)\,dx\,dt\\
&\quad + \frac{\alpha}{2(\alpha+1)}\int_{\{|u_n-v_0|\leq k\}} \varphi\left(x,|\nabla u_n|\right)\,dx\,dt + C_6(k,\alpha).
\end{aligned} \tag{29}$$

Using now (H_2) to evaluate the second term in (29)

$$\begin{aligned}
&\frac{\alpha}{\alpha+1}\int_{\{|u_n-v_0|\leq k\}} a(x,t,u_n,\nabla u_n)\frac{\alpha+1}{\alpha}\nabla v_0\,dx\,dt\\
&\leq \frac{\alpha}{\alpha+1}\Big(\int_{\{|u_n-v_0|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&\qquad - \int_{\{|u_n-v_0|\leq k\}} a\Big(x,t,u_n,\frac{\alpha+1}{\alpha}\nabla v_0\Big)\Big(\nabla u_n - \frac{\alpha+1}{\alpha}\nabla v_0\Big)\,dx\,dt\Big).
\end{aligned} \tag{30}$$

Hence, (29) becomes

$$
\begin{aligned}
&(1-\frac{\alpha}{\alpha+1})\int_{\{|u_n-v_0|\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n \,dx\,dt\\
&\le \int_{\{|u_n-v_0|\le k\}} \left|a\Big(x,t,u_n,\frac{\alpha+1}{\alpha}\nabla v_0\Big)\right|\left|\frac{\alpha+1}{\alpha}\nabla v_0\right| dx\,dt\\
&\quad+\int_{\{|u_n-v_0|\le k\}} \left|a\Big(x,t,u_n,\frac{\alpha+1}{\alpha}\nabla v_0\Big)\right|\left|\nabla u_n\right| dx\,dt\\
&\quad+\frac{\alpha}{4(\alpha+3)}\int_{\{|u_n-v_0|\le k\}} \varphi(x,|\nabla u_n|)\,dx\,dt\\
&\quad+\frac{\alpha}{2(\alpha+1)}\int_{\{|u_n-v_0|\le k\}} \varphi\left(x,|\nabla u_n|\right)\,dx\,dt + C_7(k,\alpha).
\end{aligned}
\tag{31}
$$

Using again Young's inequality as in (29) for the third term of (31) and using (H_1), we get

$$
\begin{aligned}
&(1-\frac{\alpha}{\alpha+1})\int_{\{|u_n-v_0|\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n \,dx\,dt\\
&\le \frac{\alpha}{4(\alpha+3)}\int_{\{|u_n-v_0|\le k\}} \varphi(x,|\nabla u_n|)\,dx\,dt\\
&\quad+\frac{\alpha}{4(\alpha+3)}\int_{\{|u_n-v_0|\le k\}} \varphi(x,|\nabla u_n|)\,dx\,dt\\
&\quad+\frac{\alpha}{2(\alpha+1)}\int_{\{|u_n-v_0|\le k\}} \varphi\left(x,|\nabla u_n|\right)\,dx\,dt + C_8(k,\alpha).
\end{aligned}
\tag{32}
$$

Then,

$$
\begin{aligned}
&(1-\frac{\alpha}{\alpha+1})\int_{\{|u_n-v_0|\le k\}} a(x,t,u_n,\nabla u_n)\nabla u_n \,dx\,dt\\
&\le \frac{1}{2}(\frac{\alpha}{\alpha+3}+\frac{\alpha}{(\alpha+1)})\int_{\{|u_n-v_0|\le k\}} \varphi(x,|\nabla u_n|)\,dx\,dt + C_8(k,\alpha).
\end{aligned}
\tag{33}
$$

Thanks to (H_3), it follows

$$
\Big(\alpha(1-\frac{\alpha}{\alpha+1})-\frac{1}{2}(\frac{\alpha}{(\alpha+3)}+\frac{\alpha}{(\alpha+1)})\Big)\int_{\{|u_n-v_0|\le k\}} \varphi(x,|\nabla u_n|)dxdt \le C_9(k,\alpha).
\tag{34}
$$

Since

$$
\Big(\alpha(1-\frac{\alpha}{\alpha+1})-(\frac{\alpha}{2(\alpha+3)}+\frac{\alpha}{2(\alpha+1)})\Big)=\frac{\alpha}{2(\alpha+1)}-\frac{\alpha}{2(\alpha+3)}=\frac{\alpha}{(\alpha+1)(\alpha+3)}>0,
$$

we have

$$
\int_{\{|u_n-v_0|\le k\}} \varphi(x,|\nabla u_n|)\,dx\,dt \le C(k,\alpha).
\tag{35}
$$

Finally, since $\{|u_n| \leq k\} \subset \{|u_n - v_0| \leq k + \|v_0\|_\infty\}$, one has

$$\begin{aligned} \int_{Q_T} \varphi(x, |\nabla T_k(u_n)|)\, dx\, dt &\leq \int_{\{|u_n| \leq k\}} \varphi(x, |\nabla u_n|)\, dx\, dt \\ &\leq \int_{\{|u_n - v_0| \leq k + \|v_0\|_\infty\}} \varphi(x, |\nabla u_n|)\, dx\, dt \leq C(k, \alpha). \end{aligned} \tag{36}$$

To prove (24), from (36), we have

$$\int_{Q_T} \varphi\Big(x, |\nabla T_k(u_n)|)\Big)\, dx\, dt \leq C(k, \alpha).$$

If $C(k, \alpha) \leq 1$, by Poicaré's inequality, there exists $\lambda > 0$ and δ such that

$$\int_{Q_T} \varphi\Big(x, \delta |T_k(u_n)|\Big)\, dx\, dt \leq \lambda \int_{Q_T} \varphi\Big(x, |\nabla T_k(u_n)|\Big)\, dx\, dt,$$

we obtain

$$\begin{aligned} \text{meas}\Big\{|u_n| > k\Big\} &= \frac{1}{\varphi(x, \delta k)} \int_{\{|u_n| > k\}} \varphi(x, \delta |T_k(u_n)|)\, dx\, dt \\ &\leq \frac{1}{\varphi(x, \delta k)} \int_{Q_T} \varphi(x, \delta |T_k(u_n)|)\, dx\, dt \\ &\leq \frac{\lambda}{\varphi(x, \delta k)} \int_{Q_T} \varphi\Big(x, |\nabla T_k(u_n)|\Big)\, dx\, dt \\ &\leq \frac{\lambda}{\varphi(x, \delta k)} \quad \forall n, \quad \forall k > 0, \\ &\longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \end{aligned} \tag{37}$$

If $C(k, \alpha) \geq 1$, $\dfrac{1}{C(k, \alpha)} \leq 1$, using $P \prec\prec \varphi$ appearing in assumption (H_1) which implies that $\forall \epsilon > 0$, there exist a constant $d_\epsilon : P(t) \leq \varphi(\epsilon t) + d_\epsilon$ and again Poincaré's inequality, we obtain for $\epsilon < \dfrac{1}{C(k,\alpha)} \leq 1$

$$\begin{aligned}
\text{meas}\Big\{|u_n| > k\Big\} &= \frac{1}{P(\delta k)} \int_{\{|u_n|>k\}} P(\delta |T_k(u_n)|)\,dx\,dt \\
&\leq \frac{1}{P(\delta k)} \int_{Q_T} \Big(\varphi(x, \epsilon\delta |T_k(u_n)|) + d_\epsilon\Big)\,dx\,dt \\
&\leq \frac{1}{P(\delta k)} \Big(\frac{1}{C(k,\alpha)} \int_{Q_T} \varphi\Big(x, \delta |T_k(u_n)|\Big)\,dx\,dt + d_\epsilon |Q_T|\Big) \\
&\leq \frac{\lambda}{P(\delta k)} \Big(\frac{1}{C(k,\alpha)} \int_{Q_T} \varphi\Big(x, |\nabla T_k(u_n)|\Big)\,dx\,dt + d_\epsilon |Q_T|\Big) \\
&\leq \frac{\lambda(1 + d_\epsilon |Q_T|)}{P(\delta k)} \qquad \forall n, \quad \forall k > 0, \\
&\longrightarrow 0 \quad \text{ as } k \longrightarrow \infty.
\end{aligned} \tag{38}$$

☑

Lemma 3.6. *Let u_n be a solution of the approximate problem* (22)*, then:*

(i) $u_n \longrightarrow u$ *a.e. in* Q_T*,*

(ii) $\{a(x,t,T_k(u_n), \nabla T_k(u_n))\}_n$ *is bounded in* $(L_{\overline{\varphi}}(Q_T))^N$*.*

Proof. To prove **(i)**, we proceed as in [30, 37], we take a $C^2(\mathbb{R})$ nondecreasing function Γ_k such that $\Gamma_k(s) = \begin{cases} s & \text{for } |s| \leq \dfrac{k}{2} \\ k & \text{for } |s| \geq k \end{cases}$ and multiplying the approximate problem (22) by $\Gamma_k'(u_n)$ we obtain

$$\begin{aligned}
&\frac{\partial \Gamma_k(u_n)}{\partial t} - \text{div}\,\Big(a(x,t,u_n,\nabla u_n)\Gamma_k'(u_n)\Big) + a(x,t,u_n,\nabla u_n)\Gamma_k''(u_n)\nabla u_n \\
&\quad - \text{div}\,\Big(\Gamma_k'(u_n)\Phi_n(x,t,u_n)\Big) + \Gamma_k''(u_n)\Phi_n(x,t,u_n)\nabla u_n = f_n \Gamma_k'(u_n).
\end{aligned} \tag{39}$$

Noticing that $\overline{\varphi}^{-1} \circ \varphi$ is an increasing function, $\gamma \in E_{\overline{\varphi}}(Q_T)$, $supp(\Gamma_k')$, $supp(\Gamma_k'') \subset [-k,k]$ and using Young's inequality we get

$$\begin{aligned}
&\Big|\int_{Q_T} \Gamma_k' \Phi_n(x,t,u_n)\,dx\,dt\Big| \\
&\leq \|\Gamma_k'\|_{L^\infty} \Big(\int_{Q_T} |\gamma(x,t)|\,dx\,dt + \int_{Q_T} \overline{\varphi}^{-1}(\varphi(x, |T_k(u_n)|))\,dx\,dt\Big) \\
&\leq \|\Gamma_k'\|_{L^\infty} \Big(\int_{Q_T} \Big(\overline{\varphi}(x, |\gamma(x,t)|) + \varphi(x,1)\Big)\,dx\,dt + \int_{Q_T} \overline{\varphi}^{-1}(\varphi(x,k))\,dx\,dt\Big) \\
&< C_{1,k},
\end{aligned} \tag{40}$$

and (here, we use also (36))

$$\begin{aligned}
&\Big|\int_{Q_T} \Gamma_k''\Phi_n(x,t,u_n)\nabla u_n\,dx\,dt\Big| \\
&\leq \|\Gamma_k''\|_{L^\infty}\Big(\int_{Q_T}|\gamma(x,t)|\,dx\,dt+\int_{Q_T}\overline{\varphi}^{-1}(\varphi(x,|T_k(u_n)|))|\nabla T_k(u_n)|\,dx\,dt\Big) \\
&\leq \|\Gamma_k''\|_{L^\infty}\Big[\int_{Q_T}\Big(\overline{\varphi}(x,|\gamma(x,t)|)+\varphi(x,1)\Big)\,dx\,dt+\int_{Q_T}\varphi(x,k)\,dx\,dt \\
&\qquad\qquad +\int_{Q_T}\varphi(x,|\nabla T_k(u_n)|)\,dx\,dt\Big] \\
&< C_{2,k},
\end{aligned} \tag{41}$$

where $C_{1,k}$ and $C_{2,k}$ are two positive constants independent of n. Then all above implies that

$$\frac{\partial\Gamma_k(u_n)}{\partial t}\text{is bounded in} L^1(Q_T)+W^{-1,x}L_{\overline{\varphi}}(Q_T). \tag{42}$$

Hence by theorem 2.2 and using the same techniques as in [36], we can deduce that there exists a measurable function $u \in L^\infty(0,T;L^1(\Omega))$ such that

$$u_n \longrightarrow u \text{ a.e. in } Q_T,$$

and for every $k>0$,

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W^{1,x}L_\varphi(Q_T) \text{ for } \sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}}), \tag{43}$$

and

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^1(Q_T) \text{ and a.e. in } Q_T. \tag{44}$$

For **(ii)**, we use the Banach-Steinhaus theorem. Let $\phi \in (E_\varphi(Q_T))^N$ be an arbitrary function. From (H_2) we can write

$$\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\phi)\Big)\cdot\Big(\nabla T_k(u_n)-\phi\Big)\geq 0$$

which gives

$$\begin{aligned}
&\int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\phi\,dx\,dt \\
&\leq \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt \\
&\quad+\int_{Q_T} a(x,t,T_k(u_n),\phi)(\phi-\nabla T_k(u_n))\,dx\,dt.
\end{aligned} \tag{45}$$

Let us denote by J_1 and J_2 the first and the second integral respectively in the right hand-side of (45), so that

$$J_1 = \int_{Q_T} a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt.$$

Thanks to (H_1), there exists a positive constant C_{J_1} independent of n such that $J_1 \leq C_{J_1}$.

Now we estimate the integral J_2: to this end, notice that

$$\begin{aligned} J_2 &= \int_{Q_T} a(x,t,T_k(u_n),\phi)(\phi - \nabla T_k(u_n))\,dx\,dt \\ &\leq \int_{Q_T} |a(x,t,T_k(u_n),\phi)||\phi|\,dx\,dt + \int_{Q_T} |a(x,t,T_k(u_n),\phi)||\nabla T_k(u_n)|\,dx\,dt. \end{aligned}$$

On the other hand, let η be large enough, from (H_1) and the convexity of $\overline{M}$, we get:

$$\begin{aligned} &\int_{Q_T} \overline{\varphi}\Big(x, \frac{|a(x,t,T_k(u_n),\phi)|}{\eta}\Big)\,dx\,dt \\ &\leq \int_{Q_T} \overline{\varphi}\Big(x, \frac{\beta\Big[c(x,t) + k_1\overline{\varphi}^{-1}(x, P(k_2|T_k(u_n)|) + \overline{\varphi}^{-1}(\varphi(x,k_3|\phi|))\Big]}{\eta}\Big)\,dx\,dt \\ &\leq \frac{\beta}{\eta}\int_{Q_T} \overline{\varphi}(x,c(x,t))\,dx\,dt + \frac{\beta k_1}{\eta}\int_{Q_T} \overline{\varphi}\Big(\overline{\varphi}^{-1}(x,P(k_2|T_k(u_n)|))\Big)\,dx\,dt \\ &\quad + \frac{\beta}{\eta}\int_{Q_T} \overline{\varphi}\Big(x,\overline{\varphi}^{-1}(\varphi(x,k_3|\phi|))\Big)\,dx\,dt \\ &\leq \frac{\beta}{\eta}\int_{Q_T} \overline{\varphi}(x,c(x,t))\,dx\,dt + \frac{\beta k_1}{\eta}\int_{Q_T} P(k_2 k)\,dx\,dt \\ &\quad + \frac{\beta}{\eta}\int_{Q_T} \varphi(x,k_3|\phi|)\,dx\,dt. \end{aligned} \tag{46}$$

Since $\phi \in (E_\varphi(Q_T))^N$, $c(x,t) \in E_{\overline{\varphi}}(Q_T)$, we deduce that $\{a(x,t,T_k(u_n),\phi)\}$ is bounded in $(L_{\overline{\varphi}}(Q_T))^N$ and we have that $\{\nabla T_k(u_n)\}$ is bounded in $(L_\varphi(Q_T))^N$, consequently, $J_2 \leq C_{J_2}$, where C_{J_2} is a positive constant not depending on n. And then we obtain

$$\int_{Q_T} a(x,T_k(u_n),\nabla T_k(u_n))\phi\,dx\,dt \leq C_{J_1} + C_{J_2}. \quad \text{for all } \phi \in (E_\varphi(Q_T))^N. \tag{47}$$

Finally, $\{a(x,t,T_k(u_n),\nabla T_k(u_n))\}_n$ is bounded in $(L_{\overline{\varphi}}(Q_T))^N$. ☑

3.3. Step 3: Almost everywhere convergence of the gradients.

In this step, most parts of the proof of the following proposition are the same argument as in [30, 39], we give only those details in the argument differ.

Proposition 3.7. *Let u_n be a solution of the approximate problem* (22)*. Then, for all $k \geq 0$ we have (for a subsequence still denoted by u_n): as $n \to +\infty$,*

(i) $\nabla u_n \to \nabla u$ *a.e. in Q_T,*

(ii) $a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u))$ *weakly in $(L_{\overline{\varphi}}(Q_T))^N$,*

(iii) $\varphi(x,|\nabla T_k(u_n)|) \to \varphi(x,|\nabla T_k(u)|)$ *strongly in $L^1(Q_T)$.*

Proof. Let $\theta_j \in \mathfrak{D}(Q_T)$ be a sequence such that $\theta_j \longrightarrow u$ in $W_0^{1,x}L_\varphi(Q_T)$ for the modular convergence and let $\psi_i \in \mathfrak{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Put $Z^\mu_{i,j} = T_k(\theta_j)_\mu + \mathrm{e}^{-\mu t}\, T_k(\psi_i)$ where $T_k(\theta_j)_\mu$ is the mollification with respect to time of $T_k(\theta_j)$, notice that $Z^i_{\mu,j}$ is a smooth function having the following properties:

$$\frac{\partial Z^\mu_{i,j}}{dt} = \mu(T_k(\theta_j) - Z^\mu_{i,j}), \quad Z^\mu_{i,j}(0) = T_k(\psi_i) \quad \text{and } |Z^\mu_{i,j}| \leq k,$$

$$Z^\mu_{i,j} \longrightarrow T_k(u)_\mu + \mathrm{e}^{-\mu t}\, T_k(\psi_i), \quad \text{in } W_0^{1,x}L_\varphi(Q_T) \quad \text{modularly as } j \longrightarrow \infty,$$

$$T_k(u)_\mu + \mathrm{e}^{-\mu t}\, T_k(\psi_i) \longrightarrow T_k(u), \quad \text{in } W_0^{1,x}L_\varphi(Q_T) \quad \text{modularly as } \mu \longrightarrow \infty.$$

Let now the function h_m defined on $\mathbb{R}$ for any $m \geq k$ by

$$h_m(r) = \begin{cases} 1 & \text{if } |r| \leq m \\ -|r| + m + 1 & \text{if } m \leq |r| \leq m+1 \\ 0 & \text{if } |r| \geq m+1. \end{cases}$$

Put $E_m = \Big\{(x,t) \in Q_T : m \leq |u_n| \leq m+1\Big\}$ and define $\varphi^{\mu,i}_{n,j,m} = (T_k(u_n) - Z^\mu_{i,j})h_m(u_n)$, testing the approximate problem (22) with the test function $u_n - \varphi^{\mu,i}_{n,j,m}$, we get

$$
\begin{aligned}
\Big\langle \frac{\partial u_n}{dt}, \varphi^{\mu,i}_{n,j,m} \Big\rangle &+ \int_{Q_T} a(x,t,u_n,\nabla u_n)(\nabla T_k(u_n) - \nabla Z^{\mu}_{i,j}) h_m(u_n)\, dx\, dt \\
&+ \int_{Q_T} a(x,t,u_n,\nabla u_n)(T_k(u_n) - Z^{\mu}_{i,j}) \nabla u_n h'_m(u_n)\, dx\, dt \\
&+ \int_{E_m} \Phi_n(x,t,u_n) \nabla u_n h'_m(u_n)(T_k(u_n) - Z^{\mu}_{i,j})\, dx\, dt \\
&+ \int_{Q_T} \Phi_n(x,t,u_n) \nabla u_n h_m(u_n)(\nabla T_k(u_n) - \nabla Z^{\mu}_{i,j})\, dx\, dt \\
&= \int_{Q_T} f_n \varphi^{\mu,i}_{n,j,m}\, dx\, dt \\
&+ \int_{E_m} F \nabla u_n h'_m(u_n)(T_k(u_n) - Z^{\mu}_{i,j})\, dx\, dt \\
&+ \int_{Q_T} F \nabla u_n h_m(u_n)(\nabla T_k(u_n) - \nabla Z^{\mu}_{i,j})\, dx\, dt.
\end{aligned} \tag{48}
$$

To simplify the notation, we will denote by $\epsilon(n,j,\mu,i)$ and $\epsilon(n,j,\mu)$ any quantities such that

$$\lim_{i\to+\infty} \lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \epsilon(n,j,\mu,i) = 0,$$

$$\lim_{\mu\to+\infty} \lim_{j\to+\infty} \lim_{n\to+\infty} \epsilon(n,j,\mu) = 0.$$

We have the following lemma which can be found in [30, 39]. ☑

Lemma 3.8. (cf. [39]) *Let $\varphi^{\mu,i}_{n,j,m} = (T_k(u_n) - Z^{\mu}_{i,j}) h_m(u_n)$, then for any $k \geq 0$ we have:*

$$\Big\langle \frac{\partial u_n}{dt}, \varphi^{\mu,i}_{n,j,m} \Big\rangle \geq \epsilon(n,j,\mu,i), \tag{49}$$

where $\Big\langle \cdot,\cdot \Big\rangle$ denotes the duality pairing between $L^1(Q_T) + W^{-1,x}L_{\overline{\varphi}}(Q_T)$ and $L^\infty(Q_T) \cap W^{1,x}_0 L_\varphi(Q_T)$.

To complete the proof of Proposition 3.7, we establish the results below, for any fixed $k \geq 0$, we have:

(r_1) $\displaystyle\int_{Q_T} f_n \varphi^{\mu,i}_{n,j,m}\, dx\, dt = \epsilon(n,j,\mu).$

(r_2) $\displaystyle\int_{Q_T} F_n \nabla u_n h_m(u_n)(\nabla T_k(u_n) - \nabla Z^{\mu}_{i,j})\, dx\, dt = \epsilon(n,j,\mu).$

(r_3) $\displaystyle\int_{E_m} F_n \nabla u_n h'_m(u_n)(T_k(u_n) - Z^{\mu}_{i,j})\, dx\, dt = \epsilon(n,j,\mu).$

(r_4) $\displaystyle\int_{Q_T} \Phi_n(x,t,u_n)\nabla u_n h_m(u_n)(\nabla T_k(u_n) - \nabla Z^{\mu}_{i,j})\,dx\,dt = \epsilon(n,j,\mu).$

(r_5) $\displaystyle\int_{E_m} \Phi_n(x,t,u_n)\nabla u_n h'_m(u_n)(T_k(u_n) - Z^{\mu}_{i,j})\,dx\,dt = \epsilon(n,j,\mu).$

(r_6) $\displaystyle\int_{Q_T} a(x,t,u_n,\nabla u_n)(T_k(u_n) - Z^{\mu}_{i,j})\nabla u_n h'_m(u_n)\,dx\,dt \leq \epsilon(n,j,\mu,m).$

(r_7) $$\int_{Q_T} [a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi_s)] \times[\nabla T_k(u_n) - \nabla T_k(u)\chi_s]\,dx\,dt \leq \epsilon(n,j,\mu,m,s).$$

Proof. The proofs of (r_1),(r_2), (r_3), $(r_5)(r_6)$, and (r_7) are the same as in [30, 32, 39, 37]).

To prove (r_4), for $n \geq m+1$, we have

$$\Phi_n(x,t,u_n)h_m(u_n) = \Phi(x,t,T_{m+1}(u_n))h_m(T_{m+1}(u_n)) \text{ a.e in } Q_T.$$

put $P_n = \overline{\varphi}\Big(\dfrac{|\Phi(x,t,T_{m+1}(u_n)) - \Phi(x,t,T_{m+1}(u))|}{\eta}\Big)$. Since Φ is continuous with respect to its third argument and $u_n \longrightarrow u$ a.e in Q_T, then $\Phi(x,t,T_{m+1}(u_n)) \to \Phi(x,t,T_{m+1}(u))$ a.e in Ω as n goes to infinity, besides $\overline{\varphi}(x,0) = 0$, it follows

$$P_n \longrightarrow 0, \quad \text{a.e in } \Omega \text{ as } n \to \infty. \tag{50}$$

Using now the convexity of $\overline{\varphi}$ and (H_4), we have for every $\eta > 0$ and $n \geq m+1$:

$$\begin{aligned}
P_n &= \overline{\varphi}\Big(x, \frac{|\Phi(x,t,T_{m+1}(u_n)) - \Phi(x,t,T_{m+1}(u))|}{\eta}\Big)\\
&\leq \overline{\varphi}\Big(x, \frac{2\gamma(x,t) + \overline{\varphi}^{-1}(\varphi(x,|T_{m+1}(u_n)|)) + \overline{\varphi}^{-1}(\varphi(x,|T_{m+1}(u)|))}{\eta}\Big)\\
&\leq \overline{\varphi}\Big(x, \frac{2}{\eta}|\gamma(x,t)| + \frac{2}{\eta}\overline{\varphi}^{-1}(\varphi(x,m+1))\Big)\\
&= \overline{\varphi}\Big(x, \frac{1}{2}\frac{4}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{4}{\eta}\overline{\varphi}^{-1}(\varphi(x,m+1))\Big)\\
&\leq \frac{1}{2}\overline{\varphi}(x, \frac{4}{\eta}|\gamma(x,t)|) + \frac{1}{2}\overline{\varphi}(x, \frac{4}{\eta}\overline{\varphi}^{-1}(\varphi(x,m+1))).
\end{aligned} \tag{51}$$

We put $C^{\eta}_m(x,t) = \dfrac{1}{2}\overline{\varphi}(x, \dfrac{4}{\eta}|\gamma(x,t)|) + \dfrac{1}{2}\overline{\varphi}(x, \dfrac{4}{\eta}\overline{\varphi}^{-1}(\varphi(x,m+1)))$. Since $\gamma \in E_{\overline{\varphi}}(Q_T)$, we have $C^{\eta}_m \in L^1(Q_T)$, Then by Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty}\int_{Q_T} P_n\,dx\,dt = \int_{Q_T}\lim_{n\to\infty} P_n\,dx\,dt = 0. \tag{52}$$

This implies that $\{\Phi(x,t,T_{m+1}(u_n))\}$ converges modularly to $\Phi(x,t,T_{m+1}(u))$ as $n\to\infty$ in $(L_{\overline{\varphi}}(Q_T))^N$. Moreover, $\Phi(x,t,T_{m+1}(u_n))$, $\Phi(x,t,T_{m+1}(u))$ lie in $(E_{\overline{\varphi}}(Q_T))^N$, indeed, from (H_4) we have for every $\eta>0$

$$\begin{aligned}
&\int_{Q_T}\overline{\varphi}\Big(x,\frac{|\Phi(x,t,T_{m+1}(u_n))|}{\eta}\Big)\,dx\,dt\\
&\le\int_{Q_T}\overline{\varphi}\Big(x,\frac{1}{\eta}|\gamma(x,t)|+\frac{1}{\eta}\overline{\varphi}^{-1}(\varphi(x,|T_{m+1}(u_n)|))\Big)\,dx\,dt\\
&\le\int_{Q_T}\overline{\varphi}\Big(x,\frac{1}{2}\frac{2}{\eta}|\gamma(x,t)|+\frac{1}{2}\frac{2}{\eta}\overline{\varphi}^{-1}(\varphi(x,m+1))\Big)\,dx\,dt\\
&\le\int_{Q_T}\frac{1}{2}\overline{\varphi}(x,\frac{2}{\eta}|\gamma(x,t)|)\,dx\,dt+\int_{Q_T}\frac{1}{2}\overline{\varphi}\Big(x,\frac{2}{\eta}\overline{\varphi}^{-1}(\varphi(x,m+1))\Big)\,dx\,dt\\
&<\infty \text{ since } \gamma\in E_{\overline{\varphi}}(Q_T) \text{ and } \Omega \text{ is bounded;}
\end{aligned}$$

the same for $\Phi(x,t,T_{m+1}(u))$. Thanks to Lemma 2.14, we deduce that $\Phi(x,t,T_{m+1}(u_n))\longrightarrow\Phi(x,t,T_{m+1}(u))$ strongly in $(E_{\overline{\varphi}}(Q_T))^N$. On the other hand, $\nabla T_k(u_n)\rightharpoonup\nabla T_k(u)$ weakly in $(L_\varphi(Q_T))^N$ as n goes to infinity, it follows that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{Q_T}\Phi(x,t,u_n)h_m(u_n)[\nabla T_k(u_n)-\nabla Z^\mu_{i,j}]\,dx\,dt\\
&\quad=\int_{Q_T}\Phi(x,t,u)h_m(u)[\nabla T_k(u)-\nabla Z^\mu_{i,j}]\,dx\,dt.
\end{aligned}\tag{53}$$

Using the modular convergence of $Z^\mu_{i,j}$ as $j\longrightarrow\infty$ and then $\mu\longrightarrow\infty$, we get (r_2). As a consequence of Lemma 3.1, the results of Proposition 3.7 follow. ☑

3.4. Step 4: Passing to the limit.

Now, we will pass to the limit. Let $v\in W^{1,x}L_\varphi(Q_T)\cap L^\infty(Q_T)$ such that $\dfrac{\partial v}{\partial t}\in W^{-1,x}L_{\overline{\varphi}}(Q_T)+L^1(Q_T)$. From [18], there exists a prolongation $v_p=v$ on Q_T, $v_p\in W^{1,x}L_\varphi(\Omega\times\mathbb{R})\cap L^1(\Omega\times\mathbb{R})\cap L^\infty(\Omega\times\mathbb{R})$ and

$$\frac{\partial v}{\partial t}\in W^{-1,x}L_{\overline{\varphi}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}).$$

There exists also a sequence $(\omega_j)\subset\mathfrak{D}(\Omega\times\mathbb{R})$ such that

$$\omega_j\to v_p \text{ in } W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}), \text{ and } \frac{\partial\omega_j}{\partial t}\to\frac{\partial v_p}{\partial t} \text{ in } W^{-1,x}L_{\overline{\varphi}}(\Omega\times\mathbb{R})+L^1(\Omega\times\mathbb{R}),$$

for the modular convergence and $\|\omega_j\|_{\infty,Q_T}\le(N+2)\|v\|_{\infty,Q_T}$.

Testing the approximate problem (22) with $v = u_n - T_k(u_n - \omega_j)\chi_{(0,\tau)}$ with $\tau \in [0, T]$, we get

$$\begin{aligned}
\Big\langle \frac{\partial u_n}{dt}, T_k(u_n - \omega_j) \Big\rangle_{Q_\tau} &+ \int_{Q_\tau} a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n)) \nabla T_k(u_n - \omega_j)\, dx\, dt \\
&+ \int_{Q_\tau} \Phi(x,t,T_{k_0}(u_n)) \nabla T_k(u_n - \omega_j)\, dx\, dt \\
&= \int_{Q_\tau} f_n T_k(u_n - \omega_j)\, dx\, dt \\
&\quad + \int_{Q_\tau} F \nabla T_k(u_n - \omega_j)\, dx\, dt,
\end{aligned} \tag{54}$$

where $k_0 = k + (N+2)\|v\|_{\infty, Q_T}$. This implies, with $E_{n,j} := Q_\tau \cap \{|u_n - \omega_j| \le k\}$, that

$$\begin{aligned}
\Big\langle \frac{\partial u_n}{dt}, T_k(u_n - \omega_j) \Big\rangle_{Q_\tau} &+ \int_{E_{n,j}} a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n)) \nabla u_n\, dx\, dt \\
&- \int_{E_{n,j}} a(x,t,T_{k_0}(u_n), \nabla T_{k_0}(u_n)) \nabla \omega_j\, dx\, dt \\
&+ \int_{Q_\tau} \Phi(x,t,T_{k_0}(u_n)) \nabla T_k(u_n - \omega_j)\, dx\, dt \\
&= \int_{Q_\tau} f_n T_k(u_n - \omega_j)\, dx\, dt \\
&\quad + \int_{Q_\tau} F \nabla T_k(u_n - \omega_j)\, dx\, dt.
\end{aligned} \tag{55}$$

Our aim here is to pass to the limit in each term in (56), let us start by the terms of the left-hand side:

Limit of the first term $\Big\langle \frac{\partial u_n}{dt}, T_k(u_n - \omega_j) \Big\rangle_{Q_\tau}$, we have

$$\begin{aligned}
\Big\langle \frac{\partial u_n}{dt}, T_k(u_n - \omega_j) \Big\rangle_{Q_\tau} &= \Big\langle \frac{\partial u_n}{dt} - \frac{\partial \omega_j}{dt}, T_k(u_n - \omega_j) \Big\rangle_{Q_\tau} + \Big\langle \frac{\partial \omega_j}{dt}, T_k(u_n - \omega_j) \Big\rangle_{Q_\tau} \\
&= \int_\Omega \widetilde{T}_k(u_n - \omega_j)\, dx + \Big\langle \frac{\partial \omega_j}{dt}, T_k(u_n - \omega_j) \Big\rangle_{Q_\tau} \\
&\quad - \int_\Omega \widetilde{T}_k(u_{0n} - \omega_j(0))\, dx.
\end{aligned} \tag{56}$$

Since $u_n \longrightarrow u$ in $C([0,T], L^1(\Omega))$ (see [18]), by Lebesgue's theorem we have

$$\int_\Omega \widetilde{T}_k(u_n - \omega_j)\, dx \longrightarrow \int_\Omega \widetilde{T}_k(u - \omega_j)\, dx \text{ as } n \longrightarrow \infty.$$

Passing to the limit in (57), we get

$$\lim_{n\longrightarrow\infty}\Big\langle\frac{\partial u_n}{dt},T_k(u_n-\omega_j)\Big\rangle_{Q_\tau}=\int_\Omega\widetilde{T}_k(u-\omega_j)\,dx+\Big\langle\frac{\partial\omega_j}{dt},T_k(u-\omega_j)\Big\rangle_{Q_\tau}-\int_\Omega\widetilde{T}_k(u_0-\omega_j(0))\,dx.$$

For the second and the third terms of (56), we have from (ii) of Proposition 3.7

$$a(x,t,T_{k_0}(u_n),\nabla T_{k_0}(u_n))\rightharpoonup a(x,t,T_{k_0}(u),\nabla T_{k_0}(u))\text{ weakly in }(L_{\overline{\varphi}}(Q_T))^N,$$

thus Fatou's lemma allows us to get

$$\begin{aligned}\liminf_{n\to\infty}\Big(&\int_{E_{n,j}}a(x,t,T_{k_0}(u_n),\nabla T_{k_0}(u_n))\nabla u_n\,dx\,dt\\&-\int_{E_{n,j}}a(x,t,T_{k_0}(u_n),\nabla T_{k_0}(u_n))\nabla\omega_j\,dx\,dt\Big)\\\geq&\int_{E_{n,j}}a(x,t,T_{k_0}(u),\nabla T_{k_0}(u))\nabla u\,dx\,dt\\&-\int_{E_{n,j}}a(x,t,T_{k_0}(u),\nabla T_{k_0}(u))\nabla\omega_j\,dx\,dt.\end{aligned}\tag{57}$$

Concerning the fourth term of the left-hand side of (56), we proceed as in (52) to get

$$\Phi(x,t,T_{k_0}(u_n))\to\Phi(x,t,T_{k_0}(u))\text{ as }n\to\infty.$$

And since

$$\nabla T_k(u_n-\omega_j)\rightharpoonup\nabla T_k(u-\omega_j)\text{ in }L_\varphi(Q_T)\text{ as }n\to\infty,$$

we can deduce

$$\int_{Q_\tau}F\nabla T_k(u_n-\omega_j)\,dx\,dt\to\int_{Q_\tau}F\nabla T_k(u-\omega_j)\,dx\,dt.$$

and

$$\int_{Q_\tau}\Phi(x,t,T_{k_0}(u_n))\nabla T_k(u_n-\omega_j)\,dx\,dt\to\int_{Q_\tau}\Phi(x,t,T_{k_0}(u))\nabla T_k(u-\omega_j)\,dx\,dt.$$

Finally, we turn to see the right-hand side of (56), since

$$T_k(u_n-\omega_j)\to T_k(u-\omega_j)\text{ weakly* in }L^\infty\text{ as }n\to\infty,$$

we obtain

$$\int_{Q_\tau} f_n T_k(u_n - \omega_j)\, dx\, dt \to \int_{Q_\tau} f T_k(u - \omega_j)\, dx\, dt.$$

Now, we are ready to pass to the limit as $n \to \infty$ in each term of (56) to conclude that

$$\begin{aligned}
&\int_\Omega \widetilde{T}_k(u - \omega_j)\, dx + \Big\langle \frac{\partial \omega_j}{dt}, T_K(u - \omega_j) \Big\rangle_{Q_\tau} \\
&\quad + \int_{Q_\tau} a(x,t,u,\nabla u) \nabla T_k(u - \omega_j)\, dx\, dt \\
&\quad\quad + \int_{Q_\tau} \Phi(x,t,u) \nabla T_k(u_n - \omega_j)\, dx\, dt \\
&\quad\quad \leq \int_\Omega \widetilde{T}_k(u_0 - \omega_j(0))\, dx + \int_{Q_\tau} f T_k(u - \omega_j)\, dx\, dt + \int_{Q_\tau} F \nabla T_k(u_n - v)\, dx\, dt.
\end{aligned} \tag{58}$$

Now, we pass to the limit in (59) as $j \longrightarrow \infty$, we obtain

$$\begin{aligned}
&\int_\Omega \widetilde{T}_k(u - v)\, dx + \Big\langle \frac{\partial v}{dt}, T_k(u - v) \Big\rangle_{Q_\tau} \\
&\quad + \int_{Q_\tau} a(x,t,u,\nabla u) \nabla T_k(u - v)\, dx\, dt \\
&\quad\quad + \int_{Q_\tau} \Phi(x,t,u) \nabla T_k(u_n - v)\, dx\, dt \\
&\quad\quad\quad \leq \int_\Omega \widetilde{T}_k(u_0 - v(0))\, dx + \int_{Q_\tau} f T_k(u - v)\, dx\, dt + \int_{Q_\tau} F \nabla T_k(u - v)\, dx\, dt.
\end{aligned} \tag{59}$$

It remains to show that u satisfies the initial condition of (22). To do this, recall that, $\dfrac{\partial u_n}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_{\overline{\varphi}}(Q_T)$. As a consequence, an Aubin's type Lemma (cf [40] Corollary 4) and (Lemma 2.3) implies that u_n lies in a compact set of $C^0([0,T]; L^1(\Omega))$. It follows that,$u_n(x, t = 0) = u_{0n}$ converges to $u(x, t = 0)$ strongly in $L^1(\Omega)$. Then we conclude that $u(x, t = 0) = u_0(x)$ in Ω, This concludes the proof of the main result.

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(Recibido en mayo de 2021. Aceptado en julio de 2023)

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