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Stability and Deformation Criteria in Free Boundary CMC Immersions

Criterios de estabilidad y deformación en inmersiones con CMC y frontera libre

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Abstract. Let Σ^n and M^{n+1} be smooth manifolds with smooth boundary. Given a free boundary constant mean curvature (CMC) immersion $\varphi : \Sigma \to M$, we found results related to the existence and uniqueness of a deformation family of φ, $\{\varphi_t\}_{t\in I}$, composed by free boundary CMC immersions. In addition, we give to some criteria of stability and unstability for this type of deformations. These results are obtained from properties of the eigenvalues and eigenfunctions of the Jacobi operator J_φ associated to φ and establishing conditions for this operator such as $\mathrm{Dim}(\mathrm{Ker}(J_\varphi)) = 0$, or if $\mathrm{Dim}(\mathrm{Ker}(J_\varphi)) = 1$ and, for $f \in \mathrm{Ker}(J_\varphi),\ f \neq 0,\ \int_\Sigma f\ vol_{\varphi^*(g)} \neq 0$. The deformation family is unique up to diffeomorphisms.

Key words and phrases. Free boundary constant mean curvature hypersurfaces, Deformation, Stability, Jacobi operator.

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Resumen. Sean Σ^n y M^{n+1} variedades suaves con frontera suave. Dada una inmersión $\varphi : \Sigma \to M$ con curvatura media constante (CMC) y frontera libre, encontramos resultados relacionados con la existencia y unicidad de una familia de deformación de φ, $\{\varphi_t\}_{t\in I}$ compuesta por inmersiones con curvatura media constante y frontera libre. Adicionalmente, damos algunos criterios de estabilidad e inestabilidad para este tipo de deformaciones. Estos resultados son obtenidos a partir de las propiedades de los valores propios y las funciones propias del operador de Jacobi J_φ asociado a φ, y condiciones de estabilidad para este operador, tales como, $\mathrm{Dim}(\mathrm{Ker}(J_\varphi)) = 0$, o si $\mathrm{Dim}(\mathrm{Ker}(J_\varphi)) = 1$, para $f \in \mathrm{Ker}(J_\varphi),\ f \neq 0,\ \int_\Sigma f\ vol_{\varphi^*(g)} \neq 0$. La familia de deformación es única, salvo difeomorsmos.

Palabras y frases clave. Hipersuperficies con curvatura media constante y frontera libre, deformación, estabilidad, operador de Jacobi.

1. Introduction

In different areas of science and technology, the study of the Theory of Optimization and Stability plays a very important role. Therefore, it is necessary to obtain criteria that allow finding and understanding situations such as energy conservation, minimization of materials, optimization of resources, etc. In our context,the term stability refers to the fact that surfaces or hypersurfaces with constant mean curvature (CMC) must minimize the area of enclosed volume among all nearby surfaces that enclose the same volume. A CMC hypersurface is stable if the second variation of area functional is nonnegative.

In calculus of variations there is a class of problems called *isoperimetric*; the classical isoperimetric problem consists of finding the minimum area among all hypersurfaces of a Riemannian manifold enclosing a region with prescribed volume. We know that solutions to this problem are CMC hypersurfaces. More precisely, if $\varphi : \Sigma \to M$ is an immersion of an orientable n-dimensional compact manifold Σ into the $(n+1)$-dimensional Riemannian manifold M, the condition that φ has constant mean curvature H_0 is equivalent to the fact that φ is a critical point of the area functional defined in the space of embeddings of Σ in M that bound a region of fixed volume (see for instance [3]). The solutions of the isoperimetric problem correspond to *minima* of the constrained variational problem, however, it is interesting to study all critical points of the problem. One of the interesting questions concerning general CMC hypersurfaces is establishing the non-degeneracy as constrained critical points. For the case of free boundary CMC hypersurfaces, we can find an answer of this problem in a previous article by the author (see [8]).

If φ_t is a smooth variation of φ, $t \in (-\epsilon, \epsilon)$, $\varphi_0 = \varphi$, such that $V_t = V_0$, for all $t \in (-\epsilon, \epsilon)$, where V_t is the volume of the region bounded by $\varphi_t(\Sigma)$, a standard approach for finding the solution of such a isoperimetric problem is to look for the critical points of the functional $\mathfrak{f}(t) = A_t + \lambda V_t$, A_t the area of φ_t, $\lambda =$ const., which is the classical method of Lagrange multipliers. When $\lambda = nH_0$ we have the aforementioned equivalence.

In the case where M is a manifolds with boundary ∂M and Σ is also a manifold with boundary, the isoperimetric problem can be described as follows. One wants to minimize the area among all compact hypersurfaces diffeomorphic to Σ in M with boundary contained in ∂M and whose interior lies in the interior of M, and which divide M in two regions such that the closure of one of them is compact and with prescribed volume. The solutions of this problem, called *free boundary CMC hypersurfaces*, are the so-called *normal CMC hypersurfaces*. Let H_0 be denote the value (constant) of the mean curvature of one such hypersurface. If $H_0 = 0$ then we say that $\varphi(\Sigma)$ is a *orthogonal free boundary minimal hypersurface*. A. Ros and E. Vergasta obtain results on the stability of solutions of this isoperimetric problem in the case where M is compact and convex, see [16].

We prove, using properties of eigenvalues and eigenfunctions of the Jacobi operator J_φ of φ, which is interpreted as the linearization of the mean curvature function H_{φ_t} at φ_0 (see Section 2.3), that if $\text{Dim}(\text{Ker}(J_\varphi)) = 0$, or if $\text{Dim}(\text{Ker}(J_\varphi)) = 1$ and, for $f \in \text{Ker}(J_\varphi)$, $f \neq 0$, $\int_\Sigma f \; vol_{\varphi^*(g)} \neq 0$, φ has a free boundary CMC deformation and this is unique up to diffeomorphism. Our work is inspired on the article by Miyuki Koiso (see [11]), who studies the case for CMC surfaces in $\mathbb{R}^3$ with fixed boundary. Also, we have used the results of the work of Bettiol-Piccione-Santoro ([6]) who studies deformations of hypersurfaces of constant mean curvature of free limit whose Jacobi operator is degenerate due to symmetries of the environmental space. Our first perturbation existence theorem, when the eigenvalues of J_{φ_0} are non-zero, is the following:

Theorem 4.2. *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ be a free boundary CMC immersion, with mean curvature H_0. Suppose that $dim(ker(J_{\varphi_0})) = 0$, that is, the eigenvalues of the problem $J_{\varphi_0}(f) = \lambda f$, $f \in C^{j,\alpha}_\partial(\Sigma_0)$, are nonzero. Then, there is a neighborhood $\hat{I}$ of $H_0 \in \mathbb{R}$ and a unique injective C^1 mapping, $\zeta : \hat{I} \to C^{j,\alpha}_\partial(\Sigma_0)$, such that $\zeta(H_0) = 0$ and $\varphi_{\zeta(H)}$ is a free boundary CMC immersion with mean curvature H.*

Moreover, if $\psi : \Sigma \to M$ is a free boundary CMC immersion sufficiently close to φ_0, in the topology of $C^{j,\alpha}$, then ψ must be equal (up to diffeomorphisms) to some $\varphi_{\zeta(H)}$.

Now, if J_{φ_0} has some eigenvalue equal to zero, then we have the following result:

Theorem 4.3 *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ be a free boundary CMC immersion, with mean curvature H_0. Suppose that:*

- *$dim(ker(J_{\varphi_0})) = 1$. This is, $\lambda = 0$ is an eigenvalue of multiplicity 1 for J_{φ_0}, and*
- *$\int_{\Sigma_0} f_0 \; vol_{\varphi_0^*(g)} \neq 0$ for some $f_0 \in ker(J_{\varphi_0}) - \{0\}$.*

Then there exist a neighborhood $W \subset ker(J_{\varphi_0})$ of 0 and a unique injective map C^1

$$(\xi, \eta) : W \longrightarrow (C^{j,\alpha}_\partial(\Sigma_0) \cap ker(J_{\varphi_0})^\perp) \times \mathbb{R},$$

such that $(\xi, \eta)(0) = (0, H_0)$ and such that $\varphi_{f+\xi(f)} : \Sigma \to M$, with $f \in W$, is a free boundary CMC immersion, with mean curvature $\eta(f)$.

Moreover, if $\psi : \Sigma \to M$ is a free boundary CMC immersion sufficiently close to φ_0, in the topology of $C^{j,\alpha}$, then Y must be equal (up to diffeomorphisms) to some $\varphi_{f+\xi(f)}$.

In both cases, we also obtain uniqueness in this perturbation, up to parameterizations.

Now, regarding the stability of the free boundary CMC hypersurfaces, we obtained the following result, where $\lambda_1 < \lambda_2 \leq \ldots$ are the eigenvalues of J_{φ_0} (see remark 4.1):

Theorem 5.4. *Let $\varphi_0 : \Sigma \to M$ be a free boundary CMC immersion. Let λ_i, $i \geq 1$ be the eigenvalues J_{φ_0}.*

(1) If $\lambda_1 \geq 0$, then φ_0 is stable.

(2) If $\lambda_1 < 0 < \lambda_2$, then there is a unique function $\kappa \in C^{j,\alpha}(\Sigma)$ such that $J(\kappa) = 1$ and we have that:

(2-a) If $\int_\Sigma \kappa \; vol_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.

(2-b) If $\int_\Sigma \kappa \; vol_{\varphi_0^*(g)} > 0$, then φ_0 is unstable.

(3) If $\lambda_1 < 0 = \lambda_2$, then we have:

(3-a) If there exist a λ_2-eigenfunction f_2 such that $\int_\Sigma f_2 \; vol_{\varphi_0^*(g)} \neq 0$, then φ_0 is unstable.

(3-b) If $\int_\Sigma h_2 \; vol_{\varphi_0^*(g)} = 0$ for all λ_2-eigenfunction h_2, then there exist a unique function $\bar{h}_2 \in (ker(J_{\varphi_0}))^\perp$ such that $J(\bar{h}_2) = 1$ and

(3-b-i) If $\int_\Sigma \bar{h}_2 \; vol_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.

(3-b-ii) If $\int_\Sigma \bar{h}_2 \; vol_{\varphi_0^*(g)} > 0$, then φ_0 is unstable

(4) If $\lambda_2 < 0$, then φ_0 is unstable.

For both cases (2) and (3) the calculation of eigenvalues and integrals of the eigenfunctions can be difficult. In this sense, the following criterion gives us a geometric interpretation of stability based on the existence of a deformation family.

Corollary 5.6. *Let $\varphi : \Sigma \to M$ be a free boundary CMC immersion of class $C^{j+1,\alpha}$. We assume that $\lambda_1 < 0 \leq \lambda_2$. If there exist a deformation φ_t of φ, $-\epsilon < t < \epsilon$, with $\varphi_0 = \varphi$, such that φ_t is a free boundary CMC immersion of class $C^{j,\alpha}$ for all $t \in (-\epsilon, \epsilon)$, and such that $\frac{dH_t}{dt}|_{t=0} = H_0' = constant \neq 0$, where H_t is the constant mean curvature of φ_t and V_t is the volume of φ_t, we have that:*

(1) If $H_0' V_0' \leq 0$, then φ is stable,

(2) if $H_0' V_0' > 0$, then φ is unstable.

If there is no such deformation, then φ is unstable

2. Preliminaries

Throughout this paper we will consider M as a $(n+1)$-dimensional differential manifold with smooth boundary $\partial M \neq \varnothing$ and Σ as n-dimensional differential manifold with smooth boundary $\partial\Sigma \neq \varnothing$. In this section we introduce the concepts of admissibility and orthogonality of hypersurfaces with boundary in a manifold with smooth boundary, we will give the definition of mean curvature and free boundary constant mean curvature (CMC) hypersurface.

2.1. Orthogonal submanifolds and mean curvature

Definition 2.1. Let $\varphi : \Sigma \to M$ be an embedding. We identify φ with its image $\varphi(\Sigma) \subset M$. $\vec{\eta}_{\partial M}$ is the outer unit normal field along the boundary of M. We call φ *admissible* if it satisfies (a) and (b), and *normal (orthogonal)* if it also satisfies (c), where:

(a) $\varphi(\Sigma) \cap \partial M = \varphi(\partial\Sigma)$,

(b) the normal bundle $T(\varphi(\Sigma))^{\perp}$ is orientable,

(c) and for each point $p \in \varphi(\partial\Sigma)$, $\vec{\eta}_{\partial M}(p) \in T_p\varphi(\Sigma)$,

The admissible hypersurface $\varphi(\Sigma)$ is said to *bound a finite volume* if

(d) $M \setminus \varphi(\Sigma) = \Omega_1 \cup \Omega_2$, with $\overline{\Omega}_1$ compact and $\Omega_1 \cap \Omega_2 = \varnothing$.

If $\varphi : \Sigma \to M$ is an orthogonal admissible embedding, then $\varphi(\Sigma)$ it is compact and $\varphi(\Sigma)$ and ∂M are transverse submanifolds (see Appendix 7, Definition 7.2) In this case, we say that $\varphi(\Sigma)$ is a *orthogonal submanifold* of M (see Figure 1).

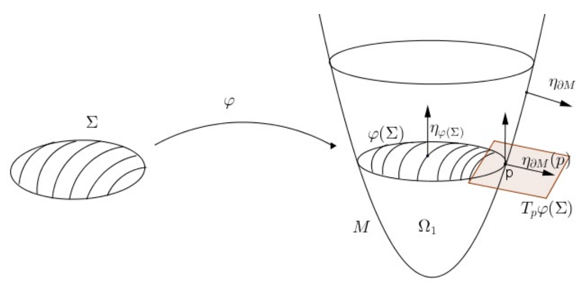

FIGURE 1. Orthogonal admissible embedding.

Let g be a C^{∞}-riemannian metric on M and $\varphi_0 : \Sigma \to M$ an orthogonal immersion. Write $\Sigma_0 := \varphi_0(\Sigma)$. We define the *second fundamental form* on Σ_0 as

$$\mathrm{II}^{\Sigma_0}(X, Y) := g(\nabla_X Y, \vec{\eta}_{\Sigma_0}), \tag{1}$$

where $\vec{\eta}_{\Sigma_0}$ is the unit normal vector field to Σ_0 in the orientable normal bundle, ∇ is the Levi-Civita connection in M, and X, Y are vector fields in $T\Sigma_0$.

The *mean curvature function* $H_{\Sigma_0}\colon \Sigma_0 \to \mathbb{R}$ is defined as the trace of the second fundamental form $\mathbb{II}^{\Sigma_0}$. The *mean curvature vector* of Σ_0 is defined as $\vec{H}_{\Sigma_0} = H_{\Sigma_0}\vec{\eta}_{\Sigma_0}$. If H_{Σ_0} is constant, Σ_0 is called a *constant mean curvature hypersurface* (or CMC hypersurface), and if $H_{\Sigma_0} = 0$, Σ_0 is a *minimal hypersurface.*

2.2. Variational problem

In the theory of variational problems it is known that the hypersurfaces with CMC of M minimize the area among all hypersurfaces enclosing a fixed volume. In the case where $\partial\Sigma$ is allowed to move freely along ∂M, the variational problem is called *free boundary CMC problem.* The solutions of this problem are orthogonal hypersurfaces with CMC which are called *free boundary CMC hypersurfaces.* We introduce the following notation:

- $\mathrm{Emb}_{\partial}(\Sigma, M)$ is the space of admissible embeddings of Σ in M,
- $\mathrm{Emb}_{\partial\perp}(\Sigma, M) \subset \mathrm{Emb}_{\partial}(\Sigma, M)$ is the subspace of normal admissible embeddings and bounding a finite volume.

We have (see Barbosa-do Carmo [3]) that $\varphi_0 \in \mathrm{Emb}_{\partial\perp}(\Sigma, M)$ have CMC H if only it if is a critical point of the funtional $\mathfrak{f}_H : \mathrm{Emb}_{\partial\perp}(\Sigma, M) \to \mathbb{R}$, defined by

$$\mathfrak{f}_H(\varphi) = \int_{\Sigma} vol_{\varphi^*(g)} - H \int_{\Omega_1} vol_g, \tag{2}$$

Note that if $H = 0$ then $\varphi_0(\Sigma)$ has the minimal volume over all hypersurfaces $\varphi(\Sigma)$, $\varphi \in \mathrm{Emb}_{\partial}(\Sigma, M)$. In this case, φ_0 is said to be a *free boundary minimal hypersurface.*

2.3. Jacobi operator

Let $\varphi_0 \in \mathrm{Emb}_{\partial\perp}(\Sigma, M)$, $\Sigma_0 = \varphi_0(\Sigma)$. $C^j(\Sigma_0)$ is the set of functions $f : \Sigma_0 \to \mathbb{R}$ with continuous derivatives to j order, where j could be infinite. The second-order linear differential operator $J_{\varphi_0} : C^j(\Sigma_0) \to C^{j-2}(\Sigma_0)$, $j \geq 2$, defined by

$$J_{\varphi_0}(f) := \Delta_{\Sigma_0} f - \big(||\mathbb{II}^{\Sigma_0}||^2_{HS} + \mathrm{Ric}_g(\eta_{\Sigma_0}, \eta_{\Sigma_0})\big) f, \tag{3}$$

is called the *Jacobi operator*. Here, Δ_{Σ_0} is the (nonnegative) laplacian of (Σ_0, γ), $||\mathbb{II}^{\Sigma_0}||^2_{HS}$ is the square of Hilbert-Schmidt norm (see Appendix 7, Definition 7.3) of the second fundamental form of φ_0 and Ric_g the Ricci tensor associated with g. A *Jacobi scalar field* along of φ_0 is a smooth function $f \in C^j(\Sigma_0)$ such that $J_{\varphi_0}(f) = 0$.

We consider a smooth variation of φ_0 as follows:

$$\Phi : \Sigma \times (-\epsilon, \epsilon) \to M, \quad \epsilon > 0, \tag{4}$$

such that $\Phi(\Sigma, s) = \varphi_s(\Sigma) = \Sigma_s \subset M$, $\varphi_s \in \mathrm{Emb}_\partial(\Sigma, M)$ with CMC H_s. Let $\Phi_{s_0} = \frac{\partial}{\partial s}\big|_{s=0}\Phi$ be the corresponding variational vector field. Then $\xi_0 = g(\Phi_{s_0}, \vec{\eta}_{\Sigma_0})$ satisfies

$$\frac{d}{ds}\Big|_{s=0} H_s = \Delta_{\Sigma_0}\xi_0 - \big(||\mathrm{I\!I}^{\Sigma_0}||^2_{HS} + \mathrm{Ric}_g(\eta_{\Sigma_0}, \eta_{\Sigma_0})\big)\xi_0 = J_{\varphi_0}(\xi_0), \tag{5}$$

Then J_{φ_0} represents the second variation $d^2\mathfrak{f}_H(\varphi_0)$ of $\mathfrak{f}_H$ at the critical point φ_0, with respect to the L^2 inner product.

***Remark* 2.2.** Note that ξ_0 is a Jacobi field exactly when $\frac{d}{ds}\Big|_{s=0} H_s = 0$.

The proof of the following lemma is found in Rodríguez [8], Lemma 2.2.

Lemma 2.3. If each φ_s is normal, that is $\varphi_s \in \mathrm{Emb}_{\partial\perp}(\Sigma, M)$, with CMC, then ξ_0 satisfies the so-called *linearized free boundary condition*

$$g(\nabla\xi_0, \vec{\eta}_{\partial M}) + \mathrm{I\!I}^{\partial M}(\vec{\eta}_{\Sigma_0}, \vec{\eta}_{\Sigma_0})\xi_0 = 0, \tag{6}$$

where $\nabla\xi_0$ is the g-gradient of ξ_0 in Σ_0.

2.4. Regularity

As we are interested in proving the existence of deformations by normal admissible hypersurfaces from a free boundary CMC hypersurface, this in a small enough neighborhood in the space of the admissible embeddings, it is necessary to establish a certain regularity condition for these embeddings. To establish this it is necessary that the Jacobi operator has the condition of being Fredholm with a certain index (see Appendix 7, Definition 7.5). This property is obtained if a regularity condition of type Hölder $C^{j,\alpha}$ (see Appendix 7, Definition 7.4) is established,with $j \geq 2$ and some $\alpha \in (0,1)$. We can endow space of functions defined from Σ to $\mathbb{R}$, $C^{j,\alpha}(\Sigma)$, with regularity $C^{j,\alpha}$, with the norm

$$\|f\|_{C^{j,\alpha}} = \|f\|_{C^j} + \max_{|\beta|=j} \left|D^\beta f\right|_{C^{0,\alpha}}, \tag{7}$$

where β ranges over multi-indices and

$$\|f\|_{C^j} = \max_{|\beta|\leq j}\sup_{x\in\Sigma}\left|D^\beta f(x)\right|, \quad |Df|_{C^{0,\alpha}} = \sup_{x\neq y\in T\Sigma}\frac{|Df(x) - Df(y)|}{||x-y||^\alpha}.$$

It is well-known that $C^{j,\alpha}(\Sigma)$ endowed with this norm is a (nonseparable) Banach space[1].

[1]Will be called C^j-Whitney type Banach space to sub-space $C^{j,\alpha}(\Sigma)$ endowed with the metric 7, with the property that the $\|\cdot\|_{C^{j,\alpha}}$-convergence of a sequence implies convergence in the weak Whitney C^j-topology, see [5] and [7]. For definition and properties of Whitney C^j-topology see [17]

We define the space

$$C_{\partial}^{j,\alpha}(\Sigma_0) := \{f \in C^{j,\alpha}(\Sigma_0) \ : \ g\big(\nabla f, \eta_{\partial M}\big) + \mathrm{II}^{\partial M}\big(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0}\big) f = 0\}. \qquad (8)$$

The restriction of J_{φ_0} $J_{\varphi_0} : C_{\partial}^{j,\alpha}(\Sigma) \to C^{j-2}(\Sigma)$ is a Fredholm operator of index zero (see [15, section 2]).

***Remark* 2.4.** When the operator $\mathfrak{f}_H$ defined in (2) is considered on the space of $C^{j,\alpha}$-embeddings, the Jacobi operator acts on the corresponding tangent space at φ_0, which can be identified with $C_{\partial}^{j,\alpha}(\Sigma_0)$ (see Proposition 2.8).

Since J_{φ_0} is the representation of the second variation of the area functional, we can define the following.

Definition 2.5. The embedding $\varphi_0 \in \mathrm{Emb}_{\partial\perp}(\Sigma, M)$ with g-CMC is called *non-degenerate* if $J_{\varphi_0}\big|_{C_{\partial}^{j,\alpha}(\Sigma)}$ is an isomorphism of Banach spaces, i.e. ker $J_{\varphi_0} \cap C_{\partial}^{j,\alpha}(\Sigma) = \varnothing$.

2.5. The smooth structure of the set of orthogonal embeddings

The appropriate setup for studying the set of submanifolds of a diffeomorphism type is obtained by considering the notion of *unparameterized embeddings*. Unparameterized embeddings are the elements of the quotient space generated by right group action of diffeomorphisms on the space of embeddings of Σ into M. The area and volume functional are invariant by this action.

Definition 2.6. Two embeddings φ_1 and φ_2 from Σ in M will be *equivalent* if there exists a diffeomorphism $\phi : \Sigma \to \Sigma$ such that $\varphi_2 = \varphi_1 \circ \phi$, i.e., if they are different parametrizations of the same submanifold of M diffeomorphic to Σ. For $\varphi \in \mathrm{Emb}(\Sigma, M)$, we denote by $[\varphi]$ the class of all embedding that are equivalent to φ. We say that $[\varphi]$ is a *unparametrized embedding* of Σ in M.

Definition 2.7. We define the following sets:

- $\mathcal{E}(\Sigma, M) := \{[\varphi] : \varphi$ is a embedding of order $C^{j,\alpha}\}$,
- $\mathcal{E}_{\partial}(\Sigma, M) := \{[\varphi] \in \mathcal{E}(\Sigma, M) : \varphi(\Sigma) \cap \partial M = \varphi(\partial\Sigma)\}$,
- Let g, as before, be a C^{∞}-riemannian metric in M,

$$\mathcal{E}_{\partial,g}^{\perp}(\Sigma, M) := \{[\varphi] \in \mathcal{E}_{\partial}(\Sigma, M) : \varphi \text{ is } g-\text{orthogonal}\}.$$

There is a smooth Banach manifold structure, of infinite dimension, for a sufficiently small neighborhood of $[\varphi_0] \in \mathcal{E}_{\partial,g}^{\perp}(\Sigma, M)$ in some suitable topology.

Proposition 2.8. [6, Proposition 4.1] Let Σ be a compact manifold with boundary and φ_0 be an admissible smooth normal embedding. Let $\mathcal{U} \subset \mathcal{E}_{\partial,g}^{\perp}(\Sigma, M)$ be a sufficiently small neighborhood of $[\varphi_0]$, then $\mathcal{U}$ can be identified with an infinite-dimensional smooth submanifold $\mathcal{N}$ of the Banach space $C^{j,\alpha}(\Sigma)$, with $0 \in \mathcal{N}$ corresponding to $[\varphi_0]$, such that $T_0\mathcal{N} = C_{\partial}^{j,\alpha}(\Sigma)$ (see 8).

By Proposition 2.8 we have a bijection $f \mapsto \varphi_f$ from a suitable neighborhood U of $0 \in C^{j,\alpha}(\Sigma_0)$ to a neighborhood V of $\varphi_0 \in C^{j,\alpha}(\Sigma, M)$ defined by

$$\varphi_f(p) := exp_{\varphi_0(p)}\big(f(p)\vec{n}_{\Sigma_0}(p)\big), \tag{9}$$

where exp is the exponential map over Σ_0 associated with g.

3. Analytical preliminaries

Following the same notation, Σ^n and M^{n+1} will be smooth manifolds, with smooth boundaries $\partial\Sigma$ and ∂M respectively. Let g be a C^∞-riemannian metric in M, $\varphi_0 : \Sigma \to M$ an free boundary CMC immersion and H_0 the mean curvature of φ_0. Let $\vec{n}_{\Sigma_0}$ be the unit normal vector field to $\Sigma_0 = \varphi_0(\Sigma)$ in the orientable normal bundle and $\vec{n}_{\partial M}$ the unit normal vector field to ∂M. Also let $C^{j,\alpha}_\partial(\Sigma_0)$ be the space of functions $C^{j,\alpha}$ that complies with the linearized free boundary condition $g(\nabla f, \vec{n}_{\partial M}) + \mathbb{I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})f = 0$, defined in (8). We have that $C^{j,\alpha}_\partial(\Sigma_0) \subset L^2(\Sigma_0)$.

The following lemma shows that there exists a family of orthogonal hypersurfaces in a suitable neighborhood of φ_0.

Lemma 3.1. Let $\Sigma_0 = \varphi_0(\Sigma)$ be a free boundary CMC hypersurface and $\bar{f} \in C^{j,\alpha}_\partial(\Sigma_0)$. Then, there is a differentiable map $o : (-\epsilon, \epsilon) \to C^{j,\alpha}(\Sigma_0)$, with $\frac{o(t)}{t} \xrightarrow[t\to 0]{} 0$, such that $\varphi_t : \Sigma_0 \to M$, defined by

$$\varphi_t(p) := \exp_{\varphi_0(p)}\Big([t\bar{f}(p) + o(t)(p)]\vec{n}_{\Sigma_0}(p)\Big),$$

is orthogonal (in the sense of definition 2.1 (c))

Proof. By Proposition 2.8 there is a bijective correspondence between a neighborhood $\mathcal{U} \subset \mathcal{E}^\perp_\partial(\Sigma, M)$ of $[\varphi_0]$ and a infinite-dimensional smooth submanifold $\mathcal{N}$ of the Banach space $C^{j,\alpha}(\Sigma)$, with $0 \in \mathcal{N}$ corresponding to $[\varphi_0]$ and $T_0\mathcal{N} = C^{j,\alpha}_\partial(\Sigma)$. So, there is a diffeomorphism, given by the Inverse Mapping Theorem (see 7.6), between $\mathcal{U}$ and a neighborhood $V \subset T_0\mathcal{N}$ of 0, such that $\varphi_t \mapsto t\bar{f}$. On the other hand, exp_{φ_0} also generates a diffeomorphism between $\mathcal{U}$ and some neighborhood $V' \subset T_0\mathcal{N}$, such that $\varphi_t \mapsto t\bar{g}_t$, with $g_0 = 0$ and $g'_0 = \bar{f}$. Since V and V' are diffeomorphic we have $g_t = t\bar{f} + o(t)$, where $o(t)$ is differentiable and $\frac{o(t)}{t} \to 0$ if $t \to 0$. ☑

Remember that the restriction of the Jacobi operator, (which we defined by the formula $J_{\varphi_0}(f) := \Delta_{\Sigma_0} f - \big(||\mathbb{I}^{\Sigma_0}||^2_{HS} + Ric_g(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})\big)f)$, to the closed subspace $C^{j,\alpha}_\partial(\Sigma_0)$ is a Fredholm operator of index zero that takes values in $C^{j-2,\alpha}(\Sigma_0)$.

To simplify the notation we set

$$J = J_{\varphi_0} : C^{j,\alpha}_\partial(\Sigma_0) \to C^{j-2,\alpha}(\Sigma_0).$$

Let $E = \mathrm{Ker}(J)$ and $E^{\perp}$ the orthogonal of E in $L^2(\Sigma)$.

By definition $\Delta_{\Sigma_0} f = Div(-\nabla f)$ and if V is a vector field, then $Div(V) = \nabla \cdot V$. Now, if h is a scalar field we have $Div(hV) = \nabla h \cdot V + hDiv(V)$. On the other hand by the Gauss theorem (divergence theorem)

$$\int_{\Sigma} Div(hV)\ d_{\Sigma} = \int_{\partial\Sigma} hV \cdot \vec{n}_{\Sigma}\ d_{\partial\Sigma}$$

Proposition 3.2. J is symmetric with the $L^2(\Sigma_0)$ inner product.

Proof. From the definition of J_{φ_0} we see that it is sufficient to prove that Δ_{Σ_0} is symmetric. Let $f, h \in C^{j,\alpha}_{\partial}(\Sigma_0)$. Then, by the divergence theorem

$$\begin{aligned}
\int_{\Sigma_0} \Delta_{\Sigma_0}(f)h\ d_{\Sigma_0} &= \int_{\Sigma_0} Div(-\nabla f)h\ d_{\Sigma_0} \\
&= -\int_{\Sigma_0} \nabla f \cdot \nabla h\ d_{\Sigma_0} - \int_{\partial\Sigma_0} h\Big(\frac{\partial f}{\partial \vec{n}_{\partial M}}\Big)\ d_{\partial\Sigma_0} \\
&= -\int_{\Sigma_0} \nabla f \cdot \nabla h\ d_{\Sigma_0} - \int_{\partial\Sigma_0} h\Big(g(\nabla f, \vec{n}_{\partial M})\Big)\ d_{\partial\Sigma_0} \\
&= -\int_{\Sigma_0} \nabla f \cdot \nabla h\ d_{\Sigma_0} + \int_{\partial\Sigma_0} h\Big(f\mathrm{I\!I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})\Big)\ d_{\partial\Sigma_0} \\
&= -\int_{\Sigma_0} \nabla h \cdot \nabla f\ d_{\Sigma_0} + \int_{\partial\Sigma_0} f\Big(h\mathrm{I\!I}^{\partial M}(\vec{n}_{\Sigma_0}, \vec{n}_{\Sigma_0})\Big)\ d_{\partial\Sigma_0} \\
&= \int_{\Sigma_0} \Delta_{\Sigma_0}(h)f\ d_{\Sigma_0}
\end{aligned}$$

☑

Lemma 3.3. Let λ be a real number. Let

$$J(f) - \lambda f = h \tag{10}$$

a equation, where $h \in C^{j-2,\alpha}(\Sigma_0)$. Then (10) has a solution in the following cases:

(a) If λ is not an eigenvalue of J, then (10) has a unique solution in $C^{j,\alpha}_{\partial}(\Sigma_0)$ for all $h \in C^{j-2,\alpha}(\Sigma_0)$

(b) If λ is an eigenvalue of J, $E_\lambda \subset C^{j,\alpha}_{\partial}(\Sigma_0)$ is the eigenspace associated to λ and $E^{\perp}_{\lambda} \subset L^2(\Sigma_0) \cap C^{j,\alpha}_{\partial}(\Sigma_0)$ the orthogonal space to E_λ, then (10) have solution in $C^{j,\alpha}_{\partial}(\Sigma_0)$ if and only if $h \in E^{\perp}_{\lambda}$, this is,

$$\int_{\Sigma_0} h\sigma\ vol_{\varphi_0^*(g)} = 0,$$

for all $\sigma \in E_\lambda$.

Proof. For (a), we have that J is a Fredholm operator of index zero. Then so is the operator $(J-\lambda) : C^{j,\alpha}_{\partial}(\Sigma_0) \to C^{j-2,\alpha}(\Sigma_0)$ (as a sum of Fredholm and compact operators). If λ is not an eigenvalue then $(J-\lambda)$ is injective and hence also surjective ($\mathrm{Codim}(J-\lambda) = 0$).

For (b), let $h \in C^{j-2,\alpha}(\Sigma_0)$ be such that $h = J(f) - \lambda(f)$ for some $f \in C^{j,\alpha}_{\partial}(\Sigma_0)$ and let $\sigma \in E_\lambda$. Then,

$$\begin{aligned}\int_{\Sigma_0} h\sigma \; vol_{\varphi_0^*(g)} &= \int_{\Sigma_0} (J(f) - \lambda f)\sigma \; vol_{\varphi_0^*(g)} \\ &= \int_{\Sigma_0} J(f)\sigma - \lambda\sigma f \; vol_{\varphi_0^*(g)} \\ &= \int_{\Sigma_0} J(f)\sigma - J(\sigma) f \; vol_{\varphi_0^*(g)} \\ &= 0.\end{aligned}$$

The equality holds because J is symmetric (by proposition 3.2). Therefore $\mathrm{Im}(J-\lambda) \subset E_\lambda^\perp$.

Again, as $(J-\lambda)$ is Fredholm of index zero, $\mathrm{Im}(J-\lambda)$ is closed and

$$\mathrm{Codim}(\mathrm{Im}(J-\lambda)) = \mathrm{Dim}(\mathrm{Ker}(J-\lambda)) = \mathrm{Dim}(E_\lambda),$$

and since $\mathrm{Im}(J-\lambda) \subset E_\lambda^\perp$, then $\mathrm{Im}(J-\lambda) = E_\lambda^\perp$. ☑

4. Existence and uniqueness of CMC deformations

Since the existence of deformations, which we deal with in this section, and the stability criteria, which are presented in section 5, depend on the eigenvalues of the Jacobi operator, it is important to note the following:

***Remark* 4.1.** In the problem (11), the eigenvalues of $J(f) = \lambda f$ are a countable set (Smale, [[18, lemma 1]), such that $\lambda_1 < \lambda_2 \leq ...$, with $\lambda_i \to \infty$ if $i \to \infty$. Each eigenfunction is in $C^{j+1,\alpha}_{\partial}(\Sigma)$ (Gilbarg-Trudinger, [9, Theorem 8.13], and Ladyzhenskaya-Ural'tseva, [13, Chap. 3, Theorem 12.1]). And we can choose an orthonormal basis $B = \{f_i\}$ for $L^2(\Sigma)$ where each f_i is associated to λ_i. The first eigenvalue λ_1, which has a special role in the spectral theory of J, is always simple, i.e., of multiplicity 1, and the λ_1-eigenfunction are positive. Let f_k be a (non zero) eigenfunction corresponding to the eigenvalue λ_k, for some $k \geq 1$. The connected components of the set $\Sigma \setminus f_k^{-1}(0)$ are called the **nodal domains of** f_k. Then, the number of nodal domains of f_k is less than or equal to k; this is known as **Courant's nodal domain theorem** (see [2]).

Next, we present our first perturbation theorem. Here we assume that the kernel of the Jacobi operator is trivial. The perturbation obtained is unique, up to parameterizations.

Theorem 4.2. *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ be a free boundary CMC immersion, with mean curvature H_0. Suppose that Dim$(E) = 0$, that is, the eigenvalues of problem*

$$\begin{cases} J(f) = \lambda f \\ f \in C^{j,\alpha}_{\partial}(\Sigma_0) \end{cases} \tag{11}$$

are nonzero. Then, there is a neighborhood $\hat{I}$ of $H_0 \in \mathbb{R}$ and a unique injective C^1 mapping, $\zeta : \hat{I} \to C^{j,\alpha}_{\partial}(\Sigma_0)$, such that $\zeta(H_0) = 0$ and $\varphi_{\zeta(H)}$ is a free boundary CMC immersion with mean curvature H.

Moreover, if $\psi : \Sigma \to M$ is a free boundary CMC immersion sufficiently close to φ_0, in the topology of $C^{j,\alpha}$, then ψ must be equal (up to diffeomorphisms) to some $\varphi_{\zeta(H)}$.

Proof. Let U be a suitably small neighborhood of $0 \in C^{j,\alpha}_{\partial}(\Sigma_0)$ such that φ_f is orthogonal to M for each $f \in U$ (see Lemma 3.1).

We define the map

$$\begin{aligned} F : U \times \mathbb{R} &\longrightarrow C^{j-2,\alpha}(\Sigma_0) \\ (f, H) &\mapsto F(f, H) = H - H_f, \end{aligned}$$

where H_f is the mean curvature of φ_f. Then we have that

i) $F(0, H_0) = 0$ and

ii) φ_f has CMC if and only if $F(f, H) = 0$ for some $H \in \mathbb{R}$.

Now,

$$\frac{\partial F}{\partial f}(0, H_0) : C^{j,\alpha}_{\partial}(\Sigma_0) \to C^{j-2,\alpha}(\Sigma_0)$$

is given by

$$\frac{\partial F}{\partial f}(0, H_0)(h) = J(h).$$

Let us see that if 0 is not an eigenvalue of $J(f) = \lambda f$ then $\frac{\partial F}{\partial f}(0, H_0)$ is *bijective*. Injectivity is immediate because Dim(Ker$J$) = Dim$(E) = 0$. Now, applying part (a) of Lemma 3.3 with $\lambda = 0$, surjectivity is obtained.

Thus, the conditions for applying the Implicit Mapping Theorem (see Appendix 7, Theorem 7.7) are satisfied for F. Then there is a neighborhood $\hat{I}$ of H_0 and a map $\zeta : \hat{I} \to U$ such that $\zeta(H_0) = 0$ and $F(\zeta(H), H) = 0$. So $\varphi_{\zeta(H)}$ is a free boundary CMC immersion with mean curvature H. ☑

A second result on existence of perturbations is given below. Here we assume that the Jacobi operator kernel is generated by a single non-zero function in $C^{j+1,\alpha}_{\partial}(\Sigma)$. Again, we get uniqueness up to parameterizations.

Theorem 4.3. *Let $\varphi_0 \in C^{j+1,\alpha}(\Sigma, M)$ be a free boundary CMC immersion, with mean curvature H_0. Suppose that:*

(1) Dim$(E) = 1$. This is, $\lambda = 0$ is an eigenvalue of multiplicity 1 for the problem (11), and

(2) $\int_{\Sigma_0} f_0 \; vol_{\varphi_0^*(g)} \neq 0$ for some $f_0 \in E - \{0\}$.

Then there exist a neighborhood $W \subset E$ of 0 and a unique injective C^1 map

$$(\xi, \eta) : W \longrightarrow (C^{j,\alpha}_{\partial}(\Sigma_0) \cap E^{\perp}) \times \mathbb{R},$$

such that $(\xi, \eta)(0) = (0, H_0)$ and such that $\varphi_{f+\xi(f)} : \Sigma \to M$, with $f \in W$, is an free boundary CMC immersion, with mean curvature $\eta(f)$.

Moreover, if $\psi : \Sigma \to M$ is an free boundary CMC immersion sufficiently close to φ_0, in the topology of $C^{j,\alpha}$, then ψ must be equal (up to diffeomorphisms) to some $\varphi_{f+\xi(f)}$.

***Remark* 4.4.** Since there is a biunivocal relation between a subset of the space of functions $C^{j,\alpha}_{\partial}(\Sigma_0)$ and a subset of the embeddings $\mathcal{E}^{\perp}_{\partial,g}(\Sigma_0, M)$, as shown in proposition 2.8, it is clear that the subscript $f + \xi(f)$ corresponds to a function in $C^{j,\alpha}_{\partial}(\Sigma_0)$, according to the definition of (ξ, η), which is a linear combination between the Jacobi field f and an orthogonal function to the kernel of J that generates a unique hypersurface $\varphi_{f+\xi(f)}$ that has constant mean curvature $\eta(f)$.

***Proof*.** We take $F : U \times \mathbb{R} \longrightarrow C^{j-2,\alpha}(\Sigma_0)$, defined as $F(f, H) = H - H_f$, as in the proof of Theorem 4.2. We have then:

i) $F(0, H_0) = 0$ and

ii) φ_f have CMC if and only if $F(f, H) = 0$ for some $H \in \mathbb{R}$.

Now we define the following map:

$$\begin{aligned} \bar{F} : (U \cap E) \times (U \cap E^{\perp}) \times \mathbb{R} &\longrightarrow C^{j-2,\alpha}(\Sigma_0) \\ (f_1, f_2, H) &\longmapsto \bar{F}(f_1, f_2, H) = F(f_1 + f_2, H). \end{aligned}$$

Thus,

$$\bar{F}(0, 0, H_0) = 0.$$

Let us see that

$$\frac{\partial \bar{F}}{\partial (f_2, H)}(0, 0, H) : (C^{j,\alpha}_{\partial}(\Sigma_0) \cap E^{\perp}) \times \mathbb{R} \longrightarrow C^{j-2,\alpha}(\Sigma_0)$$

is *bijective*.

We have that

$$\frac{\partial \bar{F}}{\partial(f_2, H)}(0,0,H)(f,H) = H - J(f)$$

and

$$\text{Ker}(\frac{\partial \bar{F}}{\partial(f_2, H)}(0,0,H)) = \{(f,H) : H - J(f) = 0\}$$

i.e. if $(f_2, H) \in \text{Ker}(\frac{\partial \bar{F}}{\partial(f_2,H)}(0,0,H))$, then $J(f_2) = H$. Let $f_0 \in E$. Then, by (b) in Lemma 3.3 we have

$$H \int_{\Sigma_0} f_0 \ vol_{\varphi_0^*(g)} = 0,$$

which implies that $H = 0$, i.e., $f_2 \in E$. But $f_2 \in E^\perp$, so $f_2 = 0$. Therefore, $\text{Ker}(\frac{\partial \bar{F}}{\partial(f_2,H)}(0,0,H)) = \{(0,0)\}$ and we have injectivity.

To prove surjectivity, we take $h \in C^{j-2,\alpha}(\Sigma_0)$ and set

$$H = \frac{\int_{\Sigma_0} h f_0 \ vol_{\varphi_0^*(g)}}{\int_{\Sigma_0} f_0 \ vol_{\varphi_0^*(g)}}.$$

Again by (b) in Lemma 3.3 there exist $f \in C_\partial^{j,\alpha}(\Sigma_0)$ such that

$$J(f) = h - H.$$

Now, we decompose $f = f_1 + f_2$ in such a way that $f_1 \in E$, $f_2 \in E^\perp$. So,

$$J(f_2) = h - H.$$

Then,

$$h = H - J(f_2) = \frac{\partial \bar{F}}{\partial(f_2, H)}(0,0,H)(f_2,H).$$

Now we can use The Implicit Mapping Theorem in $\bar{F} : (U \cap E) \times (U \cap E^\perp) \times \mathbb{R} \longrightarrow C^{j-2,\alpha}(\Sigma_0)$. Then, there exists a neighborhood $W \subset E \cap U$ of 0 and a map

$$(\xi, \eta) : W \longrightarrow (U \cap E^\perp) \times \mathbb{R},$$

such that

$$(\xi,\eta)(0) = (0, H_0)$$

and such that, for all $f \in W$,

$$0 = \psi(f, (\xi,\eta)(f)) = \psi(f, \xi(f), \eta(f)) = \phi(f + \xi(f), \eta(f)).$$

Thus, $\varphi_{f+\xi(f)}$ have CMC $\eta(f)$. If W is suitable small then (ξ, η) is unique. ☑

5. Stability Criteria

Let $\varphi_t : \Sigma \to M$ be a variation of φ_0, $-\epsilon < t < \epsilon$, such that φ_t is a free boundary CMC immersion. Let $A(t)$ and $V(t)$ be the area and volume of φ_t respectively. We will say that φ_t is volume-preserving if $V(t) = V(0)$ for all $t \in (-\epsilon, \epsilon)$. Remember that φ_t have CMC H_t if only if it is a critical point of funtional

$$\mathfrak{f}(t) = A(t) - H_t V(t) = \int_\Sigma vol_{\varphi_t^*(g)} - H_t \int_{\Omega_t} vol_g,$$

where Ω_t is the volume enclosed by φ_t. Let $\xi_t = g(\frac{\partial \varphi_t}{\partial t}, \vec{n}_{\Sigma_t})$, then we have (see [1, Appendix])

$$A'(t) = \int_\Sigma H_t \xi_t \ vol_{\varphi_t^*(g)}.$$

Then,

$$A''(0) = \int_\Sigma J(\xi_0)\xi_0 \ vol_{\varphi_0^*(g)}.$$

Also,

$$V'(t) = \int_\Sigma \xi_t \ vol_{\varphi_t^*(g)}.$$

Definition 5.1. A free boundary CMC immersion $\varphi : \Sigma \to M$ is said to be *stable* if only if $A''(0)(f) \geq 0$, for all $f \in C_\partial^{j,\alpha}(\Sigma)$ such that $\int_\Sigma f vol_{\varphi_t^*(g)} = 0$. When φ is not stable, it is said to be unstable.

For the proof of our stability criteria we need the following Smale's Lemma.

Lemma 5.2. [18, lemma 4][Smale Lemma]

a) Let $\mathcal{H}$ be the prehilbert space $\mathcal{H}_0^k(E)$ with the inner product $\langle , \rangle_L$ and B_L quadratic form on $\mathcal{H}$, $B_L(h) = B_L(h, h)$. Then B_L has a minimum λ_1 at f_1 on the unit sphere S of $\mathcal{H}$ and on $\{f_1, ..., f_{q-1}\}^\perp \cap S$ has its minimum value λ_q and f_q.

b) Let $d(v)$ be the minimum of B_L on $V^\perp \cap S$, where V is a finite dimensional subspace of $\mathcal{H}$. Then

$$\lambda_n = max_{dim V < n} d(V).$$

***Remark* 5.3.** As stated in remark 4.1, by Smale's Lemma 1 (Smale, [[18, lemma 1]), there exists an orthonormal basis $B = \{f_i\}$ for $L^2(\Sigma)$, with each f_i associated to the respective eigenvalue λ_i of J. Furthermore, these eigenvalues satisfy the inequality $\lambda_1 < \lambda_2 \leq ...$, with $\lambda_i \to \infty$ as $i \to \infty$. Now, in Lemma 5.2 (which corresponds to Smale's Lemma 4, ([18, lemma 4][Smale Lemma])), the way to generate these proper values of J is described. In this case, we defined the quadratic form

$$I(f) = \int_\Sigma J(f) f \ vol_{\varphi_0^*(g)} \tag{12}$$

and this is applied to functions of norm 1.

Thus, we can write our first stability criterion in the following way:

Theorem 5.4. *Let $\varphi_0 : \Sigma \to M$ be a free boundary CMC immersion. Let λ_i, $i \geq 1$ be the eigenvalues J_{φ_0}.*

(1) If $\lambda_1 \geq 0$, then φ_0 is stable.

(2) If $\lambda_1 < 0 < \lambda_2$, then there is a unique function $\kappa \in C^{j,\alpha}(\Sigma)$ such that $J(\kappa) = 1$ and we have that:

(2-a) If $\int_\Sigma \kappa \ vol_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.

(2-b) If $\int_\Sigma \kappa \ vol_{\varphi_0^*(g)} > 0$, then φ_0 is unstable.

(3) If $\lambda_1 < 0 = \lambda_2$, then we have:

(3-a) If there exist a λ_2-eigenfunction f_2 such that $\int_\Sigma f_2 \ vol_{\varphi_0^*(g)} \neq 0$, then φ_0 is unstable.

(3-b) If $\int_\Sigma h_2 \ vol_{\varphi_0^*(g)} = 0$ for all λ_2-eigenfunction h_2, then there exist a unique function $\bar{h}_2 \in (ker(J_{\varphi_0}))^\perp$ such that $J(\bar{h}_2) = 1$ and

(3-b-i) If $\int_\Sigma \bar{h}_2 \ vol_{\varphi_0^*(g)} \leq 0$, then φ_0 is stable.

(3-b-ii) If $\int_\Sigma \bar{h}_2 \ vol_{\varphi_0^*(g)} > 0$, then φ_0 is unstable

(4) If $\lambda_2 < 0$, then φ_0 is unstable.

Proof. By Lemma 5.2 we have that:

$$\lambda_1 = I(f_1) = \int_\Sigma J(f_1) f_1 \ vol_{\varphi_0^*(g)} = \min\Big\{ I(f) : f \in C^{j,\alpha}_\partial(\Sigma) \text{ and } \int_\Sigma f^2 \ vol_{\varphi_0^*(g)} = 1 \Big\}, \tag{13}$$

$$\lambda_i = I(f_i) = \int_\Sigma J(f_i) f_i \ vol_{\varphi_0^*(g)} = \min\Big\{ I(f) : f \in C^{j,\alpha}_\partial(\Sigma), \int_\Sigma f^2 \ vol_{\varphi_0^*(g)} = 1, \text{ and } \int_\Sigma f f_k \ vol_{\varphi_0^*(g)} = 0, \text{ for} k \in \{1, ..., i-1\} \Big\}, \quad i = 2, 3, ... \tag{14}$$

So, if $\lambda_1 \geq 0$ we have (I).

Now we assume that $\lambda_1 < 0$. We know that f_1 does not change sign (see Remark 4.1), then

$$\int_\Sigma f_1 \ vol_{\varphi_0^*(g)} \neq 0.$$

For $\kappa \in C_\partial^{j+1,\alpha}(\Sigma)$, let

$$a = -\frac{\int_\Sigma \kappa \ vol_{\varphi_0^*(g)}}{\int_\Sigma f_1 \ vol_{\varphi_0^*(g)}}, \ \text{ and } \ \xi = af_1 + \kappa.$$

Then,

$$\begin{aligned}
\int_\Sigma \xi \ vol_{\varphi_0^*(g)} &= \int_\Sigma af_1 + \kappa \ vol_{\varphi_0^*(g)} \\
&= \int_\Sigma -\frac{\int_\Sigma \kappa \ vol_{\varphi_0^*(g)} f_1 + \kappa \int_\Sigma f_1 \ vol_{\varphi_0^*(g)}}{\int_\Sigma f_1 \ vol_{\varphi_0^*(g)}} \\
&= \frac{1}{\int_\Sigma f_1 \ vol_{\varphi_0^*(g)}} \Big[-\int_\Sigma \kappa \ vol_{\varphi_0^*(g)} \int_\Sigma f_1 \ vol_{\varphi_0^*(g)} \\
&\qquad + \int_\Sigma \kappa \ vol_{\varphi_0^*(g)} \int_\Sigma f_1 \ vol_{\varphi_0^*(g)} \Big] \\
&= 0
\end{aligned}$$

By Lemma 3.3-(a) (with $\lambda = 0$ and $h = 1$) there exists $\kappa \in C_\partial^{j+1,\alpha}(\Sigma)$ such that $J(\kappa) = 1$. Using the symmetry of J we have:

$$\begin{aligned}
I(\xi) &= \int_\Sigma \xi J(\xi) \ vol_{\varphi_0^*(g)} = \int_\Sigma (af_1 + \kappa) J(af_1 + \kappa) \ vol_{\varphi_0^*(g)} \\
&= \int_\Sigma \big[a^2 f_1 J(f_1) + af_1 J(\kappa) + a\kappa J(f_1) + \kappa J(\kappa)\big] \ vol_{\varphi_0^*(g)} \\
&= a^2 \lambda_1 \int_\Sigma f_1^2 \ vol_{\varphi_0^*(g)} + 2a \int_\Sigma f_1 J(\kappa) \ vol_{\varphi_0^*(g)} \\
&\quad + a \int_\Sigma \big[\kappa J(f_1) - f_1 J(\kappa)\big] \ vol_{\varphi_0^*(g)} + \int_\Sigma \kappa \ vol_{\varphi_0^*(g)} \\
&= a^2 \lambda_1 + 2a \int_\Sigma f_1 J(\kappa) \ vol_{\varphi_0^*(g)} + \int_\Sigma \kappa \ vol_{\varphi_0^*(g)} \\
&= a^2 \lambda_1 - \int_\Sigma \kappa \ vol_{\varphi_0^*(g)}.
\end{aligned}$$

Since $\lambda_1 < 0$, then $I(\xi) < 0$ if $\int_\Sigma \kappa \ vol_{\varphi_0^*(g)} > 0$. Thus, (2-b) is satisfied.

Now, if $\kappa = f_2$, $f_2 \in E_{\lambda_2}$, then

$$\begin{aligned}
I(\xi) &= a^2 \lambda_1 \int_\Sigma f_1^2 \ vol_{\varphi_0^*(g)} + 2a \int_\Sigma f_1 J(f_2) \ vol_{\varphi_0^*(g)} \\
&\quad + a \int_\Sigma \big[f_2 J(f_1) - f_1 J(f_2)\big] \ vol_{\varphi_0^*(g)} + \int_\Sigma f_2 J(f_2) \ vol_{\varphi_0^*(g)} \\
&= a^2 \lambda_1 \int_\Sigma f_1^2 \ vol_{\varphi_0^*(g)} + 2a\lambda_2 \int_\Sigma f_1 f_2 \ vol_{\varphi_0^*(g)} + \lambda_2 \int_\Sigma f_2^2 \ vol_{\varphi_0^*(g)} \\
&= \lambda_1 \frac{(\int_\Sigma f_2 \ vol_{\varphi_0^*(g)})^2}{(\int_\Sigma f_1 \ vol_{\varphi_0^*(g)})^2} + \lambda_2,
\end{aligned}$$

therefore if $\int_\Sigma f_2\ vol_{\varphi_0^*(g)} \neq 0$ and $\lambda_2 = 0$, then $I(\xi) < 0$, so we have (3-a), e.i., φ_0 is unstable. If $\lambda_2 < 0$, (4) is fulfilled.

To proof (2-a), we define:

$$E_1 = \{af_1 : a \in \mathbb{R}\}, \qquad E_1^\perp = \Big\{ f \in C_\partial^{j+1,\alpha}(\Sigma) : \int_\Sigma f_1 f\ vol_{\varphi_0^*(g)} = 0 \Big\}.$$

Again by Lemma 3.3 -(a), there is a function $\kappa \in C_\partial^{j+1,\alpha}(\Sigma)$ such that $J(\kappa) = 1$. If $\int_\Sigma \kappa\ vol_{\varphi_0^*(g)} \leq 0$, then

$$I(\kappa) = \int_\Sigma \kappa J(\kappa)\ vol_{\varphi_0^*(g)} = \int_\Sigma \kappa\ vol_{\varphi_0^*(g)} \leq 0.$$

As

$$\lambda_2 = I(f_2) = min\Big\{ I(f) : f \in C_\partial^{j,\alpha}(\Sigma) \cap E_1^\perp, \int_\Sigma f^2\ vol_{\varphi_0^*(g)} = 1 \Big\}$$

and $\lambda_2 > 0$, we have that $\kappa \notin E_1^\perp$. Therefore, for any $\xi \in C_\partial^{j,\alpha}(\Sigma)$, where $\int_\Sigma \xi\ vol_{\varphi_0^*(g)} = 0$, we can write $\xi = b\kappa + w$, $b \in \mathbb{R}$, $w \in E_1^\perp$. So,

$$\begin{aligned}
I(\xi) &= \int_\Sigma \xi J(\xi)\ vol_{\varphi_0^*(g)} = \int_\Sigma (b\kappa + w)[bJ(\kappa) + J(w)]\ vol_{\varphi_0^*(g)} \\
&= \int_\Sigma \big[b^2 \kappa J(\kappa) + bwJ(\kappa) + b\kappa J(w) + wJ(w)\big]\ vol_{\varphi_0^*(g)} \\
&= b^2 \int_\Sigma \kappa\ vol_{\varphi_0^*(g)} + 2b \int_\Sigma wJ(\kappa)\ vol_{\varphi_0^*(g)} \\
&+ b \int_\Sigma \kappa J(w) - wJ(\kappa)\ vol_{\varphi_0^*(g)} + I(w) \\
&= b^2 \int_\Sigma \kappa\ vol_{\varphi_0^*(g)} + 2b \int_\Sigma w\ vol_{\varphi_0^*(g)} + I(w) \\
&= -b^2 \int_\Sigma \kappa J(\kappa)\ vol_{\varphi_0^*(g)} + 2b \int_\Sigma (b\kappa + w)\ vol_{\varphi_0^*(g)} + I(w) \\
&= -b^2 I(\kappa) + I(w) \\
&\geq 0.
\end{aligned}$$

Thus, φ_0 is stable.

Now, under the hypotheses of (3-b), $\lambda_2 = 0$ and $\int_\Sigma h_2\ vol_{\varphi_0^*(g)} = 0$ for all λ_2-eigenfunctions h_2. Let E_0 be the eigenspace associated with $\lambda_2 = 0$, then $E = \mathrm{Ker}(J) = E_0$. By Lemma 3.3-(b) (with $h = 1$ and $\lambda = 0$) there exist a unique function $\bar{h}_2 \in E^\perp \cap C_\partial^{j+1,\alpha}(\Sigma)$ such that $J(\bar{h}_2) = 1$. So,

(i) If $\int_\Sigma \bar{h}_2\ vol_{\varphi_0^*(g)} = 0$,

$$I(\bar{h}_2) = \int_\Sigma \bar{h}_2 J(\bar{h}_2)\ vol_{\varphi_0^*(g)} = \int_\Sigma \bar{h}_2\ vol_{\varphi_0^*(g)} = 0 = \lambda_2.$$

Assuming that $\bar{h}_2 \in E_1^{\perp}$ and since

$$\lambda_2 = I(f_2) = min\Big\{I(f) : f \in C^{j,\alpha}_{\partial}(\Sigma) \cap E_1^{\perp}, \int_{\Sigma} f^2 \ vol_{\varphi_0^*(g)} = 1\Big\},$$

then $\bar{h}_2$ is a λ_2-function, e.i., $J(\bar{h}_2) = 0$. This is a contradiction. Therefore, $\bar{h}_2 \notin E_1^{\perp}$, So, stability is proved as in (2-a). Similary, if $\int_{\Sigma} \bar{h}_2 \ vol_{\varphi_0^*(g)} < 0$, stability is also proved as in (2-a).

(ii) If $\int_{\Sigma} \bar{h}_2 \ vol_{\varphi_0^*(g)} > 0$, then the proof is the same as for (2-b). Thus, φ_0 is unstable.

☑

The following proposition shows that when the operator J_{φ_0} has an eigenvalue equal to zero, this is equivalent to the fact that the function of mean curvatures H_t, for a free boundary CMC immersions variation, has a critical point at $t = 0$.

Proposition 5.5. If $\{\varphi_t\}_{-\epsilon<t<\epsilon}$, is a perturbation of φ_0, such that $\varphi_t : M \to \Sigma$ is a $C^{j,\alpha}$ free boundary CMC immersion, and $\frac{dH_t}{dt}|_{t=0} = H'_0 = 0$, where H_t is the constant mean curvature of φ_t, then some eigenvalue λ_i of J_{φ_0} must vanish.

Conversely, if some eigenvalue λ_i of J_{φ_0} vanishes, and some λ_j-eigenfuntion f_i satisfies $\int_{\Sigma} f_i \ vol_{\varphi_0^*(g)} \neq 0$, then for every free boundary CMC perturbation φ_t of φ_0 we have $H'_0 = 0$.

Proof. Let $\xi_0 = g(\frac{\partial \varphi_t}{\partial t}|_{t=0}, \vec{n}_{\Sigma_0})$. We have that $J(\xi_0) := J_{\varphi_0}(\xi_0) = H'_0$. If $H'_0 = 0$, then $\xi_0 \in \text{Ker}(J)$, $\xi_0 \neq 0$. Thus, there is some eigenvalue $\lambda_i = 0$.

Conversely, if $\lambda_i = 0$ and f_i is a λ_i-eigenfunction such that $\int_{\Sigma} f_i \ vol_{\varphi_0^*(g)} = \langle 1, f_i \rangle_{L^2(\Sigma)} \neq 0$, then $1 \notin (\text{Ker}(J))^{\perp}$. Therefore, the only constant function in the image of J must be 0. Given any free boundary CMC perturbation φ_t of φ_0 and $\xi_0 = g(\frac{\partial \varphi_t}{\partial t}|_{t=0}, \vec{n}_{\Sigma_o})$, as $J(\xi_0) = H'_0$ is a constant function in $\text{Im}(J)$ we must have $H'_0 = 0$. ☑

The second stability criteria is given in the following corollary. It's possible to check the positivity of the first eigenvalue or the negativity of the second eigenvalue in Theorem 5.4. However, for cases (2) and (3) the calculation of the eigenvalues can be quite difficult[2]. In this sense, the following criterion facilitates the understanding of the geometric meaning for these cases and it is based on the existence of families of 1-parameter deformations.

[2]The difficulty in calculating these eigenvalues starts from the very definition of the Jacobi operator that involves the calculation of the Laplacian of f and the Hilbert-Schmidt norm of the second fundamental form of Σ_0 that is defined from the Levi-Civita connection. In addition, the calculation of the integrals can be uncertain.

Corollary 5.6. Let $\varphi : \Sigma \to M$ a free boundary CMC immersion of class $C^{j+1,\alpha}$. We assume that $\lambda_1 < 0 \leq \lambda_2$. If there exists a deformation φ_t of φ, $-\epsilon < t < \epsilon$, with $\varphi_0 = \varphi$, such that φ_t is a free boundary CMC immersion of class $C^{j,\alpha}$ for all $t \in (-\epsilon, \epsilon)$, and such that $\frac{dH_t}{dt}|_{t=0} = H_0' = constant \neq 0$, where H_t is the constant mean curvature of φ_t and V_t is the volume of φ_t, then we have that:

(1) If $H_0'V_0' \leq 0$, then φ is stable;

(2) if $H_0'V_0' > 0$, then φ is unstable.

If there is no such deformation, then φ is unstable.

Proof. In the case where $\lambda_1 < 0 < \lambda_2$, by Theorem 4.2 there exists a strictly monotonous deformation $\{\varphi_t\}_{-\epsilon<t<\epsilon}$, with $H_t = t$. So, $H_t' = 1$, in particular $H_0' = 1$. We have that $H_0' = J(\xi_0)$, where $\xi_0 = g(\frac{d\varphi_t}{dt}|_{t=0}, \vec{n}_\Sigma)$ and $V_0' = \int_\Sigma \xi_0 \ vol_{\varphi_0^*(g)}$. That way, the conditions of the Theorem 5.4-(2) is fulfilled and then we have (1) and (2).

Now, if $\lambda_1 < 0 = \lambda_2$, one of the implications of Proposition 5.5 says that: If $\lambda_i = 0$, for some $i = 1, 2, ...$, and, for some $f_i \in E_{\lambda_i}$, $\int_\Sigma f_i \ vol_{\varphi_0^*(g)} \neq 0$, then, for every perturbation $\{\varphi_t\}_{t\in I}$ of φ_0 we have that $H_0' = 0$. This is equivalent to: If $\lambda_i = 0$, for some $i = 1, 2, ...$, and there is a perturbation $\{\varphi_t\}$ of φ_0 such that $H_0' \neq 0$, then, for all $f_i \in E_{\lambda_i}$, $\int_\Sigma f_i \ vol_{\varphi_0^*(g)} = 0$. Therefore, since $H_0' = c \neq 0$, c constant, then the conditions for Theorem 5.4-(3-b) are met . So, let's take $h = \frac{1}{c}\xi_0$, we have

$$J(h) = \frac{1}{c}J(\xi_0) = \frac{1}{c}H_0' = 1.$$

Now, $J(h) = 1$ if only if $h = u + h_0$, where $u \in E^\perp \cap C_\partial^{j+1,\alpha}(\Sigma)$ is the only one such that $J(u) = 1$, (by Lemma 3.3-(b)), and $h_0 \in E = E_{\lambda_2}$. So, if we take $\bar{h}_2 = h - h_0$, and as $\int_\Sigma h_0 \ vol_{\varphi_0^*(g)} = 0$, we have

$$\begin{aligned} H_0'V_0' =& H_0' \int_\Sigma \xi_0 \ vol_{\varphi_0^*(g)} = c \int_\Sigma \xi_0 \ vol_{\varphi_0^*(g)} \\ =& c^2 \int_\Sigma \frac{1}{c}\xi_0 \ vol_{\varphi_0^*(g)} - c^2 \int_\Sigma h_0 \ vol_{\varphi_0^*(g)} \\ =& c^2 \int_\Sigma (\frac{1}{c}\xi_0 - h_0) \ vol_{\varphi_0^*(g)} \\ =& c^2 \int_\Sigma \bar{h}_2 \ vol_{\varphi_0^*(g)}. \end{aligned}$$

Thus, (1) and (2) are equivalent to (i) and (ii) in the part (3).b) of Theorem 5.4. ☑

6. Example

Among the best known examples of surfaces with CMC are the so called Delaunay surfaces, which are surfaces of revolution. Apart from the elementary cases of plane, spheres and cylinders, there are three classes of Delaunay surfaces, the catenoids, the unduloids and the nodoids, corresponding to the choice of a conic as a parabola, an ellipse or a hyperbola, respectively. When a curve rolls, without slipping, on a fixed curve, each point on the moving curve traces another curve known as a roulette. The generating curves of the Delaunay surfaces are the roulettes generated by the focus of the parabola, ellipse, and hyperbola when rolled on a straight line. The roulette associated with the parabola is the catenary, for the ellipse it is called undulary and that of the hyperbola is called nodary (see figures 2, 3 and 4).

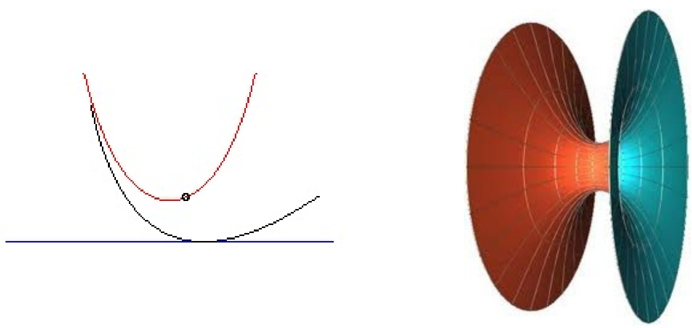

Figure 2. Roulette-parabola (catenary) and catenoid

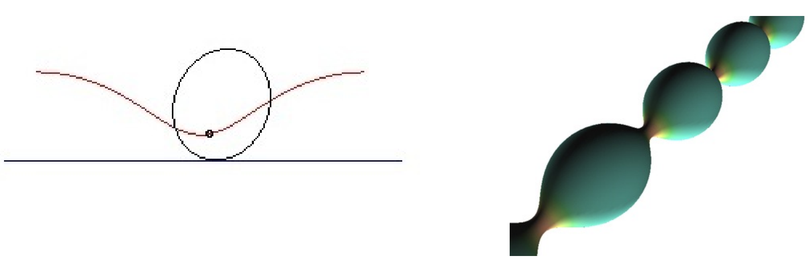

Figure 3. Roulette-ellipse (undulary) and unduloid

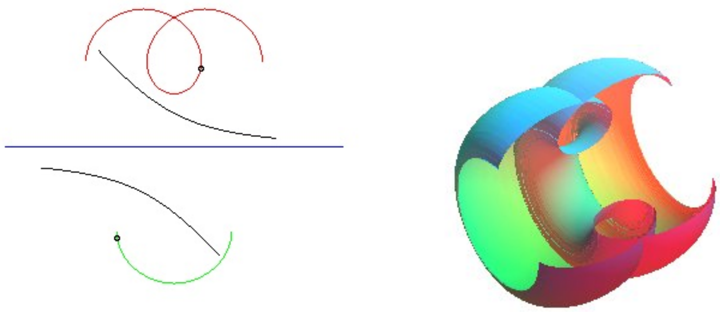

FIGURE 4. Roulette-hyperbola (nodary) and nodoid

Let us consider connected compact subsets of Delaunay surfaces, such that for each of which its boundary, if not empty, is one or two circles perpendicular to the axis of rotation (see figure 5).

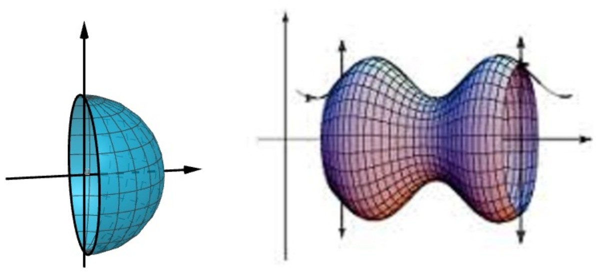

FIGURE 5. Connected compact subsets of Delaunay surfaces

Such subsets can be constrained to be free-boundary CMC surfaces within a ball in $\mathbb{R}^3$, with their boundaries within the boundary of the ball. For example, except for the plane, the catenoid is the only minimal surface between Delaunay surfaces, obtained by rotating the catenary

$$z = c\cosh(\frac{x}{c}); \quad c > 0 \tag{15}$$

around the x axis. Let φ_0 be the parametrization of the catenoid portion generated by the catenary such that $-a \leq x \leq a$, with $a = c\cosh(\frac{a}{c})$. That is a free boundary minimal surface inside the ball of radius $r = \sqrt{a^2 + c^2\cosh^2(\frac{a}{c})}$ (see figure 6).

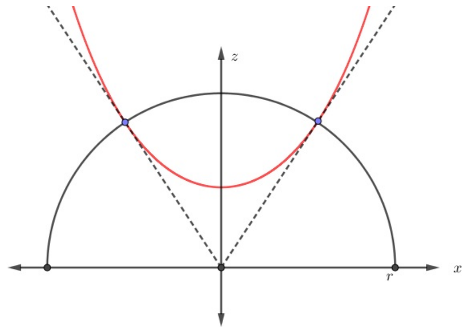

FIGURE 6. The catenary and the circle are orthogonal

We have that if the area of the image of the Gaussian function on a minimal surface is less than 2π, then the first eigenvalue of the Jacobi operator on the surface is positive, $\lambda_1 > 0$ (see [3]). So, if we take the portion of the catenoid small enough, for example by choosing $c > 0$ such that $r = 1$, we have that the eigenvalue of its Jacobi operator is positive, $\lambda_1 > 0$. Therefore, by Theorem 4.2, we have that there exists a free boundary CMC family of surfaces $\{\varphi_{\zeta(H)}\}_{H\in(-\epsilon,\epsilon)}$, $\zeta(0) = 0$, where $\zeta(H))$ is an injective map and H is a mean curvature of $\varphi_{\zeta(H)}$. Each $\varphi_{\zeta(H)}$ is a surface of revolution. By the injectivity of ζ, for $H > 0$ we have to $\varphi_{\zeta(H)}$ is part of an unduloid and for $H < 0$ we have to $\varphi_{\zeta(H)}$ it is part of a nodoid (see [4]). Furthermore, by part (1) of Theorem 5.4, φ_0 is stable.

Now, for the case of a portion of an unduloid, if we consider the boundary circles as the maximum (or minimum) circles of the unduloid, we obtain a surface orthogonal to orthogonal planes to the axis of rotation (see figure 7). Among all the possible surfaces with this characteristic, the largest stable piece is the one that corresponds exactly to a complete period of the generating curve (see [12]). In this case $\lambda_1 < 0$ and $\lambda_2 = 0$. If the limit of the surface are a maximun circle and a consecutive minimum circle, then $\lambda_1 = 0$.

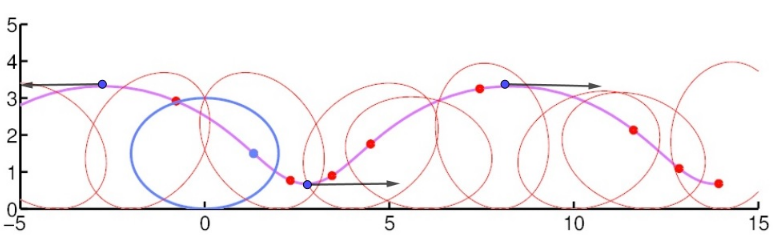

FIGURE 7. The vectors tangent to the curve at the maximum and minimum points are parallel to the axis of rotation. Figure taken from the website https://roughan.info/math/roulette/

7. Appendix A

In this appendix we will give some definitions and important results of Functional Analysis, which were used in this paper.

Definition 7.1. If $f : N \to M$ is a smooth map and $S \subset M$ is an embedded submanifold, we say that f is *transverse* to S if, for every $p \in f^{-1}(S)$, $T_{f(p)}M = T_{f(p)}S + df_p(T_pN)$.

Definition 7.2. The definition of transversality between a map $F : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \subset \mathcal{Y}$ a smooth submanifold, where $\mathcal{X}$ and $\mathcal{Y}$ are Banach manifolds, is that presented in the Definition 7.1 but with the additional assumption that $dF^{-1}_{\mathfrak{x}_0}(T_{F(\mathfrak{x}_0)}\mathcal{Z})$ is a complemented subspace of $T_{\mathfrak{x}_0}\mathcal{X}$, i.e., there is a subspace $\mathcal{V} \subset T_{\mathfrak{x}_0}\mathcal{X}$ is such that $T_{\mathfrak{x}_0}\mathcal{X} = dF^{-1}_{\mathfrak{x}_0}(T_{F(\mathfrak{x}_0)}\mathcal{Z}) \oplus \mathcal{V}$.

Definition 7.3. If A is a bounded operator in a Hilbert space H and $\{e_i : i \in I\}$ is an orthonormal bases for H, the Hilbert-Schmidt norm of A is defined as $||A||^2_{HS} = Tr(A^*A) = \sum_{i\in I} ||Ae_i||^2_H$, where $|| \cdot ||_H$ is the norm of H.

Definition 7.4. The Hölder space $C^{j,\alpha}(\Omega)$, where Ω is an open subset of some Euclidean space and $j \geq 0$ an integer, consists of those functions on Ω having continuous derivatives up to order j and such that the j-th partial derivatives are Hölder continuous with exponent α, where $0 < \alpha \leq 1$. A real valued function f on n-dimensional Euclidean space is Hölder continuous, when there are nonnegative real constants c, such that

$$|f(x) - f(y)| \leq c||x - y||^\alpha$$

Definition 7.5. A linear continuous operator $T : E \to F$ between normed spaces is Fredholm if Ker T is finite dimensional and Im T is close and finite codimensional, the index of T is ind T = dim Ker $T -$ dim coker T. A Fredholm map is a C^1 map $f : M \to N$, where M and N are differentiable Banach manifolds, such that for each $x \in M$, the derivative $df_x : T_x(M) \to T_{f(x)}(N)$ is a Fredholm operator. The index of f is defined to be the index of df_x for some x. The definition doesn't depend on x, see [10].

Theorem 7.6. [14, *The Inverse Mapping Theorem, 5.2*] *Let E, F be Banach spaces, U an open subset of E, and let $f : U \to F$ be a C^p-morphism with $p \geq 1$. Assume that for some point $x_0 \in U$ the derivative $f'(x_0) : E \to F$ is a toplinear isomorphism. Then f is a local C^p-isomorphism at x_0.*

Theorem 7.7. [14, *The Implicit Mapping Theorem, 5.9*] *Let U, V be open sets in Banach Spaces* $\mathbf{E}$, $\mathbf{F}$ *respectively, and let $f : U \times V \to G$ be a C^p mapping. Let $(a, b) \in U \times V$, and asuume that $D_2f(a, b) : \mathbf{F} \to G$ is a isomorphism. Let $f(a, b) = 0$. Then there exist a continuous map $g : U_0 \to V$ defined on an open neighborhood U_0 of a such that $g(a) = 0$ and such that $f(x, g(x)) = 0$ for all $x \in U_0$. If U_0 is taken to be a sufficiently small ball, then g is uniquely determined and is of class C^p.*

References

[1] L. C. Ambrozio, *Rigidity of area-minimizing free boundary surfaces in mean convex threemanifolds*, The Journal of Geometric Analysis **25** (2015), no. 2, 1001–1017.

[2] A. Ancona, B. Helffer, and T. Hoffmann-Ostenhof, *Nodal domain theoremsa la courant*, Documenta Mathematica **9** (2004), 283–299.

[3] J. L. Barbosa and M. do Carmo, *Stability of hypersurfaces with constant mean curvature*, athematische Zeitschrift Springer-Verlag **185** (1984), no. 3, 339–353.

[4] E. Bendito, M. J. Bowick, and A. Medina, *Delaunay Surfaces*, https://arxiv.org/abs/1305.5681. 2013.

[5] R. Bettiol and R. Giambò, *Genericity of nondegenerate geodesics with general boundary conditions*, Topological Methods in Nonlinear Analysis **35**, no. 2.

[6] R. Bettiol, P. Piccione, and B. Santoro, *Deformations of free boundary cmc hypersurfaces*, The Journal of Geometric Analysis **27** (2017), no. 4, 3254–3284.

[7] L. Biliotti, M. A. Javaloyes, and P. Piccione, *Genericity of nondegenerate critical points and Morse geodesic functional*, Indiana University Mathematics Journal (2009), 1797–1830.

[8] C. W. Rodríguez C., *Genericity of Nondegenerate Free Boundary CMC Embeddings*, Mediterranea Journal Mathematics **17** (2020), 17:188.

[9] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1983).

[10] Gohberg and Krein, *The basic propositions on defect numbers and indices of linear operators*, Transactions of the American Mathematical Society **13** (1960), 185–264.

[11] M. Koiso, *Deformation and stability of surfaces with constant mean curvature*, Tohoku Mathematical Journal, Second Series **54** (2002), no. 1, 145–159.

[12] N. Koiso, *Variational Problems*, Kyoritsu Publ., Tokyo, Japan, 1998.

[13] O. A. Ladyzhenskaya and N. U. Tseva, *linear and quasilinear elliptic equations*, Translated from the russian by scripta technica, inc. translation editor: Leon ehrenpreis, 1968.

[14] S. Lang, *Differential and riemannian manifolds*, vol. 160 of. Graduate Texts in Mathematics **185** (1995), 71–72.

[15] D. Maximo, I. Nunes, and G. Smith, *Free boundary minimal annuli in convex three-manifolds*, Journal of Differential Geometry **106** (2017), no. 1, 139–186.

[16] A. Ros and E. Vergasta, *Stability for hypersurfaces of constant mean curvature with free boundary*, Geometriae Dedicata **56** (1995), no. 1, 19–33.

[17] F. Serrano, *Whitney topology and normality*, Topology and its Applications, Elsevier **52** (1993), no. 1, 59–67.

[18] S. Smale, *On the morse index theorem*, Journal of Mathematics and Mechanics **14** (1965), no. 6, 1049–1055.

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