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Some inequalities of the Hermite-Hadamard type for two kinds of convex functions

Algunas desigualdades del tipo Hermite-Hadamard para dos tipos de funciones convexas

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Abstract. In this paper, we obtain new inequalities of the Hermite-Hadamard type, in two different classes of convex dominated functions. Several known results from the literature are obtained as particular cases of our more general perspective.

Key words and phrases. Hermite-Hadamard inequality, generalized fractional integral.

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Resumen. En este artículo, obtenemos nuevas desigualdades del tipo Hermite-Hadamard, en dos clases diferentes de funciones convexas dominadas. Varios resultados conocidos de la literatura se obtienen como casos particulares de nuestra perspectiva más general.

Palabras y frases clave. Desigualdad de Hermite-Hadamard, integral fraccionaria generalizada.

1. Introduction

A very usefull idea in mathematics is that of the convex function, which we will define below. Throughout this work, I will be denote an interval in $\mathbb{R}$.

Definition 1.1. A function $f : I \to \mathbb{R}$ is said to be **convex** on the interval $I \subset \mathbb{R}$, if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for $x, y \in I$ and $t \in [0,1]$. On the other hand, we say that f is **concave** if $-f$ is convex.

The consequent extensions of this concept, which have appeared lately, have transformed it into an extremely complex notion. For background, we suggest the reader to check the paper [20].

Definition 1.2. ([2]) Let $g : I \to \mathbb{R}$ be a convex function on the interval $I \subset \mathbb{R}$. We say the function $f : I \to \mathbb{R}$ is g**-convex dominated on** I if

$$|tf(x) + (1-t)f(y) - f(tx + (1-t)y)| \leq tg(x) + (1-t)g(y) - g(tx + (1-t)y),$$

for any $x, y \in I$ and $t \in [0,1]$.

In this work we will consider the following classes of convex functions.

Definition 1.3. ([10]) We say that a function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ belongs to the class $Q(I)$ if φ is nonnegative and

$$\varphi(tu + (1-t)v) \leq \frac{\varphi(u)}{t} + \frac{\varphi(v)}{1-t}, \quad \text{for any } u, v \in I \text{ and } t \in (0,1). \tag{1}$$

Definition 1.4. ([4]) We say that a function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ belongs to the class $P(I)$ if φ is nonnegative and

$$\varphi(tu + (1-t)v) \leq \varphi(u) + \varphi(v), \quad \text{for any } u, v \in I \text{ and } t \in (0,1).$$

In this case, we also say that φ is a P**-function**.

Definition 1.5. ([22]) Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function in $Q(I)$. We say that a function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is $(\psi, Q(I))$**-convex dominated on** I if

$$\left|\frac{\varphi(u)}{t} + \frac{\varphi(v)}{1-t} - \varphi(tu + (1-t)v)\right| \leq \frac{\psi(u)}{t} + \frac{\psi(v)}{1-t} - \psi(tu + (1-t)v), \tag{2}$$

for any $u, v \in I$ and $t \in (0,1)$.

Definition 1.6. ([22]) Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function in $P(I)$. We say that a function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is $(\psi, P(I))$**-convex dominated on** I if

$$\big|[\varphi(u) + \varphi(v)] - \varphi(tu + (1-t)v)\big| \leq [\psi(u) + \psi(v)] - \psi(tu + (1-t)v), \tag{3}$$

for any $u, v \in I$ and $t \in (0,1)$.

Integral inequalities are an active area of study in mathematics, with relevance in both pure and applied contexts. One fundamental inequality in this domain is the *Hermite-Hadamard inequality*, which provides straightforward bounds for the average value of convex functions under integration. You can find more details in references like [3, 11, 12].

For any integrable convex function $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ and any $a_1, a_2 \in I$ with $a_1 < a_2$, the inequality

$$\varphi\Big(\frac{a_1+a_2}{2}\Big) \leq \frac{1}{a_2-a_1}\int_{a_1}^{a_2} \varphi(u)du \leq \frac{\varphi(a_1)+\varphi(a_2)}{2} \tag{4}$$

always holds. Since its discovery, this inequality has received considerable attention. Some extensions and generalizations of (4), with different fractional and generalized operators and using different convexity operators, can be consulted in [5, 1, 7, 8, 16, 18, 19, 20, 21].

Fractional calculus, which involves derivatives and integrals of non-integer order, has garnered increasing attention over the past 40 years. New operators have been defined, leading to multiple applications (see [3], [26], [17], [28]). In particular, novel integral operators have emerged as natural extensions of the classical fractional Riemann-Liouville integral. In [9], the authors introduce a generalized operator that encompasses several well-known fractional integral operators as specific cases.

Definition 1.7. Let $\varphi : [0, \infty) \to \mathbb{R}$ be an integrable function. The left and right-sided k-generalized fractional Riemann-Liouville integrals of order $\alpha \in \mathbb{R}$ and $s \neq -1$ of φ on $[0, \infty)$, are defined as follows

$${}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{a_1}^{u} \frac{F(\tau,s)\varphi(\tau)d\tau}{[\mathbb{F}_s(u,\tau)]^{1-\frac{\alpha}{k}}}, \quad {}^{s}J^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(u) = \frac{1}{k\Gamma_k(\alpha)}\int_{u}^{a_2} \frac{F(\tau,s)\varphi(\tau)d\tau}{[\mathbb{F}_s(\tau,u)]^{1-\frac{\alpha}{k}}}, \tag{5}$$

where F is an integrable function on $[0, +\infty)$ with $F(\tau, 0) = 1$, and, furthermore,

$$\mathbb{F}_s(u,\tau) = \int_{\tau}^{u} F(\theta,s)d\theta \quad \text{and} \quad \mathbb{F}_s(\tau,u) = \int_{u}^{\tau} F(\theta,s)d\theta.$$

Clearly

$$\mathbb{F}_s(z,y) = -\mathbb{F}_s(y,z).$$

In (5), Γ and Γ_k, for $k > 0$, denote the Gamma and k-Gamma functions, respectively defined as follows (see [3, 23, 24, 25, 28, 29]):

$$\Gamma(z) = \int_0^\infty \tau^{z-1}e^{-\tau}\,\mathrm{d}\tau, \text{ with } \Re(z) > 0, \quad \text{and} \quad \Gamma_k(z) = \int_0^\infty \tau^{z-1}e^{-\tau^k/k}\,\mathrm{d}\tau.$$

Notice that

$$\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right) \quad \text{and} \quad \Gamma_k(z+k) = z\Gamma_k(z).$$

We have that, if $k \to 1$, then

$$\Gamma_k(z) \to \Gamma(z).$$

We also define the k-beta function as follows:

$$B_k(u,v) = \frac{1}{k}\int_0^1 \tau^{\frac{u}{k}-1}(1-\tau)^{\frac{v}{k}-1}d\tau.$$

Notice that

$$B_k(u,v) = \frac{1}{k}B\left(\frac{u}{k},\frac{v}{k}\right) \quad \text{and} \quad B_k(u,v) = \frac{\Gamma_k(u)\Gamma_k(v)}{\Gamma_k(u+v)}.$$

***Remark* 1.8.** The integral operators in Definition 1.7 contain many well-known integral operators, for example ([8]):

i) If $F \equiv 1$ and $\alpha = k$, we obtain the classic Riemann integral.

ii) If $F \equiv 1$ and $k = 1$, we have the Riemann-Liouville fractional integral.

iii) If $F \equiv 1$ and $\alpha \neq k$, we have the k-Riemann-Liouville fractional integral of [17].

iv) If $F(s,t) = t^s$ and $k = 1$, we obtain the fractional integral of Katugampola ([14]).

v) If $F(s,t) = t^s$ and $\alpha \neq k$, from Definition 1.7 we derive the (k,s)-integral of [26].

The primary objective of this paper is to utilize the generalized fractional integral operator of the Riemann-Liouville type, as defined in Definition 1.7, to establish a variety of Hermite-Hadamard type integral inequalities. Given the generality of our class of operators, we will demonstrate that several results known in the literature are special cases of our findings.

2. Main Results

Let $\varphi : I^\circ \to \mathbb{R}$ be a given function, where $a_1, a_2 \in I^\circ$ with $0 < a_1 < a_2 < \infty$ (here I° represents the interior of I). We assume that $\varphi \in L_1[a_1,a_2]$ is such that ${}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u)$ and ${}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}\varphi(u)$ are well defined. We define

$$\tilde{\varphi}(u) := \varphi(a_1 + a_2 - u), \text{ for } u \in [a_1,a_2]$$

and

$$G(u) := \varphi(u) + \tilde{\varphi}(u), \text{ for } u \in [a_1,a_2].$$

With the change of variable $w = \frac{\tau - a_1}{u - a_1}$, (5) becomes

$$ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(u) = \frac{(u-a_1)}{k\Gamma_k(\alpha)} \int_0^1 \frac{F(wu + a_1(1-w), s)\varphi\big(wu + a_1(1-w)\big)dw}{\big[\mathbb{F}_s\big(u, wu + a_1(1-w)\big)\big]^{1-\frac{\alpha}{k}}}, \quad (6) $$

where $u > a_1$.

Our first result refers to functions in the classes $Q(I)$ and $(\psi, Q(I))$-convex dominated.

Theorem 2.1. *Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function in $Q(I)$. Suppose that $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ is a $(\psi, Q(I))$-convex dominated function on I. If $a_1, a_2 \in I$, with $\varphi, \psi \in L_1[a_1, a_2]$, then we have the following inequalities:*

$$ \left| 2\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+} G(a_2) - \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \varphi\big(\frac{a_1+a_2}{2}\big) \right| $$
$$ \leq 2\ {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+} (\tilde{\psi}(a_2) + \psi(a_2)) - \psi\big(\frac{a_1+a_2}{2}\big) \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \quad (7) $$

and

$$ \left| [\varphi(a_1)+\varphi(a_2)] \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+} \tilde{\varphi}_1(a_2) - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+} \varphi_1(a_2) \right| $$
$$ \leq [\psi(a_1)+\psi(a_2)] \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+} \tilde{\psi}_1(a_2) - {}^{s}J^{\frac{\alpha}{k}}_{F,a_1^+} \psi_1(a_2), \quad (8) $$

where $\varphi_1(u) = w(1-w)\varphi(u)$ and $\psi_1(u) = w(1-w)\psi(u)$.

Proof. Since φ is a $(\psi, Q(I))$-convex dominated function, taking $t = 1/2$, $u = wa_1 + (1-w)a_2$, $v = (1-w)a_1 + wa_2$, for some $w \in [0,1]$, in (2), we have

$$ \left| 2[\varphi(wa_1 + (1-w)a_2) + \varphi((1-w)a_1 + wa_2)] - \varphi\big(\frac{a_1+a_2}{2}\big) \right| $$
$$ \leq 2\psi(wa_1 + (1-w)a_2) + 2\psi((1-w)a_1 + wa_2) - \psi\big(\frac{a_1+a_2}{2}\big). \quad (9) $$

Multiplying both sides of (9) by

$$ \frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \frac{F(wa_2 + (1-w)a_1, s)}{[\mathbb{F}_s\big(a_2, wa_2 + a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} $$

and integrating over $[0,1]$ with respect to w, we obtain

$$\begin{aligned}
&\Bigg|2\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi\big(a_1w+(1-w)a_2\big)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&+2\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi\big((1-w)a_1+a_2w\big)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&-\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\varphi\big(\frac{a_1+a_2}{2}\big)\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\Bigg| \leq\\
&2\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\psi\big(a_1w+(1-w)a_2\big)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&+2\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\psi\big((1-w)a_1+a_2w\big)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\\
&-\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\psi\big(\frac{a_1+a_2}{2}\big)\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}.
\end{aligned} \tag{10}$$

We note that

$$\int_0^1 \frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} = \frac{k[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\alpha(a_2-a_1)}.$$

Since $\tilde{\varphi}\big((1-w)a_1+a_2w\big)=\varphi(a_1w+(1-w)a_2)$, from (6) we obtain

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi(a_1w+(1-w)a_2)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} = {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}\tilde{\varphi}(a_2)$$

and

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1 \frac{F(wa_2+(1-w)a_1,s)\varphi((1-w)a_1+a_2w)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} = {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}\varphi(a_2).$$

Therefore, (10) becomes

$$\begin{aligned}
&\left|2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}(\tilde{\varphi}(a_2)+\varphi(a_2))-\frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\varphi\big(\frac{a_1+a_2}{2}\big)\right|\\
&\qquad\leq 2\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}(\tilde{\psi}(a_2)+\psi(a_2))-\psi\big(\frac{a_1+a_2}{2}\big)\frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)},
\end{aligned}$$

Thus, inequality (7) holds. On the other hand, since φ is $(\psi,Q(I))$-convex dominated, taking $u=a_1$, $v=a_2$ and $w\in[0,1]$ in (2), we obtain

$$\begin{aligned}
|(1-w)\varphi(a_1)&+w\varphi(a_2)-w(1-w)\varphi(wa_1+(1-w)a_2)|\\
&\leq (1-w)\psi(a_1)+w\psi(a_2)-w(1-w)\psi(wa_1+(1-w)a_2)
\end{aligned}$$

and

$$|w\varphi(a_1)+(1-w)\varphi(a_2)-w(1-w)\varphi((1-w)a_1+wa_2)| \\ \leq w\psi(a_1)+(1-w)\psi(a_2)-w(1-w)\psi((1-w)a_1+wa_2).$$

Then, adding the inequalities above we obtain

$$\begin{aligned}&\big|[\varphi(a_1)+\varphi(a_2)]-w(1-w)[\varphi(wa_1+(1-w)a_2)+\varphi((1-w)a_1+wa_2)]\big| \\ &\leq [\psi(a_1)+\psi(a_2)]-w(1-w)[\psi(wa_1+(1-w)a_2)+\psi((1-w)a_1+wa_2)].\end{aligned} \tag{11}$$

Next, multiplying both sides of (11) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}$$

and integrating over $[0,1]$ with respect to w, we have

$$\begin{aligned}&\Bigg|(\varphi(a_1)+\varphi(a_2))\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1\frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \\ &-\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1\frac{F(wa_2+(1-w)a_1,s)w(1-w)\varphi\big(wa_1+(1-w)a_2\big)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \\ &-\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1\frac{w(1-w)\varphi((1-w)a_1+wa_2)F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}\Bigg| \leq \\ &(\psi(a_1)+\psi(a_2))\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1\frac{F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \\ &-\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1\frac{F(wa_2+(1-w)a_1,s)w(1-w)\psi\big(wa_1+(1-w)a_2\big)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}} \\ &-\frac{(a_2-a_1)}{k\Gamma_k(\alpha)}\int_0^1\frac{w(1-w)\psi((1-w)a_1+wa_2)F(wa_2+(1-w)a_1,s)dw}{[\mathbb{F}_s\big(a_2,wa_2+a_1(1-w)\big)]^{1-\frac{\alpha}{k}}},\end{aligned}$$

which proves the inequality (8). ☑

The following remarks provide some of the generalizations obtained from Theorem 2.1.

***Remark* 2.2.** Setting $F\equiv 1$ and $\alpha=k$, we have that Theorem 3 of [22] is a particular case of Theorem 2.1. In fact, let $g: I\subseteq\mathbb{R}\to\mathbb{R}$ be a nonnegative function in the class $Q(I)$ and $f: I\subseteq\mathbb{R}\to\mathbb{R}$ a nonnegative function in $Q(I)$ that is $(g,Q(I))$-convex dominated on I. If $a,b\in I$, with $a<b$ and $f,g\in L_1[a,b]$, then we have the inequalities from Theorem 3 of [22]:

$$\left|\frac{4}{b-a}\int_a^b f(x)dx-f\left(\frac{a+b}{2}\right)\right|\leq\frac{4}{b-a}\int_a^b g(x)dx-g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b p(x)f(x)dx\right| \leq \frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b p(x)g(x)dx,$$

where $p(x)=\frac{(b-x)(x-a)}{(b-a)^2}$ for all $x \in I$.

***Remark* 2.3.** Setting $F \equiv 1$ and $\alpha = k$, we can observe that Theorem 1 of [2] is a special case of Theorem 2.1. In fact, let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ a function that is g-convex dominated. For all $a, b \in I$, with $a < b$, the following inequalities hold:

$$\left|\frac{4}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{4}{b-a}\int_a^b g(x)dx - g\left(\frac{a+b}{2}\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(x)dx.$$

These inequalities are stated in Theorem 1 of [2].

***Remark* 2.4.** If $F(t,\alpha)=1$, $k=1$, ψ is a nonnegative function and φ is a harmonically ψ-convex function, then we can derive Theorem 4.1 of [27]. In fact, let $h : (0,1) \subseteq J \to \mathbb{R}$ be a nonnegative function and $g : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonic h-convex function. Suppose that $f : I \subseteq \mathbb{R}\setminus 0 \to \mathbb{R}$ is a function that is (h,g)-harmonically convex dominated and belongs to $L_1[a,b]$. For $\alpha > 0$ and $h\left(\frac{1}{2}\right) \neq 0$, the following inequality holds:

$$\begin{aligned}&\left|\Gamma(k+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{\frac{1}{a^-}}(f\circ w)\left(\frac{1}{b}\right)+J_{\frac{1}{b^+}}(f\circ w)\left(\frac{1}{a}\right)\right\}-\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{2ab}{a+b}\right)\right|\\ &\leq \left|\Gamma(k+1)\left(\frac{ab}{b-a}\right)^{\alpha}\left\{J_{\frac{1}{a^-}}(g\circ w)\left(\frac{1}{b}\right)+J_{\frac{1}{b^+}}(g\circ w)\left(\frac{1}{a}\right)\right\}-\frac{1}{h\left(\frac{1}{2}\right)}f\left(\frac{2ab}{a+b}\right)\right|,\end{aligned}$$

which is stated in Theorem 4.1 of [27].

***Remark* 2.5.** Theorem 2.1 also extends Theorem 2.4 of [6], which states that, if φ is a differentiable function on I^o such that $\varphi' \in L_1[a_1,a_2]$ and $|\varphi'|$ is quasi-convex on $[a_1,a_2]$, then

$$\begin{aligned}&\left|\frac{\varphi(a_1)+\varphi(a_2)}{2}-\frac{\Gamma_k(\alpha+k)}{4[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}\left[\ {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+}G(a_2)+\ {}^sJ^{\frac{\alpha}{k}}_{F,a_2^-}G(a_1)\right]\right|\\ &\leq \frac{I_1+I_2+I_3+I_4}{4[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}\max\{|\varphi'(a_1)|,|\varphi'(a_2)|\},\end{aligned}$$

where

$$I_1 = \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}_s(w,a_1)]^{\frac{\alpha}{k}} dw - \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}_s(w,a_1)]^{\frac{\alpha}{k}} dw,$$

$$I_2 = \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}_s(a_2, a_1+a_2-w)]^{\frac{\alpha}{k}} dw - \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}_s(a_2, a_1+a_2-w)]^{\frac{\alpha}{k}} dw,$$

$$I_3 = \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}_s(a_2,w)]^{\frac{\alpha}{k}} dw - \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}_s(a_2,w)]^{\frac{\alpha}{k}} dw,$$

$$I_4 = \int_{a_1}^{\frac{a_1+a_2}{2}} [\mathbb{F}_s(a_2+a_1-w, a_1)]^{\frac{\alpha}{k}} dw - \int_{\frac{a_1+a_2}{2}}^{a_2} [\mathbb{F}_s(a_2+a_1-w, a_1)]^{\frac{\alpha}{k}} dw.$$

The following result refers to functions in the classes $P(I)$ and $(\psi, P(I))$-convex dominated on I.

Theorem 2.6. *Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function in $P(I)$. Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a $(\psi, P(I))$-convex dominated function on I. If $a_1, a_2 \in I$, with $\varphi, \psi \in L_1[a_1,a_2]$, then we have*

$$\Big| {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+} G(a_2) - \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \varphi\Big(\frac{a_1+a_2}{2}\Big)\Big| \leq {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+} (\tilde{\psi}(a_2) + \psi(a_2)) - \psi\Big(\frac{a_1+a_2}{2}\Big) \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \tag{12}$$

and

$$\Big| [\varphi(a_1)+\varphi(a_2)] \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} - {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+} \tilde{\varphi}(a_2) \Big| \leq [\psi(a_1)+\psi(a_2)] \frac{[\mathbb{F}_s(a_2,a_1)]^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} - {}^sJ^{\frac{\alpha}{k}}_{F,a_1^+} \tilde{\psi}(a_2). \tag{13}$$

Proof. Since φ is a $(\psi, P(I))$-convex dominated function, taking $t = 1/2$, $u = wa_1 + (1-w)a_2$, $v = (1-w)a_1 + wa_2$, for some $w \in [0,1]$, in (3), we obtain

$$\begin{aligned} &\Big| \varphi(wa_1 + (1-w)a_2) + \varphi((1-w)a_1 + wa_2) - \varphi\Big(\frac{a_1+a_2}{2}\Big)\Big| \\ &\quad \leq \psi(wa_1 + (1-w)a_2) + \psi((1-w)a_1 + wa_2) - \psi\Big(\frac{a_1+a_2}{2}\Big). \end{aligned} \tag{14}$$

Next, multiplying both sides of (14) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \frac{F(wa_2 + (1-w)a_1, s)}{[\mathbb{F}_s\big(a_2, wa_2 + a_1(1-w)\big)]^{1-\frac{\alpha}{k}}}$$

and integrating over $[0,1]$ with respect to w, we have the inequality (12).

On the other hand, since φ is $(\psi, P(I))$-convex dominated, taking $u = a_1$ and $v = a_2$ and $w \in [0,1]$ in (3), we have

$$|\varphi(a_1)+\varphi(a_2)-\varphi(wa_1+(1-w)a_2)| \leq \psi(a_1)+\psi(a_2)-\psi(wa_1+(1-w)a_2). \quad (15)$$

Now, multiplying both sides of (15) by

$$\frac{(a_2-a_1)}{k\Gamma_k(\alpha)} \frac{F(wa_2+(1-w)a_1,s)}{[\mathbb{F}_s(a_2, wa_2+a_1(1-w))]^{1-\frac{\alpha}{k}}}$$

and integrating over $[0,1]$ with respect to w, the inequality (13) is proved. ☑

***Remark* 2.7.** If we take $F \equiv 1$ and $\alpha = k$ in Theorem 2.6, then we obtain Theorem 4 of [22]. In fact, let $g : I \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function in $P(I)$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ a function that is $(g, P(I))$-convex dominated on I. If $a, b \in I$, with $a < b$ and $f, g \in L_1[a,b]$, then we obtain the inequalities:

$$\left|\frac{2}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{2}{b-a}\int_a^b g(x)dx - g\left(\frac{a+b}{2}\right)$$

and

$$\left|[f(a)+f(b)] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq [g(a)+g(b)] - \frac{1}{b-a}\int_a^b g(x)dx,$$

which are stated in Theorem 4 of [22].

3. Concluding Remarks

This paper gives new inequalities of the Hermite-Hadamard type, in two different classes of convex dominated functions, some related inequalities (fractional or not) are also obtained as particular cases of our results.

The inequalities proved here represents an extension of the existing theory. Our results improves the theoretical framework for this class of functions and generalizes the theory for the case of generalized fractional integrals used.

Finally, with the operators of the Definition 1.7, we can generalize different results already reported in the literature, obtained for different classes of dominated convex functions, such as those of [13, 15, 27], which open up new research possibilities.

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