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The Plykin and Solenoid attractor are homoclinic classes

Los atractores Plykin y Solenoide son clases homoclínicas

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Abstract. A homoclinic class is the closure of the transverse intersection points of the stable and unstable manifolds of a hyperbolic periodic orbit. In this paper, we prove, using the techniques presented in [1], that the Plykin and the Solenoid attractors are a homoclinic class.

Key words and phrases. Attractor, hyperbolic attractors, homoclinic classes, periodic orbits.

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Resumen. Una clase homoclínica es la clausura de puntos de intersecciones transversales entre las variedades estables e inestables sobre una órbita periódica hiperbólica. En este trabajo, usamos la técnica presentada en [1], para probar que los atractores Plykin y Solenoide son clases homoclínicas

Palabras y frases clave. Atractores, atractores hiperbólicos, clases homoclínicas, órbita periódica.

1. Introduction

Hyperbolicity is one of the fundamental tools for studying dynamical systems. Lately this has been developed by the Smale's hyperbolic theory. In the context of homoclinic class for dynamical systems, many people have been publishing works studying these ideas. In fact, López and Arbieto [3] proved robustly finiteness of homoclinic class on small perturbations of a sectional hyperbolic set on a compact manifold. Yang and Gan [10] showed that every expansive homoclinic class of a C^1 generic diffeomorphism of a compact smooth manifold

M is hyperbolic. Lee [2] proves that for a generically C^1 vector field X of a compact smooth manifold M, if a homoclinic class $H(\gamma, X)$ which contains a hyperbolic closed orbit γ is measure expansive for X then $H(\gamma, X)$ is hyperbolic.

Our focus is a bit simpler, the Plykin and solenoid attractors are two basic examples, i.e., attractors with a *hyperbolic structure*. Both are *the closure of the transverse intersection points of the stable and unstable manifolds of a hyperbolic periodic orbit*, in other words a *homoclinic class*. This fact is a consequence of the Smale's spectral decomposition Theorem [9]. In [1] it was proved that the geometric Lorenz attractor, which is not a hyperbolic set, is a homoclinic class. In this paper, we will use the techniques presented in [1] in order to prove that the Plykin and solenoid attractors are homoclinic class. The key part of the technique is to use the uniform expansion in the direction of the unstable manifold, resulting in an expansive dynamic system. It is known that this type of expansive dynamical systems are topologically exact and many properties of the system are consequences of these properties (see [4]).

The technique can be extended to a family of hyperbolic attractors with the properties that Plykin and solenoid attractors have in common. It may be interesting in the future to consider this type of technique on partially hyperbolic dynamics or to apply it to systems whose unstable direction is not uniformly expansive.

This paper is organized as follows: In Section 2, we summarize some basic notions from dynamical systems that we will use throughout the paper. In Section 3, we present the basic properties of the Plykin attractor and the trapping region for this hyperbolic set, which will be used in order to prove our result. Similarly, in Section 4 we will present the basic properties of the solenoid and its correspondent trapping region. Finally, in Section 5 we prove the main result of this work.

2. Basic definitions

Let M be a compact riemannian manifold and $f : M \to M$ a diffeomorphism of class C^r, with $r \geq 1$. This diffeomorphism represents the evolution law of a discrete dynamical system on M, which is the action of the additive group $\mathbb{Z}$ on M, i.e., $f^0 = id_M$ and $f^m \circ f^n = f^{n+m}$ for all $n, m \in \mathbb{Z}$.

For each $x \in M$ the set $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$ is the *orbit of* x through f. A point $x \in M$ is called periodic for f if there exists $n \geq 1$ such that, $f^n(x) = x$. The *minimum* n satisfying this property is called the period of x. The set of all periodic points of f is denoted by $Per(f)$. If the period of some $x \in M$ is $n = 1$, then x is a *fixed point of* f. The set of all fixed points of f is denoted by $Fix(f)$.

Definition 2.1. Consider a continuous map $f : M \to M$. A compact subset $\Lambda \subset M$ is said to be:

(1) *Invariant under* f, if $f(\Lambda) \subseteq \Lambda$. Therefore, we can define the restriction of f on the set Λ, i. e. $f\,|_\Lambda\colon \Lambda \to \Lambda$.

(2) *Transitive*, if there exists $x \in \Lambda$ such that its positive orbit defined as $\mathcal{O}_f^+(x) = \{f^n(x); n \in \mathbb{N}\}$ is dense in Λ.

(3) *Trapping region*, provided that $f(\Lambda) \subset int(\Lambda)$.

(4) *Attracting*, if there exists a trapping region N such that

$$\Lambda = \bigcap_{n \geq 0} f^n(N).$$

(5) *An attractor*, if it is an attracting set and $f\,|_\Lambda$ is transitive.

Definition 2.2. We say that $\Lambda \subset M$ has a *hyperbolic structure* or simply referred as *hyperbolic* (see [8]) for $f : M \to M$ if:

- For each $x \in \Lambda$, the tangent space of M splits as the direct sum of E_x^u and E_x^s, $T_x(M) = E_x^u \oplus E_x^s$, where E_x^u and E_x^s are subspaces of T_xM.
- The splitting is invariant under the action of the derivative map. That is, $Df_xE_x^u = E_{f(x)}^u$ and $Df_xE_x^s = E_{f(x)}^s$.
- There exists $0 < \lambda < 1$ and $C \geq 1$ not dependent of x, such that for all $n \geq 0$,

  $$\|Df_x^n v^s\| \leq C\lambda^n \|v^s\| \text{ for } v^s \in E_x^s,$$

  and

  $$\|Df_x^{-n} v^u\| \leq C\lambda^n \|v^u\| \text{ for } v^u \in E_x^u.$$

The subspaces E_x^s and E_x^u are called the *stable* and *unstable* subspaces at x, respectively.

Definition 2.3. When $\Lambda \subset M$ is a hyperbolic set for $f : M \to M$, we can define for every $x \in \Lambda$ the sets:

$$W^s(x, f) = \left\{ y \in M : d\left(f^n(y), f^n(x)\right) \xrightarrow{n \to +\infty} 0 \right\},$$
$$W^u(x, f) = \left\{ y \in M : d\left(f^n(y), f^n(x)\right) \xrightarrow{n \to -\infty} 0 \right\}.$$

These sets are called, respectively, the *stable* and *unstable manifolds* of the point x.

By the Invariant Manifold Theory (see [5]), $W^s(x, f)$ and $W^u(x, f)$ are injectively immersed C^r submanifolds tangent at x to E_x^s and E_x^u, respectively. Furthermore, they satisfy

$$f\left(W^s(x, f)\right) \subset W^s\left(f(x), f\right) \text{ and } f^{-1}\left(W^u(x, f)\right) \subset W^u\left(f^{-1}(x), f\right).$$

Definition 2.4. Let $\varepsilon > 0$ and $x \in M$. Consider the *local stable manifold*, $W^s_\varepsilon(x)$, and the *local unstable manifold*, $W^u_\varepsilon(x)$, which are defined as

$$W^s_\varepsilon(x) = \left\{ y \in M : d\left(f^n(x), f^n(y)\right) \xrightarrow{n\to+\infty} 0, \text{ and } d\left(f^n(x), f^n(y)\right) \le \varepsilon, \forall n \ge 0 \right\}$$

and

$$W^u_\varepsilon(x) = \left\{ y \in M : d\left(f^{-n}(x), f^{-n}(y)\right) \xrightarrow{n\to+\infty} 0, \text{and } d\left(f^{-n}(x), f^{-n}(y)\right) \le \varepsilon, \forall n \ge 0 \right\}.$$

We have

$$W^s(x,f) = \bigcup_{n\ge 0} f^{-n}\left(W^s_\varepsilon(f^n(x))\right) \text{ and } W^u(x,f) = \bigcup_{n\ge 0} f^{n}\left(W^u_\varepsilon(f^{-n}(x))\right).$$

Let $p \in \Lambda \subset M$ be a fixed point of f. For any $\varepsilon > 0$, let $B_\varepsilon(p) \subset M$ be the open ball centered at p of radius ε. The sets

$$W^s_\varepsilon(p) = \left\{ q \in B_\varepsilon(p) : f^n(q) \in B_\varepsilon(p),\ \forall n \ge 0 \right\},$$
$$W^u_\varepsilon(p) = \left\{ q \in B_\varepsilon(p) : f^{-n}(q) \in B_\varepsilon(p),\ \forall n \ge 0 \right\},$$

are called *local stable and unstable manifolds of size ε at p.*

Finally, given $x \in \Lambda$ and $\mathcal{O}_f(x)$, we define the *stable manifold*, $W^s(\mathcal{O}_f(x))$ and the *unstable manifold*, $W^u(\mathcal{O}_f(x))$ as:

$$W^s\left(\mathcal{O}_f(x)\right) = \bigcup_{n\in\mathbb{Z}} W^s\left(f^n(x), f\right) \text{ and } W^u\left(\mathcal{O}_f(x)\right) = \bigcup_{n\in\mathbb{Z}} W^u\left(f^n(x), f\right)$$

Definition 2.5. The *homoclinic class* $H_f(\mathcal{O})$ associated to a hyperbolic periodic orbit $\mathcal{O} := \mathcal{O}_f(p)$ of f, with a periodic point p, is the closure of the set consisting of points with intersection transversal between $W^s(\mathcal{O})$ and $W^u(\mathcal{O})$, i.e.,

$$H_f(\mathcal{O}) := CL\left(W^s(\mathcal{O}) \pitchfork W^u(\mathcal{O})\right).$$

3. The Plykin attractor

In Section 5 we will prove that both the Plykin and the Solenoid attractors are homoclinic classes, using the Bautista techniques developed in [1]. In order to prove these facts, we need to define the trapping region N for both attractors. In this section, we will define the Plykin attractor and we will find its trapping region. In this case, N is a region with three holes showed in Figure 3.1.

In fact, let $N = A \cup B \cup C \cup D$ be the trapping region in the plane with three holes, foliated by line segments as shown in Figure 3.1. These segments define the foliations $\mathcal{F}^s$ on N, and on N we define the map $f : N \to N$ satisfying the conditions in Figure 3.2.

Now, on N we can define an equivalence relation $\sim$: two points $p, q \in N$ are equivalent if they are in the same component of the stable manifolds, i.e.,

$$q \sim p \iff q \in \ \mathrm{comp}_p(W^s \cap N).$$

Consider the quotient space $K = N/\sim$.

Condition 3.1. Consider the foliations $\mathcal{F}^s$ on N consisting of the segments of line as showed in Figure 3.1.

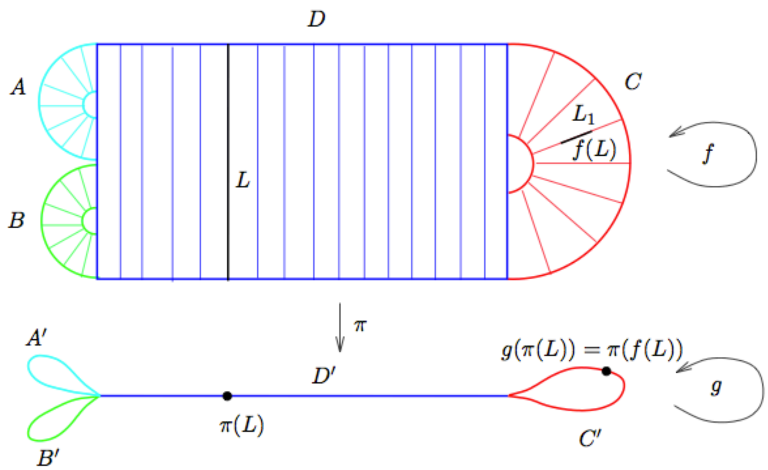

FIGURE 3.1. The trapping region for the Plykin attractor

- If L is a leaf of $\mathcal{F}^s$, then $f(L)$ is contained in a other leaf L_1 of $\mathcal{F}^s$. i.e., the foliations $\mathcal{F}^s$ are invariant under f.
- It contracts in parallel directions and expands in transverse directions to the leaf.
- N is an invariant set, i.e., $f(N) \subseteq N$.

In Figure 3.2 we show the first iterated of N under f.

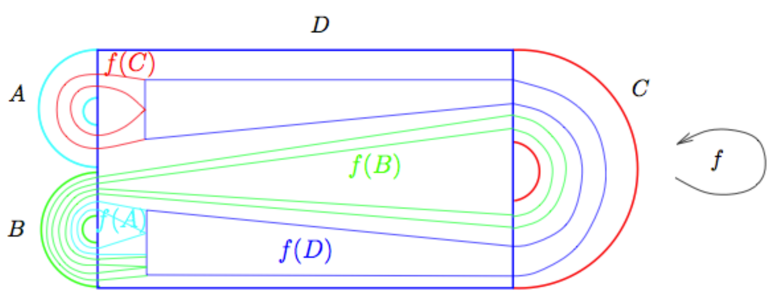

FIGURE 3.2. The first iterated of f.

Definition 3.1. The set $\Lambda := \bigcap_{n\geq 0}^{\infty} f^n(N)$ is the *Plykin attractor* associated to f.

The space of leaves K is the union $K = A' \cup B' \cup C' \cup D'$, as defined in Figure 3.1. The map f induces an application $g : K \to K$ (see [7]). Let us denote by $\pi : N \to K$ the *projection* on K. As shown in Figure 3.1, each $x \in N$ is on a line segment and this is represented by a point in K. Therefore, the projection π is a onto map.

Next, we will show the application g is *locally eventually* onto (LEO), i.e., given any open subset I of K, there exists $n \in \mathbb{N}$, such that $g^n(I) = K$. In fact, note that

$$B' \subset g(A'),\ A' \subset g(C'),\ B' \cup C' \cup D' \subset g(B') \text{ and } C' \subset g(D').$$

Therefore,

$$K \subset g^2(B').$$

Moreover, since $B' \subset g(A')$, then $K \subset g^3(A')$. Finally, we have $K \subset g^4(C')$ and therefore $K \subset g^5(D')$. Hence we have:

Lemma 3.2. *The map $g : K \to K$ is LEO.*

Proof. Let $I \subset K$ be any open subset of K. Since g is an expansion, there exists $n_I \geq 0$ such that $g^n(I)$ covers at least one of the sets A', B', C' or D'. Therefore, there exists an $m \in \mathbb{Z}^+$ such that $K \subset g^{n+m}(I)$ with $2 \leq m \leq 5$. So, by definition of g, we have that $g^{n+m}(I) = K$. ☑

Moreover, g has periodic points:

Lemma 3.3. *The application $g : K \to K$ has periodic points.*

Proof. Let I be an open subset of K. Since g is LEO, there exists $n > 0$ such that $g^n(I) = K \subset I$. Thus, g^n has a fixed point in I. Therefore, g has a periodic point in I. ☑

Hence, if b is a periodic point for g, then $L_b = \pi^{-1}(b)$ is a periodic point for f. Therefore, since f preserves foliations and contracts in parallel directions to these foliations, we have that f has periodic points.

Lemma 3.4. *For all $x \in K$, we have*

$$K = CL\left(\bigcup_{n\geq 0} g^{-n}\{x\}\right)$$

Proof. Let I be any open subset of K and $x \in I$. Since g is LEO, there exists $n \geq 0$, such that $g^n(I) = K$. Therefore, $x \in g^n(I)$. Thus $I \cap g^{-n}\{x\} \neq \varnothing$, which implies that $\bigcup_{n\geq 0} g^{-n}\{x\} \cap I \neq \varnothing$. ☑

4. The Solenoid attractor

In order to build the solenoid, we consider the solid torus $N = \mathbb{S}^1 \times \mathbb{D}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{S}^1 = \{t \in \mathbb{R} \bmod \equiv 1\}$. The embedding $f : N \to N$ (into N, not onto) is defined as

$$f(t,z) = \left(2t \bmod \equiv 1, \frac{1}{4}z + \frac{1}{2}\exp(2\pi t i)\right).$$

The first component $2t$ mod 1 with $t \in \mathbb{S}^1$ is called the doubling map and allows the function f stretch in the direction $\mathbb{S}^1$. On the other hand, in the second component, the constants $1/4$ allows f to contract on $\mathbb{D}$ or on the transverse discs $\mathbb{D}(t) = \{t\} \times D$ for $t \in \mathbb{S}^1$. We have, that $f(N) \subset int(N)$. The intersection $f(N) \cap D(t)$ for some $t \in \mathbb{S}^1$, consists of two disks of radius $1/4$ centered at diametrically opposite points at distance $1/2$ from the center of the intersection. In general, $f^n(N) \cap D(t)$ consists of 2^n disks of radius $1/4^n$.

Definition 4.1. The set $\Lambda = \bigcap_{n=0}^{\infty} f^n(N)$ is called solenoid and $N = \mathbb{S}^1 \times \mathbb{D}$ is a trapping region for f.

The set Λ is an attracting set. f satisfies the Condition 3.1. In Figure 4.1 we show the first iteration of f.

We can write $N = \bigcup_{t \in \mathbb{S}^1} \{t\} \times \mathbb{D} = \bigcup_{t \in \mathbb{S}^1} D(t)$, hence the leafs in $\mathcal{F}^s$ are defined as $L_t := D(t)$ for every $t \in \mathbb{S}^1$. Then, we will have invariance over them, i.e., $f(L_t) \subset L_{2t}$. Therefore, $(L_t)_{t \in \mathbb{S}^1}$ forms an invariant family of submanifolds of N which is contracted under f. Hence, they are the stable manifolds.

For the specific case, given $p = (t,z) \in \Lambda$, we obtain that $W^s(p) \supset \{t\} \times \mathbb{D} = L_t$. It can be shown that Λ is a hyperbolic expanding attractor and has the next properties: it is connected, not locally connected, it is not path connected and its the topological dimension is one and $f|_\Lambda : \Lambda \to \Lambda$ has the following properties:

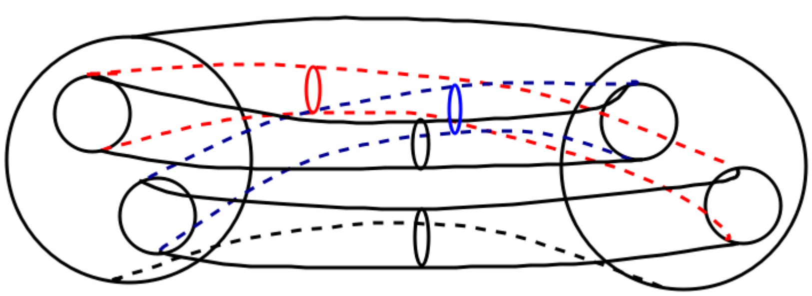

FIGURE 4.1. First iteration of f.

- The periodic points of f are dense in Λ,
- f is topologically transitive on Λ,
- f has a hyperbolic structure on Λ.

For details of the proofs, see [8]. The unstable manifold, $W^u(p)$ is an immersed line contained in Λ and crosses L_t in a countable number of points.

We now take the quotient space $K = N/\sim$, given by equivalence classes of points that are in the same components of the stable manifold in N, i. e. given $q = (t, z_0)$ and $p = (t, z_1)$, then $q \sim p$ if $q \in comp_p\left(W^s(p) \cap N \cap (\{t\} \times \mathbb{D})\right) = L_t$. Therefore, f induces the function $g : K \to K$ which is an expansion. This fact implies that g is locally eventually onto. It is well known, that expansions are locally eventually onto, see [6].

5. Plykin and Solenoid are homoclinic classes

Finally, in this section we will prove that both the Plykin and the Solenoid attractors are homoclinic classes. In order to prove this result, we will need the next facts:

Lemma 5.1. *Let N be the trapping region of the attractors Plykin or Solenoid, such that N is leafed by L_x for $x \in N$. If $x \nsim y (L_x \cap L_y = \varnothing)$. Then, the projection $\pi : N \to N/\sim$ is an open map.*

Proof. Let $J \subset N$ be an open subset. Given $x, y \in J$ with $x \nsim y$, we have $L_x \cap L_y = \varnothing$. By the definition of π, we have $\pi(J) = \bigcup_{x\in J}[x]$. We will show that $\pi^{-1}(\pi(J)) = \pi^{-1}\left(\bigcup_{x\in J}[x]\right)$ is an open subset of N. Taking $z \in \pi^{-1}\left(\bigcup_{x\in J}[x]\right)$, there exists $x \in J$ such that $\pi(z) \in [x]$, i.e.. $z \sim x$. Since J is an open set, there exists $\varepsilon > 0$, such that $B_\varepsilon(x) \subset J$. As $L_z = L_x$, we obtain that for every $y \in B_\varepsilon(x)$, we have $L_y \cap B_\varepsilon(z) \neq \varnothing$. Therefore, there exists a neighborhood $B_\varepsilon(z) \subset \pi^{-1}\left(\bigcup_{x\in J}[x]\right)$. This fact implies that $\pi(J)$ is an open set. ☑

Proposition 5.2. *For every periodic point $p \in \Lambda$ of f we have*

$$N \subset \bigcup_{q \in W^u(\mathcal{O}_f(p), f)} L_q,$$

where L_q is a leaf in $\mathcal{F}^s$.

Proof. Let $p \in \Lambda$ be a periodic point of f and $x \in N$. Since $\mathcal{O}_f(p) \subset \Lambda$ and f is hyperbolic on Λ there exist $W^u(p, f)$ and $W^s(p, f)$ (see Figure 5.1). Let $\pi(x) \in K$ be the equivalence class of x for π. Let $J = W^u_\varepsilon(p, f) \times W^s_\varepsilon(p, f)$ (see Figure 5.2) for some $\varepsilon > 0$ small enough such that $\pi(J) \subset K$ is an open set. Since $\pi(J)$ is open, $g : K \to K$ is locally eventually onto.

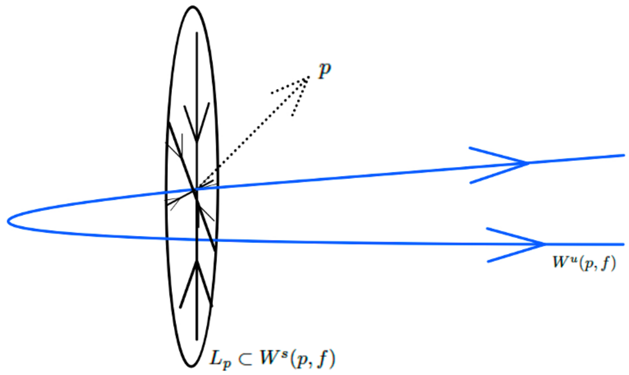

FIGURE 5.1. Local stable and unstable manifold at point p.

Thus there exists $m \geq 0$ such that $g^m(\pi(J)) = K$. On the other hand, the diagram

$$\begin{array}{ccc} N & \xrightarrow{f} & N \\ \pi \downarrow & & \downarrow \pi \\ K & \xrightarrow{g} & K \end{array}$$

commutes, therefore $g^n(\pi(J)) = \pi(f^n(J))$ for all $n \geq 1$. In particular, if $n = m$ then $\pi(x) \in \pi(f^m(J))$, so $L_x \cap \pi(f^m(J)) \neq \varnothing$. The invariance implies that $\pi(f^m(J)) \subset W^u(\mathcal{O}_f(p), f)$, then there exists $q \in W^u(\mathcal{O}_f(p), f) \cap L_x$, so $L_q = L_x$. Then, $x \in L_q$ with $q \in W^u(\mathcal{O}_f(p), f)$. ☑

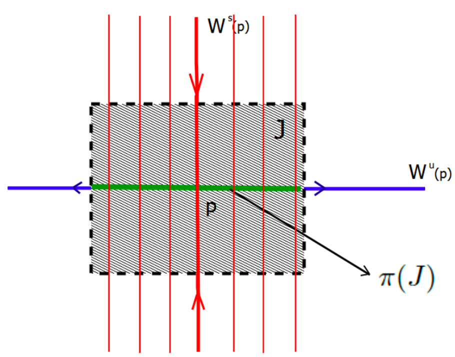

FIGURE 5.2. J local open set.

Proposition 5.3. *For every periodic point $p \in \Lambda$ of f we have*

$$CL\,(W^u\,(\mathcal{O}_f(p), f)) = \Lambda.$$

Proof. It is sufficient to prove the conclusion for a point of a periodic orbit p. Let $\varepsilon > 0$ be small enough such that

$$W^u_\varepsilon\left(f^k(p), f\right) \subset Int(N) \subset N \quad \text{for } k \in \mathbb{Z}.$$

Thus for each $k \geq 0$, we have

$$W^u\left(f^{-k}(p), f\right) = \bigcup_{j\geq 0} f^j\left(W^u_\varepsilon\left(f^{-j-k}(p), f\right)\right).$$

Note that $W^u_\varepsilon\left(f^{-j-k}(p), f\right) \subset N$, therefore

$$f^j\left(W^u_\varepsilon\left(f^{-j-k}(p), f\right)\right) \subset f^j(N) \subset N \ \forall j \geq 0.$$

Thus, $W^u\left(f^{-k}(p), f\right) \subset N$. Since

$$f^k\left(W^u\left(f^{-k}(p), f\right)\right) = W^u(p, f) \ \text{ and } \ W^u\left(f^{-k}(p), f\right) \subset N,$$

then $W^u(p, f) \subset f^k(N)$. Next, taking intersections for $k \geq 0$, we have

$$W^u(p, f) \subset \bigcap_{k\geq 0} f^k(N).$$

Therefore,

$$CL\left(W^u(p, f)\right) \subset CL\left(\bigcap_{k\geq 0} f^k(N)\right) = \Lambda.$$

Now, we prove the other inclusion, i.e., $\Lambda \subset CL\left(W^u(\mathcal{O}_f(p), f)\right)$. Fix $x \in \Lambda$. By assumption, f contracts the leaf of $\mathcal{F}^s$, thus for each $\varepsilon > 0$ there exists $n_\varepsilon \geq 0$ such that $d(x, w) < \varepsilon$ with $w \in f^{n_\varepsilon}(L)$, where

$$L \in f^{-n_\varepsilon}(L_x) = \{L' \in \mathcal{F}^s : f^{n_\epsilon}(L') \subset L_x\} \ \text{ and } x \in f^{n_\varepsilon}(L).$$

Since

$$N \subset \bigcup_{q\in W^u(\mathcal{O}_f(p), f)} L_q$$

there exists $q \in W^u(\mathcal{O}_f(p), f) \cap L$. This fact implies that $d(x; f^{n_\varepsilon}(q)) < \varepsilon$. Since $\varepsilon > 0$ arbitrary and $f^{n_\varepsilon}(q) \in W^u(\mathcal{O}_f(p), f)$, then $x \in CL\left(W^u\left(\mathcal{O}_f(p), f\right)\right)$. ☑

Now, we are ready to show the main result of this paper.

Theorem 5.4. *If* $\Lambda := \cap_{j=0}^{\infty} f^j(N)$ *is the Plykin attractor or Solenoid, then* $\Lambda = H_f\left(\mathcal{O}\right)$.

Proof. We know that,

$$W^s(\mathcal{O}_f(p), f) \pitchfork W^u(\mathcal{O}_f(p), f) \subset W^u(\mathcal{O}_f(p), f),$$

taking the closure, we have

$$CL(W^s(\mathcal{O}_f(p), f) \pitchfork W^u(\mathcal{O}_f(p), f)) \subset CL(W^u(\mathcal{O}_f(p), f))$$

and applying Proposition 5.3 we obtain $H_f(\mathcal{O}_f(p)) \subset \Lambda$.

In order to show that $\Lambda \subset H_f(\mathcal{O}_f(p))$ it is enough to prove that $W^u(\mathcal{O}_f(p), f) \subset H_f(\mathcal{O}_f(p))$, since by Proposition 5.3 we have that $CL(W^u(\mathcal{O}_f(p), f)) = \Lambda$. Take $q \in W^u(p, f)$ and choose a small neighborhood J on N, such that $q \in \pi(J) \subset W^u(p, f)$ (see Figure 5.3 for the Plykin attractor and Figure 5.4 for the solenoid). Let $L_p \in \mathcal{F}^s$ be the stable leaf that contains p. Given that g is locally eventually onto, there exists $m \geq 0$, such that $g^m(\pi(J)) = K$. Then $\pi(f^m(J)) \cap L_p \neq \varnothing$ and hence $f^{-m}(L_p) \cap \pi(J) \neq \varnothing$. Therefore, there exists $y \in f^{-m}(L_p) \cap \pi(J)$. Since $\pi(J) \subset W^u(p, f)$ and $f^{-m}(L_p) \subset W^s(p, f)$, we have $y \in W^s(\mathcal{O}_f(p), f) \pitchfork W^u(\mathcal{O}_f(p), f)$. Therefore q is a limit point of the homoclinic points set, which proves the theorem. ☑

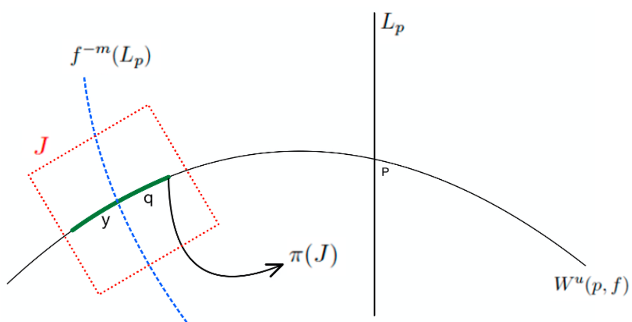

FIGURE 5.3. Diagram of the Plykin attractor

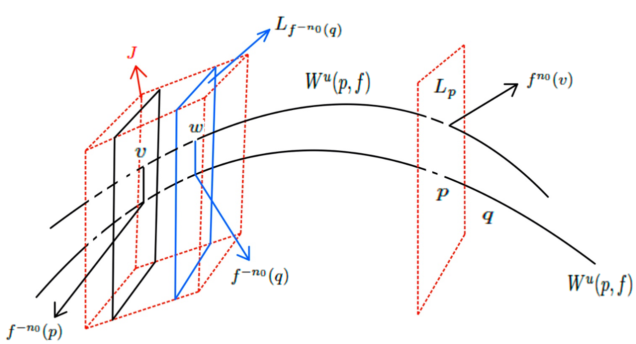

FIGURE 5.4. Diagram of the Solenoid attractor

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