

The Principal Symbol Map for Lagrangian Distributions with Complex Phase

Símbolo principal para distribuciones Lagrangianas con función de
fase compleja

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ABSTRACT. The present paper focus on Lagrangian distributions with complex phase. We provide an alternative construction of their principal symbol map, which allows us to compute the principal symbol after clean composition of Fourier integral operators with complex phase.

Key words and phrases. Lagrangian distribution, complex phase function, principal symbol.

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RESUMEN. El presente artículo se centra en el estudio de las distribuciones lagrangianas con fase compleja. Proponemos una construcción alternativa del símbolo principal, que nos permite calcular el símbolo principal de la distribución resultante de la composición de operadores integrales de Fourier con fase compleja.

Palabras y frases clave. distribuciones Lagrangianas, función de fase compleja, símbolo principal.

1. Introduction

Lagrangian distributions and, more generally, Fourier integral operators (FIOs) provide a powerful framework for the study of systems of partial differential equations. One can, for instance, represent the solution of certain type of initial boundary value problem as a Lagrangian distribution. This facilitates the study of the solution operator by using the theory of FIOs. The need for complex-valued phase functions comes from dealing with operators with non-real principal symbol, as they lead to a Lagrangian distribution with complex phase.

For this reason, many authors have studied Lagrangian distributions and FIOs with complex phase. One of the most systematic and complete approaches to the study of such distributions can be found in [5] and [6]. Our work builds upon these two articles, to provide a detailed description of the principal symbol map. We also focus on the composition of FIOs under the assumption of *clean intersection*.

Given a smooth manifold X , we denote by $\Omega^{\frac{1}{2}}(X)$, or simply by $\Omega^{\frac{1}{2}}$, the half-density bundle over X . The term *Lagrangian distribution with complex phase* refers to distributions $A \in \mathcal{D}'(X, \Omega^{\frac{1}{2}})$, which are microlocally of the form

$$\int e^{i\phi(x,\theta)} a(x,\theta) d\theta, \quad (x,\theta) \in X \times \mathbb{R}^n, \quad (1)$$

where the amplitude $a : X \times \mathbb{R}^n \rightarrow \Omega^{\frac{1}{2}}$ is a symbol in the space S_{cl}^m , for some $m \in \mathbb{R}$, and the phase function ϕ is assumed to have non-negative imaginary part, $\Im\phi \geq 0$.

In the real-valued setting, ϕ is often assumed to be *non-degenerate* in the sense of Hörmander (see [2] for details on the real-valued theory). For the purpose of this paper, we follow the definition of *complex-valued non-degenerate phase function* introduced in [5]. It should be noted that for the integral (1) to be defined in the sense of oscillatory integrals, one only needs to assume that $\Re\phi$ is a non-degenerate phase function. Indeed the contribution of $\Im\phi$ to the integral is just an exponentially decreasing factor that does not affect the local computations. However, by ignoring the contributions of this factor globally, we are omitting important geometric information. Moreover, by considering complex-valued phases one obtains a broader class of distributions that can be used to model the solution of systems of PDEs for which the real-valued theory falls short. Examples of applications of this theory to systems of PDEs can be found in [5] Section 8, [8] Chapter XI and [7].

This paper is concerned with the principal symbol of Lagrangian distributions with complex phase, and the composition of Fourier integral operators (also with complex phase) under the assumption of clean intersection. Specifically, our main theorem provides an explicit description of the principal symbol $\sigma^m(A)$ of a Lagrangian distribution $A \in I_{\text{cl}}^m(X, \Lambda, \Omega^{\frac{1}{2}})$. Here Λ is a positive Lagrangian sub-manifold (see 2.7) of the complexification of the cotangent bundle of X .

Theorem 1.1. *Let $A = I(\phi, a)$, with $\phi(x, \theta)$ a complex-valued non-degenerate phase function and $a(x, \theta) \in S_{\text{cl}}^{m+(n-2N)/4}$. Then, the principal symbol $\sigma^m(A)$ of $A \in I_{\text{cl}}^m(X, \Lambda, \Omega^{\frac{1}{2}})$ defines a section of the virtual line bundle \mathcal{L} over Λ . In admissible local coordinates, $\sigma^m(A)$ takes the form*

$$\sigma^m(A) \sim \tilde{a}_0 \sqrt{d\phi} \in S^{(m+n/4)}(\Lambda, \mathcal{L}), \quad (2)$$

where a_0 is the top order term of the asymptotic expansion of a and $\sqrt{d\phi}$ is the section defined in 2.16.

Readers familiar with the real-valued theory might recognize the above description of the principal symbol. However, due to the geometric objects involved in the construction, the meaning of equation (2) differs slightly from the classical real-valued definition of the principal symbol. When ϕ is assumed to be real-valued, $\Lambda \subset T^*X \setminus 0$ is a Lagrangian manifold that can be parametrized by the phase function ϕ , and \mathcal{L} is the tensor product of the so-called Maslov line bundle and the space of half-densities $\Omega^{\frac{1}{2}}$. However, this cannot be directly carried to the complex domain. As mentioned above, we followed the approach proposed in [5] and [6]. There, the authors use *almost analytic extensions* to define Λ as an abstract, complex-valued object called *positive Lagrangian manifold*, even though it is not a Lagrangian manifold in the usual sense. In the same way, \mathcal{L} is only a virtual line bundle over Λ . Thus, here $S^{(d)}(\Lambda, \mathcal{L})$, $d = d_1 + d_2$, denotes the space of almost analytic functions of the form fs , such that s is a section of \mathcal{L} which homogeneous of degree d_1 , and $f = f(x, \theta)$ is an homogeneous function of degree d_2 in θ .

It should be noted that working with almost analytic extensions introduces some errors, and that the constructions are highly technical. For this reason, the authors in [5] define the principal symbol of a Lagrangian distribution $A \in I_{cl}^m(X, \Lambda, \Omega^{\frac{1}{2}})$ as the pre-image of A under a, rather complicated, bijection \mathcal{P} , between $\Gamma^{m+n/4}(\Lambda, \mathcal{L})$, the space of homogeneous sections of \mathcal{L} , and the set of oscillatory integrals (1) with amplitude a equal to some function defined on $\mathbb{C}^N \times \mathbb{C}^n$ and homogeneous of degree m .

To improve this description, we extend to the complex-valued case, the method presented in [1] to compute the principal symbol in the real-valued theory. First, the leading order term of the asymptotic expansion gets identified with the principal symbol of a given Lagrangian distribution. Then, we show that the resulting expression defines a section of \mathcal{L} , which is equivalent to the section $\mathcal{P}^{-1}(A)$ defined in [5]. This improved description allows us to see that, similar to the real-valued case, the principal symbol map σ^m fits into a short exact sequence

$$0 \rightarrow I_{cl}^{m-1}(X, \Lambda; \Omega^{\frac{1}{2}}) \rightarrow I_{cl}^m(X, \Lambda; \Omega^{\frac{1}{2}}) \xrightarrow{\sigma^m} S^{(m+n/4)}(\Lambda, \mathcal{L}) \rightarrow 0.$$

Theorem 1.2. *Let $A_1 \in I_{cl}^{m_1}(X \times Y, C'_1; \Omega^{\frac{1}{2}}(X \times Y))$, $A_2 \in I_{cl}^{m_2}(Y \times Z, C'_2; \Omega^{\frac{1}{2}}(Y \times Z))$ be such that the clean composition $B = A_1 \circ A_2$ defines a distribution in $I_{cl}^{m_1+m_2+e/2}(X \times Z, (C_1 \circ C_2)'; \Omega^{\frac{1}{2}}(X \times Z))$. Then,*

$$\sigma(B)^{m+e/2} \sim \int_{C_\gamma} \left(\tilde{a}_{10} \tilde{a}_{20} (|\tilde{\theta}|^2 + |\tilde{\sigma}|^2)^{\frac{-n_Y}{2}} \sqrt{d\Phi} \right) d\omega'' \in S^{(m-e/2+n/4)}(\Lambda, \mathcal{L}),$$

with $m = m_1 + m_2$, $n = n_X + n_Z$, e is the excess of the intersection and C_γ is the compact image of a point $\gamma \in (C_1 \circ C_2)_{\mathbb{R}}$ in $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap D$. Here a_{10} , a_{20} are the principal parts of the amplitudes of A_1 and A_2 , respectively.

Since the publication of [5] and [6], other authors have contributed to the theory. For instance, in [4], the author presents an alternative approach using *complex Lagrangian ideals* instead of almost analytic extensions. In [3], the same author proves necessary and sufficient conditions for an operator of order zero to be L^2 continuous. In [8], one can find a general method for solving complex eikonal equations, and thus finding the phase function of the solution operators associated to hyperbolic problems. But, to the best of our knowledge, there is no explicit formulation of the principal symbol of a Lagrangian distribution with complex phase in the literature. We were also not able to find any mention of complex-valued clean phase functions or clean composition. With this paper, we attempted to fill these gaps in the literature.

The present document is divided into three sections. The first of them consists of an overview of the theory of Lagrangian distributions with complex phase. There we collect, without proofs, the results from [5] that are necessary to achieve our goals. In the next section, we present our construction of the principal symbol map (1.1), which is complementary to the one in [5]. In the last section, we consider the case of *clean composition* and proof our main result, Theorem 1.2.

2. Previously known results

In this section, a summary of the theory of Lagrangian distributions with complex phase is presented. Since this theory is broad and technical, we focused on the definitions and theorems necessary to understand our main results. The content is completely taken from [5], but some of the statements have been reformulated to facilitate the reading process. For a complete presentation, the interested reader can see the original document [5]. For those interested in the theory due its applications, [8] is a good source. There, a nice exposition of the theory can be found, however, it does not include the construction of the principal symbol map.

We begin by clarifying some notation and recalling the almost analytic machinery. Let $\bar{\partial}_z$ denote the Cauchy-Riemann operator for $z = x + iy \in \mathbb{C}$. If f is a smooth function in \mathbb{C}^n , denote by ∂f and $\bar{\partial} f$ the operators

$$\partial f = \sum_{j=1}^n \partial_{z_j} f dz_j \quad \text{and} \quad \bar{\partial} f = \sum_{j=1}^n \bar{\partial}_{z_j} f d\bar{z}_j.$$

Definition 2.1. Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $\Omega_{\mathbb{R}} := \Omega \cap \mathbb{R}^n$. We say that:

- (1) $f \in \mathcal{C}^\infty(\Omega)$ is almost analytic if $\bar{\partial}f$ vanish to the infinite order near $\Omega_{\mathbb{R}}$. This means that, for all $z \in \Omega$, there exist a constant $C_N > 0$ such that

$$|\bar{\partial}f(z)| \leq C_N |\Im z|^N, \quad \forall N \in \mathbb{Z}_+.$$

- (2) $f_1, f_2 \in \mathcal{C}^\infty(\Omega)$ are equivalent if $f_1 - f_2$ is almost analytic. In this case, we write $f_1 \sim f_2$.
- (3) An almost analytic extension of a function $f \in \mathcal{C}^\infty(\Omega_{\mathbb{R}})$, is an almost analytic function $\tilde{f} \in \mathcal{C}^\infty(\Omega)$ such that $\tilde{f}|_{\Omega_{\mathbb{R}}} = f$.

As the equivalence relation suggests, when working with almost analytic extensions, we care about the equivalence class, not about any particular representative. It may be useful to think about these classes as the germs, in the sense of algebraic topology, of complex-valued functions at real points.

It is not hard to see that every $f \in \mathcal{S}(\mathbb{R})$ admits an almost analytic extension, see for example [9, Theorem 3.6]. In [5], a more general result is proven: Given a function $a \in S_{cl}^m(\Gamma)$, defined in a conic set $\Gamma \subseteq \mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$, there exists a unique up to equivalence almost analytic extension $\tilde{a} \in S_{cl}^m(\tilde{\Gamma})$ defined in $\tilde{\Gamma} \subseteq \mathbb{C}^n \times (\mathbb{C}^N \setminus 0)$, a complex extension of Γ .

Remark 2.2. Throughout this document, the tilde ($\tilde{\cdot}$ or $\cdot \sim$) denotes an almost analytic extension of a given object. The notation is used indiscriminately for functions (\tilde{f}), sets ($\tilde{\Gamma}$), manifolds (\tilde{M}) and vector bundles ($(T^*X \setminus 0) \sim$).

Definition 2.3. Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $M \subseteq \Omega$ a closed submanifold of real dimension $2k$. We say that M is an almost analytic manifold if for every real point $z_0 \in M$, one can find an open neighborhood \mathcal{O} of z_0 in Ω and almost analytic functions f_{k+1}, \dots, f_n such that, in \mathcal{O} ,

- M is defined by $f_{k+1} = \dots = f_n = 0$,
- and the differentials $\partial f_{k+1}(z), \dots, \partial f_n(z)$ are linearly independent over \mathbb{C} .

The following theorem gives an useful description of almost analytic manifolds. It corresponds to the third statement of a series of equivalent definitions proved in [5, Theorem 1.4].

Theorem 2.4. *Let $\Omega \subseteq \mathbb{C}^n$ be an open set and $M \subseteq \Omega$ an almost analytic manifold. Then, for every real point $z_0 \in M$, one can find a neighborhood $\mathcal{O} = \mathcal{O}' \times \mathcal{O}'' \subseteq \mathbb{C}^k \times \mathbb{C}^{n-k}$ of z_0 in M and an almost analytic function h on \mathcal{O}' such that, for all $z = (z', z'') \in \mathcal{O}$, $z'' = h(z')$.*

We also need the notion of equivalent almost analytic manifolds. This equivalence is defined *locally*, by considering only neighborhoods of real points.

Definition 2.5. Let $M_1, M_2 \subseteq \Omega \subseteq \mathbb{C}^n$ be almost analytic manifolds of the same dimension with $M_{1\mathbb{R}} = M_{2\mathbb{R}}$. Let h_1, h_2 be the defining functions from Theorem 2.4. Then, $M_1 \sim M_2$ if any of the following equivalent conditions is satisfied:

- (1) For all compact set $K \Subset \Omega$ and $N \in \mathbb{Z}_+$ there is a constant $C_{N,K} > 0$ such that,

$$|h_1(x') - h_2(x')| \leq C_{N,K} |\Im h_2(x')|^N, \quad (x', h_j(x')) \in K, \quad x' \in \mathbb{R}^k.$$

- (2) For all compact set $K \Subset \Omega$ and $N \in \mathbb{Z}_+$ there is a constant $C_{N,K} > 0$ such that,

$$|h_1(z') - h_2(z')| \leq C_{N,K} |\Im h_2(z')|^N, \quad (z', h_j(z')) \in K.$$

We now consider a key result: a complex-valued version of the so called *stationary phase formula*. This is a generalization of the well-known *stationary phase method*. For a complete presentation of the usual real-valued method see [1, Section 1.2].

For the remaining of the section, let $F(x, v)$ be a smooth function of $(x, v) \in \mathbb{R}^n \times \mathbb{R}^k$, defined in a neighborhood of $(0, 0)$. Assume that $\Im F \geq 0$ with equality only at the origin, and that

$$\partial_x F(0, 0) = 0, \quad \det(\partial_x^2 F(0, 0)) \neq 0.$$

Let $\tilde{F}(z, \omega)$, $z = x + iy$, $\omega \in \mathbb{C}^k$, be an almost analytic extension of F to a complex neighborhood of $(0, 0)$. Then, it can be shown ([5, Lemma 2.1]) that the equation $\partial_z \tilde{F}(z, \omega) = 0$ defines an almost analytic manifold M of the form $z = Z(v)$, such that there is exists $C > 0$ satisfying

$$\Im \tilde{F}(z, \omega) \geq C |\Im z|^2, \quad (z, \omega) \in M, \quad v \in \mathbb{R}^k.$$

Theorem 2.6. ([5, Theorem 2.3]) *Let F and Z be as above. Then, there are neighborhoods of the origin $U \in \mathbb{R}^n$, $V \in \mathbb{R}^k$ and differential operators $C_{\nu,w}(D)$, which are \mathcal{C}^∞ functions of $w \in V$, and have order at most 2ν , such that*

$$\int e^{itF(x,w)} u_t(x) dx \sim \sum_{\nu=0}^{\infty} t^{-\nu-n/2} e^{it\tilde{F}(Z(w),w)} (C_{\nu,w}(D) \tilde{u}_t) Z(w), \quad t \rightarrow +\infty, \tag{3}$$

in $S_{\text{cl}}^{-n/2}(V \times \mathbb{R}_+)$. Here $u_t(x) \in S^0(\mathbb{R}^n \times \mathbb{R}_+)$ is supported in $U \times \mathbb{R}_+$ and the function $(2\pi)^{-n/2} C_{0,w}$ is the branch of the square root of $(\det \frac{1}{i} \partial_z^2 F(Z(w), w))^{-1}$ which continuously deform into 1 under the homotopy

$$[0, 1] \ni s \mapsto \frac{1}{i} (1-s) \partial_z^2 F + sI \in GL(n, \mathbb{C}).$$

2.1. Positive Lagrangian manifolds and complex-valued phase functions

Let M be a real symplectic manifold of dimension $2n$, fix a point $\rho_0 \in M$ and consider a coordinate neighborhood $W \subseteq \mathbb{R}^{2n}$ of ρ_0 . Assuming that $\Lambda \subseteq \widetilde{M}$ is an almost analytic manifold containing ρ_0 , the goal is to extend the symplectic structure of M to \widetilde{M} . Note that, given symplectic coordinates (x, ξ) near ρ_0 in M , we can have coordinates in Λ by taking almost analytic extensions $(\tilde{x}, \tilde{\xi})$ to \widetilde{W} .

Definition 2.7. The manifold Λ is called positive (almost) Lagrangian if, near every real point (x_0, ξ_0) , it is equivalent to a manifold of the form

$$\tilde{\xi} = \frac{\partial h}{\partial \tilde{x}}(\tilde{x}), \quad \tilde{x} \in \mathbb{C}^n,$$

where h is an almost analytic function satisfying $\Im h \geq 0$ on \mathbb{R}^n , with equality at x_0 .

So far, we have no information on the symplectic form σ . In fact, the manifolds on which σ vanish, represent a special case.

Definition 2.8. An almost analytic manifold $\Lambda \subseteq \widetilde{M}$, of real dimension $2n$, is called strictly positive Lagrangian if

- (1) $\Lambda_{\mathbb{R}}$ is a submanifold of M .
- (2) $\sigma_{\alpha}|_{\Lambda_{\alpha}} \sim 0$ for all local representatives Λ_{α} and all local almost analytic extensions σ_{α} of σ .
- (3) $i^{-1}\sigma(v, v) > 0$ for all $v \in T_{\rho}(\Lambda) \setminus (T_{\rho}(\Lambda_{\mathbb{R}}))^{\sim}$, $\rho \in \Lambda_{\mathbb{R}}$.

In practice, we will consider $M = T^*X \setminus 0$, for some manifold X , and $\Lambda \subseteq (T^*(X) \setminus 0)^{\sim}$ positive Lagrangian. Under these conditions, it can be shown that Λ is of the form $\tilde{x} = H(\tilde{\xi})$, where H is positive homogeneous of degree 0 with $\Im H(\xi) \leq 0$ for ξ real.

Definition 2.9. A complex-valued function $\phi(x, \theta)$, smooth in a conic set $V \subseteq \mathbb{R}^n \times \mathbb{R}^N \setminus 0$, is called a non-degenerate phase function of positive type if $\Im \phi \geq 0$, $d\phi \neq 0$ and

- ϕ is homogeneous of degree 1 in θ ,
- the differentials $\left\{d\left(\frac{\partial \phi}{\partial \theta_j}\right)\right\}_{j=1}^N$ are linearly independent over \mathbb{C} on the real set

$$C_{\phi_{\mathbb{R}}} = \{(x, \theta) \in V : \phi'_{\theta} = 0\}.$$

Let $\tilde{\phi}(\tilde{x}, \tilde{\theta})$ be an almost analytic extension of ϕ , defined in a conic neighborhood $W \subseteq \mathbb{C}^n \times (\mathbb{C}^N \setminus 0)$ of the real point $(x_0, \theta_0) \in V$. Then the critical set

$$C_{\tilde{\phi}} = \left\{ (\tilde{x}, \tilde{\theta}) \in W : \partial_{\tilde{\theta}} \tilde{\phi}(\tilde{x}, \tilde{\theta}) = 0 \right\}$$

is a conic almost analytic manifold of dimension $2n$. The image $\Lambda_{\tilde{\phi}}$ of $C_{\tilde{\phi}}$ under the map

$$C_{\tilde{\phi}} \ni (\tilde{x}, \tilde{\theta}) \mapsto \left(\tilde{x}, \partial_{\tilde{x}} \tilde{\phi}(\tilde{x}, \tilde{\theta}) \right) \in \Lambda_{\tilde{\phi}} \quad (4)$$

is locally, near $\rho_0 = (x_0, \phi'_x(x_0, \theta_0))$, a conic positive Lagrangian manifold of dimension $2n$. Moreover the image of $C_{\tilde{\phi}_{\mathbb{R}}}$ is precisely $\Lambda_{\tilde{\phi}_{\mathbb{R}}}$. It is important to mention that $\Lambda_{\tilde{\phi}_{\mathbb{R}}} \subseteq \mathbb{R}^n \times (\mathbb{R}^N \setminus 0)$ does not need to be a closed Lagrangian manifold in the usual sense. In some cases, it may not even be a manifold. This highlights the advantages of allowing complex-valued phase functions over the usual real-valued theory.

Throughout this document, we should keep in mind that we are dealing with an equivalent class of almost analytic manifolds, thus it is always possible to choose a different representative for the class $\Lambda_{\tilde{\phi}}$. We refer to this process as a *re-parametrization*, because it amounts to finding a new non-degenerate phase function equivalent to $\tilde{\phi}$. In fact, a positive Lagrangian manifold $\Lambda \subseteq (T^*(X) \setminus 0)^\sim$ can always be parametrized by a non-degenerate phase function of the form

$$\psi(x, \xi) = x \cdot \xi - g(\xi),$$

for some almost analytic function g which homogeneous of degree 1 and satisfies $\Im g(\xi) \leq 0$, for ξ real. Moreover, in the process described above, any other choice of almost analytic extension $\tilde{\phi}$ defines an equivalent almost analytic manifold. For this reason, we often omit the tilde and simply write Λ_{ϕ} instead of $\Lambda_{\tilde{\phi}}$.

2.2. Lagrangian distributions and their principal symbol

Let $V \subseteq \mathbb{R}^n \times \mathbb{R}^N \setminus 0$ be a conic open set, $\phi \in \mathcal{C}^\infty(V)$ be a complex-valued non-degenerate phase function, and $a \in S_{\text{cl}}^m(\mathbb{R}^n \times \mathbb{R}^N \setminus 0)$ be supported in a closed conic subset of V . We can formally define the distribution $A = I(\phi, a) \in \mathcal{D}'(\mathbb{R}^n)$ by

$$I(\phi, a) = \int e^{i\phi(x, \theta)} a(x, \theta) d\theta. \quad (5)$$

Note that the contribution of $\Im \phi$ to the integral is an exponentially decreasing factor, which does not contribute to the singularities of the distribution. Thus we get, directly from the real-valued case, a description of the wave front set of A :

$$\text{WF}(A) \subseteq \{(x, \phi'_x(x, \theta)) : (x, \theta) \in \text{supp}(a) \cap C_{\phi_{\mathbb{R}}}\} \subseteq \Lambda_{\phi_{\mathbb{R}}}. \quad (6)$$

Let X be a \mathcal{C}^∞ paracompact manifold of dimension n and denote by $\mathcal{D}'(X; \Omega^{\frac{1}{2}})$ be the space of 1/2-densities in X . From this point forward, unless stated otherwise, $\Lambda \subseteq (T^*(X) \setminus 0)^\sim$ denotes a positive Lagrangian manifold.

Definition 2.10 (Lagrangian distributions). A distribution $A \in \mathcal{D}'(X; \Omega^{\frac{1}{2}})$ belongs to the class of Lagrangian distributions of order m , $I_{cl}^m(X, \Lambda; \Omega^{\frac{1}{2}})$, if $\text{WF}(A) \subseteq \Lambda_{\mathbb{R}}$ and there are functions $a \in S_{cl}^{m+(n-2N)/4}(\mathbb{R}^n \times \mathbb{R}^N)$ and $\phi \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^N)$ such that

- for every $\rho_0 = (x_0, \xi_0) \in \Lambda_{\mathbb{R}}$ and every choice of local coordinates, A is microlocally of the form $I(\phi, a)$ near ρ_0 ,
- Λ is parametrized by ϕ , that is $\Lambda \sim \Lambda_\phi$ near ρ_0 ,
- $\text{supp}(a)$ is contained in a small conic neighborhood of $(x_0, \theta_0) \in C_{\phi_{\mathbb{R}}}$.

For the global theory of Lagrangian distributions with complex phase to be complete, one would have to define the principal symbol of $A \in I^m(X, \Lambda; \Omega^{\frac{1}{2}})$ invariantly. In analogy with the real case, one might be tempted to define the principal symbol as section of the tensor product of the bundle of 1/2-densities in Λ and the Maslov line bundle. But it turns out that it is impossible to replicate this construction for complex manifolds in a way that is invariant under coordinate changes. To avoid this difficulty, the authors in [5] introduce *admissible coordinates* and define a *virtual* line bundle over Λ . This is an intricate construction, so we first introduce the linear situation. The content of this sub-section is a summary of [5, Section 6].

Let \widetilde{M} be the complex extension of a real symplectic vector space M of dimension $2n$, and denote by \mathcal{L}^- the set of negative definite Lagrangian planes in M . Let $F \subseteq M$ be a fixed real Lagrangian plane and \widetilde{F} its complex extension. We denote by $B(F)$ the set of all real bases of F .

Definition 2.11 (Admissible basis). Let $N \subseteq \widetilde{M}$ be a positive semi-definite Lagrangian plane. A basis $e = \{e_1, \dots, e_n\}$ of N is said to be admissible if there exist a basis $f = \{f_1, \dots, f_n\}$ of F and a plane $L \in \mathcal{L}^-$ such that, for each j , e_j is the projection of f_j along L . We write

$$e = E(f, L) = E_N(f, L), \quad (f, L) \in B(F) \times \mathcal{L}^-$$

and denote by $\mathcal{B}(N)$ the set of all admissible bases for N .

Proposition 2.12. *The set $\mathcal{B}(N)$ is the union of two disjoint arcwise-connected subsets. Two admissible bases $e = E(f, L)$, $e' = E(f', L')$ belong to the same set if and only if $f, f' \in B(F)$ have the same orientation. Moreover, there exists a unique function $s = s_N : \mathcal{B}(N) \times \mathcal{B}(N) \rightarrow \mathbb{C} \setminus 0$ with the following properties*

- (1) *For all compact set $K \subseteq B(F) \times \mathcal{L}^-$, $s(e, e')$ restricted to $E_N(K) \times E_N(K)$ is a continuous function of e, e' and N .*

- (2) If $e, e', e'' \in \mathcal{B}(N)$, then $s(e, e')s(e', e'') = s(e, e'')$.
- (3) If e, e' have the same $L \in \mathcal{L}^-$, then $s(e, e') > 0$.
- (4) $s^2(e, e') = \pm e/e'$ with the plus sign precisely when f, f' have the same orientation. Here,

$$e/e' = e_1 \wedge \dots \wedge e_n / e'_1 \wedge \dots \wedge e'_n, \quad e = E(f, L), e' = E(f', L') \in \mathcal{B}(N).$$

Consider now a positive Lagrangian manifold $\Lambda \subseteq (T^*(X) \setminus 0)^\sim$ and $\rho \in \Lambda_{\mathbb{R}}$. Then, $M = T_\rho(T^*X)$ and $N = T_\rho(\Lambda)$ satisfy the conditions of Definition 2.11. Namely, $T_\rho(T^*X)$ is a symplectic vector space and $T_\rho(\Lambda) \subset T_\rho(T^*X)$ is a positive semi-definite Lagrangian plane. Taking $F \subseteq M$ as the tangent space to the fiber, we can define $\mathcal{B}(T_\rho(\Lambda))$ as above.

Keeping in mind the previous notation, we now define admissible coordinate systems on Λ . Denote by $E_{T_\rho(\Lambda)}(S)$ the set of all admissible bases $e = E(f, L)$ of $T_\rho(\Lambda)$ with $(f, L) \in S \subseteq B(F) \times \mathcal{L}^-$.

Definition 2.13 (Admissible coordinate systems). Let $\lambda = \{\lambda_1, \dots, \lambda_n\}$ be almost analytic functions on Λ , defined in some complex neighborhood \mathcal{U}^λ of a real point. We say that $\lambda_1, \dots, \lambda_n$ are admissible coordinates on Λ if

- (1) The differentials $d\lambda_1, \dots, d\lambda_n$ are linearly independent over \mathbb{C} at real points.
- (2) $\delta\lambda = \{\delta\lambda_1, \dots, \delta\lambda_n\}$ belongs locally to $E_{T_\rho(\Lambda)}(K)$ with $\rho \in \mathcal{U}^\lambda \cap \Lambda_{\mathbb{R}}$, for some compact set $K \subseteq B(F) \times \mathcal{L}^-$. Here $\delta\lambda$ is the dual basis of $d\lambda$ in $T_\rho(\Lambda)^*$.

We refer to the neighborhoods \mathcal{U}^λ as admissible coordinate systems.

One can show that it is always possible to find admissible coordinates locally, however, this construction is not unique. It is precisely this property what allows us to define the almost analytic version of the Maslov line bundle. At this point, we are only missing a description of the transition functions between two coordinate systems \mathcal{U}^λ and \mathcal{U}^μ . Thanks to Proposition 2.12, we know that

- $s(\delta\lambda, \delta\mu)$ is continuous in $\mathcal{U}^\lambda \cap \mathcal{U}^\mu \cap \Lambda_{\mathbb{R}}$.
- $s^2 = \pm \frac{d\mu}{d\lambda} = \pm \det \left[\left(\frac{\partial \mu_j}{\partial \lambda_k} \right)_{j,k} \right]$, where $\partial \mu_j / \partial \lambda_k$ is defined by

$$d\mu_j = \sum_k \left(\frac{\partial \mu_j}{\partial \lambda_k} \right) d\lambda_k + \sum_k \left(\frac{\partial \mu_j}{\partial \bar{\lambda}_k} \right) d\bar{\lambda}_k.$$

Consider now an almost analytic extension \mathbf{S} of $s(\delta\lambda, \delta\mu)$, defined in a small complex neighborhood of $\mathcal{U}^\lambda \cap \mathcal{U}^\mu \cap \Lambda_{\mathbb{R}}$ in Λ . Thanks to the previous properties, \mathbf{S} can be chosen to satisfy

$$(\mathbf{S}_{\lambda,\mu})^2 \sim \pm \frac{d\mu}{d\lambda}, \tag{7}$$

$$\mathbf{S}_{\lambda,\lambda} \sim 1, \quad \mathbf{S}_{\lambda,\mu} \mathbf{S}_{\mu,\omega} \sim \mathbf{S}_{\lambda,\omega}. \tag{8}$$

Additionally, the functions $\mathbf{S}_{\lambda,\mu}$ are continuous under small perturbations of λ, μ for which $\delta\lambda, \delta\mu$ stay in the same component of $E_{T_\rho(\Lambda)}(K)$. Therefore, the functions $\mathbf{S}_{\lambda,\mu}$ are the ideal choice of transition functions in the new almost analytic Maslov line bundle.

Definition 2.14. The virtual line bundle $\mathcal{L} \rightarrow \Lambda$ is defined as the family of admissible coordinate systems \mathcal{U}^λ on Λ with transition functions $\mathbf{S}_{\lambda,\mu}$. A section $f \in \Gamma(\Lambda, \mathcal{L})$ is an almost analytic function on Λ such that, the restriction to each \mathcal{U}^λ satisfy

$$f|_{\mathcal{U}^\lambda} \sim \mathbf{S}_{\lambda,\mu} f|_{\mathcal{U}^\mu}.$$

The space of homogeneous section of degree m is denote by $\Gamma^m(\Lambda, \mathcal{L})$.

Remark 2.15. Given $t \in \mathbb{R}_+$, denote by $\mathbf{t} : \Lambda \rightarrow \Lambda$ the multiplication by t in the second coordinate. Then, if $\lambda = \{\lambda_1, \dots, \lambda_n\}$ are admissible coordinates near $\mathbf{t}\rho \in \Lambda_{\mathbb{R}}$, the pullback

$$\mathbf{t}^*\lambda = \{\lambda_1 \circ \mathbf{t}, \dots, \lambda_n \circ \mathbf{t}\}$$

defines admissible coordinates near ρ . The following facts are important when considering homogeneous sections of \mathcal{L} :

- (1) By definition, every section $f \in \Gamma(\Lambda, \mathcal{L})$ satisfies

$$f_{\mathbf{t}^*\lambda} \sim t^{n/2} f_\lambda.$$

- (2) A section $f \in \Gamma(\Lambda, \mathcal{L})$ is homogeneous of degree m if for all $\rho \in \Lambda_{\mathbb{R}}$, all $t \in \mathbb{R}_+$ and all coordinates λ near $\mathbf{t}\rho$ it holds

$$f_{\mathbf{t}^*\lambda} \sim t^m \mathbf{t}^*(f_\lambda), \text{ near } \rho.$$

A particularly important homogeneous section is the one determined by a phase function ϕ . The following lemma is a crucial step in defining the principal symbol of a distribution in $I_{cl}^m(X, \Lambda; \Omega^{\frac{1}{2}})$.

Lemma 2.16. *Let $\phi(x, \theta)$ be a non-degenerate phase function that parametrizes Λ near $\rho_0 \in \Lambda_{\mathbb{R}}$. Then, there is a section $\sqrt{d\phi} \in \Gamma^{N/2}(\Lambda, \mathcal{L})$, defined by*

$$(\sqrt{d\phi})_\tau \sim \left[\det \frac{1}{i} \begin{pmatrix} \tilde{\phi}''_{xx} - \tilde{\psi}''_{xx} & \tilde{\phi}''_{x\theta} \\ \tilde{\phi}''_{\theta x} & \tilde{\phi}''_{\theta\theta} \end{pmatrix} \right]^{-1/2}, \tag{9}$$

where $\tau = \tilde{\xi} - \tilde{\psi}'_x$ is an admissible coordinate system on Λ , and $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ satisfy $\tilde{\psi}''_{xx} < 0$. The branch of the square root is chosen as in Theorem 2.6.

This Lemma can be found in [5], as part of the proof of Theorem 6.4.

We now recall the definition of Fourier integral operator with complex phase. Let X, Y be paracompact \mathcal{C}^∞ manifolds of dimension n_X, n_Y respectively. As with the real case, if $C \subseteq (T^*X \setminus 0)^\sim \times (T^*Y \setminus 0)^\sim$ is an arbitrary submanifold, we denote by C' the manifold

$$\{(x, y, \xi, -\eta) : (x, \xi, y, \eta) \in C\} \subseteq (T^*(X \times Y) \setminus 0)^\sim.$$

Definition 2.17. The submanifold $C \subseteq ((T^*X \setminus 0) \times (T^*Y \setminus 0))^\sim$ is a (strictly) positive canonical relation, if $C' \subseteq (T^*(X \times Y) \setminus 0)^\sim$ is a closed conic (strictly) positive Lagrangian manifold and $C_{\mathbb{R}} \in (T^*X \setminus 0) \times (T^*Y \setminus 0)$.

Here *closed* means that $C'_{\mathbb{R}}$ is a closed set of $T^*(X \times Y) \setminus 0$. Finally, we recall the definition of Fourier integral operator with complex phase.

Definition 2.18. An operator $A : \mathcal{C}_0^\infty(Y; \Omega^{\frac{1}{2}}) \rightarrow \mathcal{D}'(X; \Omega^{\frac{1}{2}})$ is called a Fourier integral operator with complex phase if its distributional kernel K_A belongs to the class of Lagrangian distributions $I_{cl}^m(X \times Y, \Lambda; \Omega^{\frac{1}{2}})$, where the closed conic positive Lagrangian manifold $\Lambda \subseteq (T^*(X \times Y) \setminus 0)^\sim$ satisfies $C' = \Lambda$ for some $C \subseteq ((T^*X \setminus 0) \times (T^*Y \setminus 0))^\sim$. We write $A \in I_{cl}^m(X \times Y, C; \Omega^{\frac{1}{2}})$.

It follows from the real case that, whenever C is a canonical relation, the operator A maps

$$\mathcal{C}_0^\infty(Y; \Omega^{\frac{1}{2}}) \rightarrow \mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}).$$

Moreover, A can be extended to a continuous operator from $\mathcal{E}'(Y; \Omega^{\frac{1}{2}})$ to $\mathcal{D}'(X; \Omega^{\frac{1}{2}})$.

3. The principal symbol map

The following construction shows how the complex-valued stationary phase formula can be used to provide an explicit description of the principal symbol of a Lagrangian distribution with complex phase. In particular, given a distribution $A \in I_{cl}^m(X, \Lambda; \Omega^{\frac{1}{2}})$, which is microlocally of the form $I(\phi, a)$ near some real point ρ , we are able to see the relation between the amplitude a and the action of the map \mathcal{P} defined in [5, Theorem 6.4].

We follow the ideas from [1] for the real-valued case and adapt them to the complex domain. Namely, we use the asymptotic expansion in Theorem 2.6 to provide a local description of the principal symbol. Later, we show that this description is equivalent to the definition provided in [5].

Lemma 3.1. *Let $\phi \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ be a non-degenerate phase function and $(x_0, \theta_0) \in C_{\phi_{\mathbb{R}}}$ fixed. If $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is real valued and*

$$\psi(x_0) = 0, \quad \psi'_x(x_0) = \phi'_x(x_0, \theta_0), \quad \psi''_{xx} < 0.$$

Then, the function $F(x, \theta) = \phi(x, \theta) - \psi(x)$ satisfies the assumptions of Theorem 2.6 around (x_0, θ_0) .

Proof. It is clear that $F(x, \theta)$ is smooth with $\Im F \geq 0$. From the definition of $C_{\phi\mathbb{R}}$, one sees that $\Im F(x_0, \theta_0) = 0$. Thus, we only need to show that

$$\partial_x F(x_0, \theta_0) = 0, \quad \det(\partial_x^2 F(x_0, \theta_0)) \neq 0.$$

A short computation shows that

$$\partial_x F(x_0, \theta_0) = \phi'_x(x_0, \theta_0) - \psi'_x(x_0) = 0, \quad \partial_x^2 F = \begin{pmatrix} \phi''_{xx} - \psi''_{xx} & \phi''_{x\theta} \\ \phi''_{\theta x} & \phi''_{\theta\theta} \end{pmatrix}.$$

This matrix is the same one in the definition of $\sqrt{d\phi}$ (Lemma 2.16). Then, the result follows by the same arguments, see [5], Theorem 6.4. \square

Let X be a smooth manifold of dimension n and consider a distribution $A \in I_{cl}^m(X, \Lambda; \Omega^{\frac{1}{2}})$, which is of the form $I(\phi, a)$ microlocally near $\rho_0 = (x_0, \xi_0)$, for $\xi_0 = (x_0, \phi'_x(x_0, \theta_0))$, where ϕ is a non-degenerate phase function and $a(x, \theta) \in S_{cl}^{m+(n-2N)/4}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$. Let $u \in C_c^\infty$ be supported in a neighborhood of (x_0, θ_0) and ψ be as in the previous lemma. We want to understand the asymptotic behaviour of $I := (I(\phi, a), v)_{L^2}$, as $t \rightarrow \infty$, with $v(x) = e^{-it\psi(x)}u(x)$. By definition,

$$\begin{aligned} (I(\phi, a), v)_{L^2} &= \int e^{i\phi(x, \theta)} a(x, \theta) e^{-it\psi(x)} u(x) \, dx d\theta \\ &= \int e^{i[\phi(x, \theta) - t\psi(x)]} a(x, \theta) u(x) \, dx d\theta. \end{aligned}$$

After the change of variables $\theta = t\eta$, we obtain

$$I = t^N \int e^{itF(x, \eta)} u_t(x, \eta) \, d\eta \, dx,$$

with $F = \phi - \psi$ and $u_t = a(x, t\eta)u(x)$. Thanks to the previous lemma, we know that the complex-valued stationary formula can be applied here. Thus we get from Theorem 2.6,

$$e^{-it\tilde{F}(Z(\tilde{\eta}), \tilde{\theta})} I \sim \sum_{\nu=0}^{\infty} t^{-\nu-(n+N)/2} (C_{\nu, \eta}(D) \tilde{u}_t) Z(\tilde{\eta}),$$

where $x = Z(\tilde{\eta})$ is the almost analytic manifold described by $\partial_x \tilde{F}(\tilde{x}, \tilde{\eta}) = 0$. We can now describe the principal symbol of A as the map that assigns to each ψ the top order term of the asymptotic expansion of I . From the proof of

Theorem 2.6, we know that $C_{0,w}(D) = (2\pi)^{\frac{n+N}{2}} \sqrt{d\phi}$. We now use this to write the principal symbol map explicitly

$$\mathcal{T}_A : \psi \mapsto (2\pi)^{\frac{n+N}{2}} \tilde{a}_0(Z(\tilde{\eta}), \tilde{\eta}) \tilde{u}(Z(\tilde{\eta})) \sqrt{d\phi}, \quad (10)$$

where $\sqrt{d\phi} \in \Gamma^{N/2}(\Lambda, \mathcal{L})$ and a_0 , the highest order term in the asymptotic sum of a , is and homogeneous function of degree $m + (n - 2N)/4$ in η . The final step is to relate this expression with the definition of the principal symbol given in [5]. Or, equivalent, we prove our main result Theorem 1.1, which is stated again below.

Let $s \in \Gamma^{d_1}(\mathcal{L}, \Lambda)$ and b an homogeneous function of degree d_2 in θ . In the rest of the document, we denote by $S^{(d)}(\Lambda, \mathcal{L})$, with $d = d_1 + d_2$, the space of almost analytic functions $f \sim bs$.

Theorem 3.2. *Let $A = I(\phi, a)$, with $\phi(x, \theta)$ a non-degenerate phase function and $a(x, \theta) \in S_{cl}^{m+(n-2N)/4}$. Then, the principal symbol of $A \in I_{cl}^m(X, \Lambda, \Omega^{\frac{1}{2}})$ is equivalent to the homogeneous section*

$$\sigma^m(A) \sim \tilde{a}_0 \sqrt{d\phi} \in S^{(m+n/4)}(\Lambda, \mathcal{L}), \quad (11)$$

where a_0 is the top order term of the asymptotic expansion of a .

Proof. Let be \mathcal{P} be the bijection defined [5, Theorem 6.4]. The result would follow after showing that, $\mathcal{T}_A(\psi) \sim \mathcal{P}^{-1}(A)$ as sections of \mathcal{L} . But first, we need to verify that $\mathcal{T}_A(\psi) \in \Gamma^{m+n/4}(\Lambda, \mathcal{L})$.

It is not hard to see that almost analytic homogeneous functions define homogeneous sections of \mathcal{L} . Indeed, let g be an almost analytic function in Λ , homogeneous of degree m in ξ . We know that for each $\rho \in \Lambda_{\mathbb{R}}$, it is possible to find an admissible coordinate system \mathcal{U}^λ , locally near ρ . Then, we can define a function f such that $f_\lambda = g|_{\mathcal{U}^\lambda}$, which is, by definition, a section of \mathcal{L} . Consider now $\rho = (x, \xi) \in \Lambda_{\mathbb{R}}$ fixed and let $t \in \mathbb{R}_+$. Then, near $t\rho$, points in Λ are of the form $(x, t\xi)$. Let $\lambda = \{\lambda_1, \dots, \lambda_n\}$ and $\mu = (\mathbf{t}^{-1})^* \lambda$ be admissible coordinates near ρ and $t\rho$, respectively. Explicitly, for each $j = 1, \dots, n$,

$$\mu_j(x, t\xi) = (\lambda_j \circ \mathbf{t}^{-1})(x, t\xi) = \lambda_j(x, \xi).$$

Then, near ρ , we have $(\mathbf{t}^{-1})^*(f_\lambda)(x, \xi) \sim f(x, \frac{1}{t}\xi) \sim t^{-n/2} f_\lambda(x, \xi)$ and

$$\begin{aligned} f_{(\mathbf{t}^{-1})^* \lambda} &\sim f(x, t\xi) = t^m f(x, \xi) \\ &= t^{n/2} t^{-n/2} t^m f(x, \xi) \sim t^{m-n/2} f_\lambda(x, \xi) \sim t^m (\mathbf{t}^{-1})^*(f_\lambda). \end{aligned}$$

Which, by definition, means that $f \in \Gamma^m(\Lambda, \mathcal{L})$. This, together with the local identification of C_ϕ and $\Lambda \sim \Lambda_\phi$, allow us to interpret \tilde{a}_0 and \tilde{u} as elements of $\Gamma^{m+(n-2N)/4}(\Lambda, \mathcal{L})$ and $\Gamma^0(\Lambda, \mathcal{L})$, respectively. Then, it follows that the right hand side of (10) defines an homogeneous section of degree $m + n/4$.

Recall that a distribution $A \in I_{cl}^m$ is locally given by an oscillatory integral

$$A = I(\phi, a) = \int e^{i\phi(x,\eta)} a(x, \eta) \, d\eta,$$

where $a \in S_{cl}^{m+(n-2N)/4}$. We can assume, without loss of generality, that $\mathcal{P}(s) = I(\phi, \bar{b}_0)$ where b_0 is the top-order term of the asymptotic sum of some $b \in S_{cl}^{m+(n-2N)/4}$ and s is some section in $\Gamma^{N/2}(\Lambda, \mathcal{L})$. Thus, it remains to show that $\mathcal{P} \circ \mathcal{T}_A$ is the identity in the set of oscillatory integrals $I(\phi, b_0)$, with b_0 as above.

The action of \mathcal{P} , tells us that if $s \sim b\sqrt{d\phi} \in \Gamma^{m+n/4}(\Lambda, \mathcal{L})$, then $\mathcal{P}(s) = I(\phi, \bar{b})$ with \bar{b} an extension of b to $\mathbb{C}^n \times \mathbb{C}^N$. Then, taking $s = \mathcal{T}_A(\psi) \sim \tilde{a}_0 \tilde{u} \sqrt{d\phi}$, we have $\mathcal{P}(s) = I(\phi, \bar{b})$ with $b = \tilde{a}_0 \tilde{u}$. Since this is valid for any $u \in C_c^\infty$ and any almost analytic extensions, we conclude that

$$\mathcal{P}(s) \sim I(\phi, \tilde{a}_0).$$

Or, equivalently, $\mathcal{P}(\mathcal{T}_A(\psi)) \sim I(\phi, \tilde{a}_0)$, where a_0 is the highest order term of the amplitude a in the local representation $A = I(\phi, a)$, which concludes the proof. \(\checkmark\)

It is now easy to see that the principal symbol map fits into a short exact sequence. First note that σ^m maps

$$\sigma^m : I_{cl}^m(X, \Lambda; \Omega^{\frac{1}{2}}) \rightarrow S^{(m+n/4)}(\Lambda, \mathcal{L}). \tag{12}$$

Moreover, the fact that \mathcal{P} is bijective, implies that σ^m is surjective. Thus, similarly to the real-valued case, the map σ^m fits into a short exact sequence

$$0 \rightarrow I_{cl}^{m-1}(X, \Lambda; \Omega^{\frac{1}{2}}) \rightarrow I_{cl}^m(X, \Lambda; \Omega^{\frac{1}{2}}) \xrightarrow{\sigma^m} S^{(m+n/4)}(\Lambda, \mathcal{L}) \rightarrow 0.$$

4. Clean composition

As mentioned before, for the calculus of FIOs to be complete, one needs to provide conditions for the composition to stay in the class. In [5], the case of *transverse composition* was considered. We wish to relax this condition, so we consider the case of *clean composition*. To do so, we first need to introduce a more general type of phase function that we were not able to find in the existing literature, even though it is a natural generalization of the non-degenerate case.

Definition 4.1. A complex-valued function $\phi(x, \theta)$, smooth in a conic set $V \subset \mathbb{R}^n \times \mathbb{R}^N \setminus 0$, is called clean phase function of positive type if $\Im\phi \geq 0$, $d\phi \neq 0$ and

- ϕ is homogeneous of degree 1 in θ ,

- there exist $M \leq N$, such that M of the differentials $\left\{d\left(\frac{\partial\phi}{\partial\theta_j}\right)\right\}_{j=1}^N$ are linearly independent over \mathbb{C} on

$$C_{\phi\mathbb{R}} = \{(x, \theta) \in V : \phi'_\theta = 0\}.$$

The number $e = N - M$ is called the excess of ϕ .

It is possible to re-organize the variables so that θ can be splitted as $(\theta', \theta'') \in \mathbb{R}^M \times \mathbb{R}^e$, where the differentials $\left\{d(\partial\phi/\partial\theta'_j)\right\}_{j=1}^M$ are the ones satisfying the definition. We will sometimes refer to θ'' as the *excess variables*. As usual, we denote by Λ_ϕ the manifold

$$\left\{(\tilde{x}, \partial_{\tilde{x}}\tilde{\phi}(\tilde{x}, \tilde{\theta})) \in \mathbb{C}^n \times \mathbb{C}^N \setminus 0 : (\tilde{x}, \tilde{\theta}) \in C_{\tilde{\phi}}\right\},$$

for some almost analytic extension $\tilde{\phi}$ of ϕ to a complex extension of V . One can easily verify that Λ_ϕ is a positive Lagrangian manifold of real dimension $2n$. As such, it can be parameterized by a non-degenerate phase function,

$$\tilde{\psi}(\tilde{x}, \tilde{\xi}) = \tilde{x}\tilde{\xi} - g(\tilde{\xi}), \quad (\tilde{x}, \tilde{\xi}) \in \mathbb{C}^n \times (\mathbb{C}^n \setminus 0) \quad (13)$$

for some almost analytic function g with $\Im g \leq 0$ at $\xi \in \mathbb{R}^n \setminus 0$. Denoting by ψ the restriction of $\tilde{\psi}$ to the real domain, we see that Λ_ϕ and Λ_ψ are equivalent in the sense of almost analytic manifolds. This equivalence allows us to associate distributions in I_{cl}^m with a microlocal representation $I(\phi, a)$, where ϕ is a clean phase function instead of a non-degenerate one.

Proposition 4.2. *Let $\phi(x, \theta) \in \mathcal{C}^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ be a clean phase function of excess e . Then, for $a \in S^{m+(n-2N)/4}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$, the oscillatory integral $I(\phi, a)$ defines a Fourier distribution of order $m + e/2$.*

Proof. Fix a point $(x_0, \theta_0) \in C_{\phi\mathbb{R}}$ and let $\xi_0 = \phi'_x(x_0, \theta_0)$. We know that, near (x_0, ξ_0) , the almost analytic manifold $\Lambda \sim \Lambda_\phi$ is equivalent to a manifold Λ_ψ , with ψ the non-degenerate phase function (13). We wish to follow the construction in [5, Theorem 4.2] to show that there exist an amplitude $b \in S^{m+(n-2N')/4}(\mathbb{R}^n \times (\mathbb{R}^{N'} \setminus 0))$, $N' = n$, such that the oscillatory integrals $I(\phi, a)$ and $I(\psi, b)$ are microlocally equivalent near (x_0, ξ_0) . Which implies that $I(\phi, a)$ defines a Fourier distribution.

The construction is based on the proof of the complex-valued stationary phase formula (Theorem 2.6). There, it was possible to apply the result with respect to the variables (x, θ) , because the phase functions were assumed to be non-degenerate. This is no longer true for a clean phase function ϕ . Instead, we need to consider $\theta = (\theta', \theta'') \in (\mathbb{R}^M \times \mathbb{R}^e) \setminus 0$, and

$$I(\phi, a) \sim \int \left(\int e^{i\phi(x, \theta', \theta'')} a(x, \theta', \theta'') d\theta' \right) d\theta''. \quad (14)$$

Since the differentials $\{d(\partial\phi/\partial\theta'_j)\}$ are linearly independent over \mathbb{C} at real points, we can apply the stationary phase formula to the inner integral in (14). The rest of the argument in [5, Theorem 4.2] applies without further modification. The desired conclusion follows after integrating out the excess variables θ'' .

Finally, note that applying the stationary formula in e variables less, increases the order of the distribution by $e/2$. Indeed, the asymptotic sum (3) would belong to $S^{-(n+N)/2+e/2}$ instead of $S^{-(n+N)/2}$, as it was the case in [5]. \square

Since the construction that leads to our description of the principal symbol is also based on the stationary phase formula, equation (11) applies only to the inner integral in (14). In other words, the principal symbol of the distribution $A = I(\phi, a)$ above should be given by the integral, with respect to θ'' , of the principal symbol of the inner distribution. But, for this to be correctly defined, we first need to modify the definition of $\sqrt{d\phi}$.

Lemma 4.3. *Let $\phi(x, \theta)$ be a clean phase function with excess e that parameterizes Λ . Then, there is a section $\sqrt{d\phi} \in \Gamma^{(N-e)/2}(\mathcal{L}, \Lambda)$, defined by*

$$(\sqrt{d\phi})_\tau \sim \left[\det \frac{1}{i} \begin{pmatrix} \tilde{\phi}''_{xx} - \tilde{\psi}''_{xx} & \tilde{\phi}''_{x\theta'} \\ \tilde{\phi}''_{\theta'_x} & \tilde{\phi}''_{\theta'\theta'} \end{pmatrix} \right]^{-1/2}, \tag{15}$$

where θ'' are the excess variables in the splitting $\theta = (\theta', \theta'')$. Here τ, ψ and the branch of the square root are chosen as in Lemma 2.16.

Proof. Note that, for θ'' fixed, ϕ defines a non-degenerate phase function with respect to the variables (x, θ') . Then, it follows from Lemma 2.16, that $\sqrt{d\phi}$ defines a section of \mathcal{L} . Since the matrix in (15) is now of dimension $(n + N - e) \times (n + N - e)$, we see that $\sqrt{d\phi}$ is homogeneous of degree $(N - e)/2$, which completes the proof. \square

With this new meaning for $\sqrt{d\phi}$, we can apply Theorem 1.1 to the inner integral in equation (14). It follows that $\tilde{a}_0\sqrt{d\phi}$ defines an element of $S^{(m-e+n/4)}(\Lambda, \mathcal{L})$ but we still need to integrate out the excess variables θ'' . In principle, this integral may not be defined. Thus, similar to the real case, we restrict the domain of integration.

Let $\pi : \Lambda_\phi \rightarrow \mathbb{C}^n$ be the projection $\pi(\tilde{x}, \tilde{\xi}) = \tilde{\xi}$. The composition of π with the map (4) defines a fiber bundle over Λ with fiber

$$C_{\tilde{\xi}} = \left\{ (\tilde{x}, \tilde{\theta}) : \partial_{\tilde{\theta}} \tilde{\phi}(\tilde{x}, \tilde{\theta}) = 0, \partial_{\tilde{x}} \tilde{\phi}(\tilde{x}, \tilde{\theta}) = \tilde{\xi} \right\}.$$

The fiber $C_{\tilde{\xi}}$ can be interpreted as an almost analytic manifold of dimension $2e$, if the differentials $\left\{ d\left(\frac{\partial\phi}{\partial x_j}\right) \right\}_{j=1}^n$ are linearly independent at real points. In any

case, we can compute $\int_{C_{\xi}} \tilde{a}_0 \sqrt{d\phi} d\theta''$ if we assume that $C_{\xi\mathbb{R}}$, the restriction of C_{ξ} to the real domain, is compact.

Definition 4.4. Let $\phi(x, \theta) \in \mathcal{C}^{\infty}(\mathbb{R}^n \times (\mathbb{R}^N \setminus 0))$ be a clean phase function with excess e , such that the set $C_{\xi\mathbb{R}}$ is compact, and $a(x, \theta) \in S_{\text{cl}}^{m+(n-2N-2e)/4}$. Then, the principal symbol of $A \in I_{\text{cl}}^m(X, \Lambda; \Omega^{\frac{1}{2}})$ is

$$\sigma^{m+e/2}(A) = \int_{C_{\xi\mathbb{R}}} \tilde{a}_0 \sqrt{d\phi} d\theta'' \in S^{(m-e+n/4)}(\Lambda, \mathcal{L}). \quad (16)$$

where $A = I(\phi, a)$ and a_0 denotes the top order term of the asymptotic expansion of a .

Now let X, Y, Z be paracompact \mathcal{C}^{∞} manifolds of dimension n_X, n_Y, n_Z respectively. Suppose that

$$A_1 \in I_{\text{cl}}^{m_1}(X \times Y, \Lambda_1; \Omega^{\frac{1}{2}}) \quad \text{and} \quad A_2 \in I_{\text{cl}}^{m_2}(Y \times Z, \Lambda_2; \Omega^{\frac{1}{2}}),$$

are properly supported operators, where $C_j = \Lambda_j'$ are positive canonical relations. Denote by Δ_Y the subspace

$$\{(y, y) \in Y \times Y\}.$$

Set $D = T^*X \times \Delta_{T^*Y} \times T^*Z$ and let \tilde{D} be its almost analytic extension. We consider the following condition

Assumption 1. Suppose that:

- (1) The intersection $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap D$ is clean with excess e .
- (2) The natural projection $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \rightarrow (T^*X \setminus 0) \times (T^*Z \setminus 0)$ is injective and proper.

Note that, whenever the excess is equal to zero, the intersection is actually transverse. In other words, after showing that under Assumption 1, the composition of Lagrangian distributions stays in the class; the composition theorem in [5] follows as a particular case.

The following results follow from arguments similar to those in [5, Proposition 7.1], so we only present the parts where the proofs are different.

Proposition 4.5. Let $C_1 \in (T^*(X \times Y) \setminus 0)^{\sim}$, $C_2 \in (T^*(Y \times Z \setminus 0))^{\sim}$ be positive canonical relations satisfying Assumption 1. Then, there exist a manifold $(C_1 \circ C_2)'$, parameterized by a clean phase function Φ , such that $(C_1 \circ C_2)_{\mathbb{R}} = (C_{1\mathbb{R}} \circ C_{2\mathbb{R}})$. The excess of Φ is equal to the excess of the intersection.

Proof. Since the $\Lambda_j = C'_j$, $j = 1, 2$, are almost Lagrangian manifolds, there are coordinates such that, near the real points $(x_0, \xi_0, y_0, -\eta_0) \in \Lambda_{1\mathbb{R}} = C'_{1\mathbb{R}}$ and $(y'_0, \eta'_0, z_0, \zeta_0) \in \Lambda_{2\mathbb{R}} = C'_{2\mathbb{R}}$, the manifolds are given by the vanishing of

$$\tilde{x} - \frac{\partial H_1}{\partial \tilde{\xi}}(\tilde{\xi}, \tilde{\eta}), \quad \tilde{y} + \frac{\partial H_1}{\partial \tilde{\eta}}(\tilde{\xi}, \tilde{\eta}); \quad \tilde{y}' - \frac{\partial H_2}{\partial \tilde{\eta}'}(\tilde{\eta}', \tilde{\zeta}), \quad \tilde{z} + \frac{\partial H_2}{\partial \tilde{\zeta}}(\tilde{\eta}', \tilde{\zeta}).$$

Moreover, the intersection $(C_1 \times C_2) \cap \tilde{D}$ is completely describe by the previous functions and

$$\tilde{y} = \tilde{y}' \quad \tilde{\eta} = \tilde{\eta}'.$$

Its tangent plane is given by the vanishing of the differentials of all these functions. Clean intersection means that $T(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap TD$ is described by the equations

$$\begin{aligned} d\left(x - \frac{\partial H_1}{\partial \xi}(\xi, \eta)\right) &= 0, & d\left(y + \frac{\partial H_1}{\partial \eta}(\xi, \eta)\right) &= 0, & d(y - y') &= 0, \\ d\left(y' - \frac{\partial H_2}{\partial \eta'}(\eta', \zeta)\right) &= 0, & d\left(z + \frac{\partial H_2}{\partial \zeta}(\eta', \zeta)\right) &= 0, & d(\eta - \eta') &= 0, \end{aligned} \tag{17}$$

and has dimension $n_X + n_Z + e$, where e is the excess of the intersection. As in the transversal case, define $C_1 \circ C_2$ as the manifold that satisfy $(C_1 \circ C_2)_{\mathbb{R}} = (C_{1\mathbb{R}} \circ C_{2\mathbb{R}})$ where

$$C_{1\mathbb{R}} \circ C_{2\mathbb{R}} = \{((x, y, \xi, \eta), (y', z, \eta', \zeta)) \in C_{1\mathbb{R}} \times C_{2\mathbb{R}} : y = y', \eta + \eta' = 0\}$$

can be identified with $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap D$. The main difference is that now the manifold $\Lambda = (C_1 \circ C_2)'$ is of dimension $n_X + n_Z + e$. Suppose now that Λ_1 and Λ_2 are parameterized by the regular phase functions

$$\phi_1(x, y, \xi, \eta) = x \cdot \xi - y \cdot \eta + H_1(\xi, \eta), \quad \phi_2(y, z, \eta, \zeta) = y \cdot \eta - z \cdot \zeta + H_2(\eta, \zeta).$$

Consider now the function

$$\Phi(x, z, \omega) = \phi_1(x, y, \xi, \eta) + \phi_2(y, z, \eta, \zeta), \quad \omega = \omega(y, \xi, \eta, \zeta),$$

where ω is some function homogeneous of degree 1. The previous analysis shows that Φ defines a clean phase function with excess e , because the differentials $d(\partial\Phi/\partial\omega_j)$ are exactly those in (17). Then, the excess of the phase function Φ is

$$\dim \Lambda - (n_X + n_Z) = e.$$

Finally, note that there is a one-to-one correspondence between $C_{1\mathbb{R}} \circ C_{2\mathbb{R}}$ and

$$C_{\Phi\mathbb{R}} = \{((x, y, \xi, \eta), (y, z, \eta, \zeta)) \in C_{\phi_1} \times C_{\phi_2} : \partial_y(\phi_1 + \phi_2) = 0\},$$

so the manifold $\Lambda = (C_1 \circ C_2)'$ can be parameterized by a the clean phase function Φ . \(\checkmark\)

We now present an extension of the composition theorem to the case of clean intersection. The proof is omitted because it follows the same arguments as Theorem 7.3 in [5], but using the canonical transformation given by Proposition 4.5. The order of the distribution follows from Proposition 4.2.

To make our notation consistent with the one used in [5], from now on we put $\theta := (\xi, \eta)$ and $\sigma := (\eta, \zeta)$. Then, we consider Φ as a clean phase function depending on $(x, z, y, \theta, \sigma)$, that is

$$\Phi(x, z, \omega) = \phi_1(x, y, \theta) + \phi_2(y, z, \sigma), \quad \omega = \omega(y, \theta, \sigma). \tag{18}$$

An standard choice for the function ω is $\omega(y, \theta, \sigma) = (y(\theta^2 + \sigma^2)^{1/2}, \theta, \sigma)$.

Theorem 4.6. *Let $C_1 \subset (T^*(X \times Y) \setminus 0)^\sim$, $C_2 \subset (T^*(Y \times Z) \setminus 0)^\sim$ be positive canonical relations satisfying Assumption 1. Suppose that the operators $A_1 \in I_{cl}^{m_1}(X \times Y, C_1'; \Omega^{\frac{1}{2}})$ and $A_2 \in I_{cl}^{m_2}(Y \times Z, C_2'; \Omega^{\frac{1}{2}})$ are properly supported. Then*

$$A_1 \circ A_2 \in I_{cl}^m(X \times Z, (C_1 \circ C_2)'; \Omega^{\frac{1}{2}}), \quad m = m_1 + m_2 + e/2,$$

where e is the excess of the intersection.

Assuming that A_1 and A_2 have local representations $I(\phi_1, a_1)$ and $I(\phi_2, a_2)$, respectively, one can see that $B = A_1 \circ A_2$ is, modulo \mathcal{C}^∞ , locally given by $I(\Phi, b)$. Where Φ is the clean phase function in equation (18), and the amplitude $b \in S_{cl}^{m+(n_X+n_Z-2N)/4}(X \times Z \times \mathbb{R}^N \setminus 0)$ is of the form

$$b(x, z, \omega) = a_1(x, y, \theta)a_2(y, z, \sigma)(\theta^2 + \sigma^2)^{-n_Y/2},$$

with $m = m_1 + m_2$ and $N = N_1 + N_2 + n_Y$.

Remark 4.7. For the amplitude b , the estimates of the symbol class $S_{cl}^{m'}(X \times Z \times \mathbb{R}^N \setminus 0)$, are taken with respect to (x, z, ω) , with ω the homogeneous function above.

It is then clear that we should be able to write the principal symbol of B in terms of $\sigma(A_1)$ and $\sigma(A_2)$. The next theorem shows precisely this in the case of transverse composition.

Theorem 4.8. *Let $A_1 \in I_{cl}^{m_1}(X \times Y, C_1'; \Omega^{\frac{1}{2}})$ and $A_2 \in I_{cl}^{m_2}(Y \times Z, C_2'; \Omega^{\frac{1}{2}})$ satisfy Assumption 1 with excess $e = 0$, and denote by B the composition $A_1 \circ A_2 \in I_{cl}^{m_1+m_2}(X \times Z, (C_1 \circ C_2)'; \Omega^{\frac{1}{2}})$. Then,*

$$\sigma(B) \sim \tilde{a}_{10}\tilde{a}_{20}(|\tilde{\theta}|^2 + |\tilde{\sigma}|^2)^{\frac{-n_Y}{2}} \sqrt{d\Phi} \in \Gamma^{m+n/4}(\Lambda, \mathcal{L}), \tag{19}$$

with $m = m_1 + m_2$ and $n = n_X + n_Z$. Here $\tilde{a}_{10}, \tilde{a}_{20}$ are the principal parts of the amplitudes of A_1 and A_2 , respectively.

Proof. Since $e = 0$, the intersection is transverse and Φ is a non-degenerate phase function. Then $\sqrt{d\Phi}$ is defined according to Lemma 2.16. The definition of the principal symbol applied to $B = A_1 \circ A_2$, tells us that

$$\sigma^m(B) \sim \tilde{b}_0 \sqrt{d\Phi}, \tag{20}$$

with $b = (a_1)(a_2)(\theta^2 + \sigma^2)^{-\frac{n_Y}{2}}$ the amplitude of B . The difficulty of the proof lies in verifying that $\sqrt{d\Phi}$ defines a section in $\Gamma(\mathcal{L}, \Lambda)$, for $\Lambda = (C_1 \circ C_2)'$. Luckily, all the necessary steps were proved in [5]. First of all, notice that in general, the square of a section $\sigma \in \Gamma(\mathcal{L}, \Lambda)$ defines, up to a sing, an almost analytic form on Λ of maximal degree. That is

$$\sigma^2 \sim \pm \omega, \quad \text{for some } n\text{-form } \omega.$$

By construction (see Lemma 2.16) we know that, given a phase function ϕ , there is a n -form $\omega = d\phi$, moreover

$$\sigma^2 \sim \pm d\phi, \quad \text{for } \sigma = \sqrt{d\phi}. \tag{21}$$

On the other hand, the proof of Theorem 7.5 in [5] states that

$$d\phi_1 \wedge d\phi_2 \sim \pm (\theta^2 + \sigma^2)^{-n_Y} d\Phi \wedge \Omega$$

defines an almost analytic form on $\Lambda = (C_1 \circ C_2)'$, and that

$$\alpha_1^2 \wedge \alpha_2^2 \sim \pm \alpha^2 \wedge \Omega, \quad \text{for some } \Omega,$$

here $\alpha_1, \alpha_2, \alpha$ are the principal symbols of A_1, A_2 , and B , respectively. Combing these two facts with the expressions for α_1, α_2 according to Theorem 1.1, we get

$$\begin{aligned} \alpha_1^2 \wedge \alpha_2^2 &\sim \pm (\tilde{a}_{10}^2 d\phi_1) \wedge (\tilde{a}_{20}^2 d\phi_2) \sim \pm (\tilde{a}_{10}\tilde{a}_{20})^2 d\phi_1 \wedge d\phi_2 \\ &\sim \pm \left(\tilde{a}_{10}\tilde{a}_{20}(\theta^2 + \sigma^2)^{-\frac{n_Y}{2}} \right)^2 d\Phi \wedge \Omega, \end{aligned}$$

Then, $\alpha^2 \sim \pm b_0^2 d\Phi$. From equation (21), we see that $\alpha \sim b_0 \sqrt{d\Phi}$. The result follow from equation (20) and the definition of b_0 . \square

Whenever Assumption 1 is satisfied with $e > 0$, the principal symbol of the resulting operator is given as in Definition 4.4, with $\sqrt{d\Phi}$ defined according to Lemma 4.3. However, to compute the principal symbol we need further assumptions. We take advantage of the identification between $C_1 \circ C_2$ and $(C_1 \times C_2) \cap \tilde{D}$. Denote by C the positive canonical relation $C_1 \circ C_2 \subseteq (T^*X \setminus 0) \sim \times (T^*Z \setminus 0) \sim$. Then, the image C_γ of a point $\gamma \in C_{\mathbb{R}}$ in $(C_{1\mathbb{R}} \times C_{2\mathbb{R}}) \cap D$, defines a fiber of dimension e over γ . We can now prove our second main result, Theorem 1.2.

Theorem 4.9. Let $A_1 \in I_{\text{cl}}^{m_1}(X \times Y, C_1; \Omega^{\frac{1}{2}})$ and $A_2 \in I_{\text{cl}}^{m_2}(Y \times Z, C_2; \Omega^{\frac{1}{2}})$ satisfy the assumptions of Theorem 4.6, and suppose that, for $\gamma \in C_{\mathbb{R}}$, the set C_γ is compact. Then, the principal symbol of $B = A_1 \circ A_2 \in I_{\text{cl}}^{m+e/2}(X \times Z, C; \Omega^{\frac{1}{2}})$, $m = m_1 + m_2$, is

$$\sigma^{m+e/2}(B) \sim \int_{C_\gamma} \tilde{a}_{10} \tilde{a}_{20} (\theta^2 + \sigma^2)^{-\frac{n_Y}{2}} \sqrt{d\Phi} dy'' d\theta'' d\sigma'' \in S^{(m-e/2+n/4)}(\Lambda, \mathcal{L}), \quad (22)$$

with $n = n_X + n_Z$ and $\sqrt{d\Phi}$ defined as in Lemma 4.3.

Proof. This is a direct consequence of Theorem 4.8 and Definition 4.4. We only need to compute the order of homogeneity. Since $\sqrt{d\Phi}$ is defined according to Lemma 4.3, it is homogeneous of degree $(N - e)/2$, with $N = n_Y + N_1 + N_2$ and e the excess of the intersection. Then,

$$\tilde{a}_{10} \tilde{a}_{20} (\theta^2 + \sigma^2)^{-\frac{n_Y}{2}} \sqrt{d\Phi} \in S^{(m'')}(\Lambda, \mathcal{L})$$

for $m'' = m' + (N - e)/2 = m_1 + m_2 - e/2 + (n_X + n_Z)/4$. Equation (22) follows after integration with respect to the excess variables $\omega'' = \omega''(y, \theta, \sigma)$. Note that, it is possible to organize the variables in a way that $\omega'' = (y'', \theta'', \sigma'') \in \mathbb{R}^e$, for some splitting $y = (y', y'')$, $\theta = (\theta', \theta'')$ and $\sigma = (\sigma', \sigma'')$. \square

Remark 4.10. Proposition 4.5 implies that $\Lambda \sim (C_1 \circ C_2)'$ is a positive Lagrangian manifold. As such, it can be parametrized by a non-degenerate phase function. This re-parametrization could potentially save us the difficulty of working with the excess variables, but formula (22) would no longer be valid. We need to keep in mind that working with a new phase function changes the amplitude in the local representation of the distribution. Instead of the previous formulation of the principal symbol, we would obtain from Theorem 4.8 a section of \mathcal{L} which is equivalent (as section of the virtual line bundle) to the one presented above.

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