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Vainikko operator on discrete Morrey spaces

El operador de Vainikko en espacios de Morrey discretos

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Abstract. We prove boundedness of a discrete version of Vainikko operator on discrete Morrey spaces. We also show that the commutator of this Vainikko operator with a multiplication operator by an element of a discrete version of BMO is bounded on these spaces.

Key words and phrases. discrete Morrey; Vainikko operator; commutator.

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Resumen. Probamos que una versión discreta del operador de Vainikko en espacios de Morrey discretos es acotado. También probamos que el conmutador de este operador de Vainikko con un operador de multiplicación discreto de tipo BMO es acotado en espacios de Morrey discretos.

Palabras y frases clave. Espacios de Morrey discretos, operador de Vainikko, conmutador.

1. Introduction

Discrete Morrey spaces were introduced by Gunawan, Kikianty and Schwanke in [6]. Since then, these spaces have been studied by several authors like [2], [5], [7], [8] and [9]. In all these articles have been obtained important results about boundedness of some operators acting on these spaces: discrete versions of the Hardy-Littlewood maximal operator, discrete Riesz potentials, discrete Hilbert transform, convolution and multiplication operators, and commutators.

Our goal in this article is to show the boundedness of a discrete version of the Vainikko operator introduced in [12] and [13], on discrete Morrey spaces. Additionally, we also consider the commutator of the Vainikko operator and a multiplication operator by an element of a discrete BMO-type space. We prove under appropriate conditions that this commutator is also bounded on discrete Morrey spaces. All of this work is done in the last two sections of this paper.

We will use standard notation along this note, and as usual, we shall denote by C a constant that could be changing line by line.

2. Discrete Morrey Spaces

Throughout this section we will asume that p and q are two real numbers such that $1 \leq p \leq q < \infty$.

The discrete Morrey spaces for dimension $n = 1$ were defined by Gunawan et al. (see [6]) in the following way:

The set

$$l_q^p = \left\{ x = (x_k)_{k\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}} : \|x\|_{l_q^p} < \infty \right\}, \tag{1}$$

where

$$\|x\|_{l_q^p} = \sup_{m\in\mathbb{Z},\ N\in\mathbb{N}_0} \frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}} \left(\sum_{k=m-N}^{m+N} |x_k|^p \right)^{1/p},$$

is called a discrete Morrey space. Here $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$.

As in [6], we will denote by $S_{m,N}$ the set $\{m-N, m-N+1, ..., m+N\}$.

The authors of [6] also consider other generalizations of discrete Morrey spaces, however, we will focus our attention in the version given by (1).

It turns out that $\left(l_q^p, \|\cdot\|_{l_q^p}\right)$ is a Banach space such that $l_p^p = l^p$ (see [6]), the classical space of p-summable sequences indexed by $\mathbb{Z}$. Notice here that the equality $l_p^p = l^p$ is an isometric isomorphism, since it is very easy to check that the norms $\|\cdot\|_{l_p^p}$ and $\|\cdot\|_{l^p}$ coincide. Also, the space l_q^p can be viewed as a subspace of the continuous Morrey space $M_q^p(\mathbb{R})$ equipped with the norm

$$\|f\|_{M_q^p} = \sup_{a\in\mathbb{R},\ r>0} \frac{1}{r^{\frac{1}{p}-\frac{1}{q}}} \left(\int_{a-r}^{a+r} |f(y)|^p \, dy \right)^{1/p},$$

if we consider a sequence $(x_k)_{k\in\mathbb{Z}} \in l_q^p$ as a step function

$$\sum_{k\in\mathbb{Z}} x_k \chi_{\left[k-\frac{1}{2}, k+\frac{1}{2}\right)}.$$

However, as in [6], we will consider here l_q^p as a space on its own.

In the paper [7], the authors consider the Hardy-Littlewood maximal operator M on the space l_q^p. This operator M is defined for each $m \in \mathbb{Z}$ by

$$Mx(m) = \sup_{N\in\mathbb{N}_0} \frac{1}{2N+1} \sum_{k=m-N}^{m+N} |x_k|. \tag{2}$$

They also proved that this operator is bounded from l_q^p into itself for $1 < p \leq q < \infty$.

3. Vainikko operator on l_q^p spaces

The Vainikko operator V_φ is formally defined as follows:

For Lebesgue measurable functions $f, \varphi : (0, \infty) \to \mathbb{R}$

$$V_\varphi f(x) = \int_0^\infty f(tx)\, \varphi(t)\, dt. \tag{3}$$

Under appropriate conditions for f and φ, this operator acts continuously on different functional spaces (see for example, [3], [12], [13]).

Here, we will consider the following discrete version of (3):

Let $f : \mathbb{Z} \to \mathbb{R}$ and $\varphi : \mathbb{N} \to (0, \infty)$ be sequences indexed by $\mathbb{Z}$ and $\mathbb{N}$, respectively. The discrete Vainikko operator V_φ^d is formally defined as

$$\left(V_\varphi^d f\right)(k) = \sum_{n=1}^{\infty} \varphi(n)\, f(kn)$$

for each $k \in \mathbb{Z}$.

We are interested in the behavior of this operator on discrete Morrey spaces. The corresponding result is as follows.

Theorem 3.1. *Assume $1 < p < q < \infty$ and let $\varphi : \mathbb{N} \to (0, \infty)$ be a sequence of positive real numbers. Then, the operator V_φ^d is bounded from l_q^p into itself if*

$$\sum_{n=1}^{\infty} \varphi(n)\, n^{\frac{1}{p} - \frac{1}{q}} < \infty. \tag{4}$$

In such case, the norm of the operator satisfies

$$\sum_{n=1}^{\infty} \varphi(n)\, n^{-\frac{1}{q}} \leq \left\| V_\varphi^d \right\|_{l_q^p \to l_q^p} \leq \sum_{n=1}^{\infty} \varphi(n)\, n^{\frac{1}{p} - \frac{1}{q}}.$$

Proof. Let us assume first that condition (4) is true. Given any $m \in \mathbb{Z}$, any $N \in \mathbb{N}_0$ and $f \in l_q^p$, we can use Minkowski's inequality for integrals to obtain

$$\begin{aligned}
&\frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}}\left[\sum_{k=m-N}^{m+N}\left|\left(V_\varphi^d f\right)(k)\right|^p\right]^{1/p} \\
&= \frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}}\left[\sum_{k=m-N}^{m+N}\left|\sum_{n=1}^{\infty}\varphi(n)\, f(kn)\right|^p\right]^{1/p} \\
&\leq \sum_{n=1}^{\infty}\frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{k=m-N}^{m+N}|f(kn)|^p\right)^{1/p}\varphi(n) \\
&\leq \sum_{n=1}^{\infty}\frac{n^{\frac{1}{p}-\frac{1}{q}}}{(2Nn+n)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l=mn-Nn}^{mn+Nn}|f(l)|^p\right)^{1/p}\varphi(n) \\
&\leq \sum_{n=1}^{\infty}\frac{1}{(2Nn+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l=mn-Nn}^{mn+Nn}|f(l)|^p\right)^{1/p} n^{\frac{1}{p}-\frac{1}{q}}\varphi(n) \\
&\leq \|f\|_{l_q^p}\sum_{n=1}^{\infty}\varphi(n)\, n^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

It follows that

$$\left\|V_\varphi^d f\right\|_{l_q^p} \leq \left(\sum_{n=1}^{\infty}\varphi(n)\, n^{\frac{1}{p}-\frac{1}{q}}\right)\|f\|_{l_q^p}. \tag{5}$$

To prove the second assertion, for $\varepsilon > 0$ let us consider the sequence $f_\varepsilon : \mathbb{Z} \to \mathbb{R}$ defined by

$$f_\varepsilon(n) = \begin{cases} |n|^{-\varepsilon} & \text{if} \quad n \neq 0 \\ 0 & \text{if} \quad n = 0 \end{cases}.$$

Now, setting $\alpha = \frac{1}{p} - \frac{1}{q}$, notice that for every $m \in \mathbb{Z}$, $N \in \mathbb{N}_0$, and $\varepsilon \neq \frac{1}{p}$

$$\begin{aligned}
&(2N+1)^{-\alpha p} \sum_{k=m-N}^{m+N} |(f_\varepsilon)(k)|^p \\
&\leq 2(2N+1)^{-\alpha p} \sum_{k=1}^{N} k^{-\varepsilon p} \\
&= 2(2N+1)^{-\alpha p} + 2(2N+1)^{-\alpha p} \sum_{k=2}^{N} k^{-\varepsilon p} \\
&\leq 2(2N+1)^{-\alpha p} + 2(2N+1)^{-\alpha p} \sum_{k=2}^{N} \int_{k-1}^{k} x^{-\varepsilon p} dx \\
&\leq 2(2N+1)^{-\alpha p} + 2(2N+1)^{-\alpha p} \int_{1}^{N} x^{-\varepsilon p} dx \\
&= 2(2N+1)^{-\alpha p} + 2(2N+1)^{-\alpha p} \frac{1}{1-\varepsilon p} \left(N^{1-\varepsilon p} - 1\right). \qquad (6)
\end{aligned}$$

The expression in (6) is bounded by a constant M when $1 - \varepsilon p - \alpha p \leq 0$, that is, when $\varepsilon \geq \frac{1}{q}$.

The previous calculations show that $f_\varepsilon \in l_q^p$ for $\varepsilon \geq \frac{1}{q}$ and $\varepsilon \neq \frac{1}{p}$. Moreover, for $n \in \mathbb{Z} - \{0\}$

$$\begin{aligned}
V_\varphi^d f_\varepsilon(n) &= \sum_{k=1}^{\infty} \varphi(k) f_\varepsilon(kn) \\
&= \left(\sum_{k=1}^{\infty} \varphi(k) k^{-\varepsilon}\right) |n|^{-\varepsilon} \\
&= \left(\sum_{k=1}^{\infty} \varphi(k) k^{-\varepsilon}\right) f_\varepsilon(n).
\end{aligned}$$

Hence, for any $\varepsilon \geq \frac{1}{q}$ and $\varepsilon \neq \frac{1}{p}$

$$\left\|V_\varphi^d\right\|_{l_q^p \to l_q^p} \geq \frac{\left\|V_\varphi^d f_\varepsilon\right\|_{l_q^p}}{\left\|f_\varepsilon\right\|_{l_q^p}} = \sum_{k=1}^{\infty} \varphi(k) k^{-\varepsilon}. \qquad (7)$$

In particular, taking $\varepsilon = \frac{1}{q}$ we have,

$$\left\|V_\varphi^d\right\|_{l_q^p \to l_q^p} \geq \sum_{k=1}^{\infty} \varphi(k) k^{-\frac{1}{q}}$$

From (5) and (7) it follows that

$$\sum_{k=1}^{\infty} \varphi(k)\, k^{-\frac{1}{q}} \leq \left\| V_{\varphi}^{d} \right\|_{l_q^p \to l_q^p} \leq \sum_{k=1}^{\infty} \varphi(k)\, k^{\frac{1}{p}-\frac{1}{q}}.$$

This concludes the proof. ☑

Question: Does the converse hold in the previous theorem?

4. Commutators

In this section we prove the boundedness of the commutator $\left[b, V_{\varphi}^{d}\right]$ on discrete Morrey spaces.

We remind the reader that if T_1 and T_2 are operators acting on a space of functions X, the commutator $[T_1, T_2]$ is defined by

$$[T_1, T_2](f) = T_1 T_2(f) - T_2 T_1(f),$$

for every $f \in X$.

We will assume as in the previous section that φ is a sequence of positive real numbers indexed by $\mathbb{N}$. The function b will be a sequence of real numbers indexed by $\mathbb{Z}$ that belongs to the space $BMO^s(\mathbb{Z})$ for some $1 < s < \infty$, that is,

$$\|b\|_{BMO^s(\mathbb{Z})} := \sup_{m\in\mathbb{Z},\ N\in\mathbb{N}_0} \left(\frac{1}{2N+1} \sum_{k\in S_{m,N}} \left| b(k) - b_{S_{m,N}} \right|^s \right)^{1/s} < \infty,$$

where

$$b_{S_{m,N}} := \frac{1}{2N+1} \sum_{k\in S_{m,N}} b(k).$$

Notice that for $1 < s \leq t < \infty$ we have that $BMO^t(\mathbb{Z}) \subset BMO^s(\mathbb{Z})$ since

$$\left(\frac{1}{2N+1} \sum_{k\in S_{m,N}} \left| b(k) - b_{S_{m,N}} \right|^s \right)^{1/s} \leq \left(\frac{1}{2N+1} \sum_{k\in S_{m,N}} \left| b(k) - b_{S_{m,N}} \right|^t \right)^{1/t}.$$

There are many references for these spaces in the continuous case, see for example, [11] or [4].

The following Lemma will be very useful to prove the desired boundedness.

Lemma 4.1. Let $1 < r < \infty$, $\varphi : \mathbb{N} \to (0, \infty)$ and assume that

$$\sum_{n=1}^{\infty} n^{1+1/r} \varphi(n) < \infty.$$

Let $b \in BMO^{r'}(\mathbb{Z})$, where $\frac{1}{r} + \frac{1}{r'} = 1$. Then, for any $f : \mathbb{Z} \to \mathbb{R}$ we have the inequality

$$M\left(\left[b, V_\varphi^d\right](f)\right)(m) \leq \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} \left(n^{1/r} + n + n^{1+1/r}\right) \varphi(n) \left(M\left(f^r\right)(mn)\right)^{1/r}, \tag{8}$$

for every $m \in \mathbb{Z}$. Here M denotes the discrete Hardy-Littlewood maximal operator defined in (2).

Proof. Let $k \in \mathbb{Z}$. Then

$$\begin{aligned} \left[b, V_\varphi^d\right](f)(k) &= b(k) V_\varphi^d(f)(k) - V_\varphi^d(bf)(k) \\ &= \sum_{n=1}^{\infty} \left(b(k) - b(kn)\right) f(kn) \varphi(n). \end{aligned}$$

Thus, for any $N \in \mathbb{N}_0$ and $m \in \mathbb{Z}$

$$\begin{aligned} \frac{1}{2N+1} \sum_{k \in S_{m,N}} \left|\left[b, V_\varphi^d\right](f)(k)\right| &\leq \frac{1}{2N+1} \sum_{n=1}^{\infty} \sum_{k \in S_{m,N}} \left|b(k) - b(kn)\right| \left|f(kn)\right| \varphi(n) \\ &\leq I_1 + I_2 + I_3 \end{aligned}$$

where

$$I_1 := \frac{1}{2N+1} \sum_{n=1}^{\infty} \sum_{k \in S_{m,N}} \left|b(k) - b_{S_{m,N}}\right| \left|f(kn)\right| \varphi(n)$$

$$I_2 := \frac{1}{2N+1} \sum_{n=1}^{\infty} \sum_{k \in S_{m,N}} \left|b_{S_{m,N}} - b_{S_{mn,Nn}}\right| \left|f(kn)\right| \varphi(n)$$

and

$$I_3 := \frac{1}{2N+1} \sum_{n=1}^{\infty} \sum_{k \in S_{m,N}} \left|b_{S_{mn,Nn}} - b(kn)\right| \left|f(kn)\right| \varphi(n).$$

Let us estimate I_1.

Using Hölder's inequality we get

$$
\begin{aligned}
I_1 \le \sum_{n=1}^{\infty} & \left(\frac{1}{2N+1} \sum_{k \in S_{m,N}} \left| f(kn) \right|^r \right)^{1/r} \\
& \times \left(\frac{1}{2N+1} \sum_{k \in S_{m,N}} \left| b(k) - b_{S_{m,N}} \right|^{r'} \right)^{1/r'} \varphi(n) \\
& \le \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} n^{1/r} \varphi(n) \left(\frac{1}{(2N+1)\,n} \sum_{l \in S_{mn,Nn}} \left| f(l) \right|^r \right)^{1/r} \\
& \le \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} n^{1/r} \varphi(n) \left(\frac{1}{(2Nn+1)} \sum_{l \in S_{mn,Nn}} \left| f(l) \right|^r \right)^{1/r} \\
& \le \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} n^{1/r} \varphi(n) \left(M(f^r)(mn) \right)^{1/r}. \qquad (9)
\end{aligned}
$$

Similarly, for I_3 we obtain

$$
\begin{aligned}
I_3 \le \sum_{n=1}^{\infty} & n^{\frac{1}{r} + \frac{1}{r'}} \varphi(n) \left(\frac{1}{2Nn+1} \sum_{l \in S_{mn,Nn}} \left| f(l) \right|^r \right)^{1/r} \\
& \times \left(\frac{1}{2Nn+1} \sum_{k \in S_{m,N}} \left| b(k) - b_{S_{m,N}} \right|^{r'} \right)^{1/r'} \\
& \le \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} n \varphi(n) \left(M(f^r)(mn) \right)^{1/r}. \qquad (10)
\end{aligned}
$$

Concerning I_2 we have

$$\begin{aligned}
I_2 &\leq \sum_{n=1}^{\infty}\left(\frac{1}{2N+1}\sum_{k\in S_{m,N}}|f(kn)|^r\right)^{1/r} \\
&\quad\times\left(\frac{1}{2N+1}\sum_{k\in S_{m,N}}\left|b_{S_{m,N}}-b_{S_{mn,Nn}}\right|^{r'}\right)^{1/r'}\varphi(n) \\
&\leq \sum_{n=1}^{\infty} n^{1/r}\varphi(n)\left(\frac{1}{2Nn+1}\sum_{l\in S_{mn,Nn}}|f(l)|^r\right)^{1/r} \\
&\quad\times\left(\frac{1}{2N+1}\sum_{k\in S_{m,N}}\left|b_{S_{m,N}}-b_{S_{mn,Nn}}\right|^{r'}\right)^{1/r'} \\
&\leq \sum_{n=1}^{\infty}\left(n^{1/r}\varphi(n)\left(M(f^r)(mn)\right)^{1/r}\right) \\
&\quad\times\left(\frac{1}{2N+1}\sum_{k\in S_{m,N}}\left|b_{S_{m,N}}-b_{S_{mn,Nn}}\right|^{r'}\right)^{1/r'}. \qquad (11)
\end{aligned}$$

Now, notice that

$$\begin{aligned}
\left|b_{S_{m,N}}-b_{S_{mn,Nn}}\right| &\leq \frac{1}{2N+1}\sum_{k\in S_{m,N}}\left|b(k)-b_{S_{mn,Nn}}\right| \\
&\leq \frac{n}{2Nn+1}\sum_{k\in S_{mn,Nn}}\left|b(k)-b_{S_{mn,Nn}}\right| \\
&\leq n\left(\frac{1}{2Nn+1}\sum_{k\in S_{mn,Nn}}\left|b(k)-b_{S_{mn,Nn}}\right|^{r'}\right)^{1/r'} \\
&\leq n\|b\|_{BMO^{r'}(\mathbb{Z})}. \qquad (12)
\end{aligned}$$

It follows from (12) that the right hand side of the inequality (11) can be estimated by

$$\sum_{n=1}^{\infty}\left(n^{1+1/r}\varphi(n)\left(M(f^r)(mn)\right)^{1/r}\right)\|b\|_{BMO^{r'}(\mathbb{Z})}$$

and therefore

$$I_2 \leq \sum_{n=1}^{\infty}\left(n^{1+1/r}\varphi(n)\left(M(f^r)(mn)\right)^{1/r}\right)\|b\|_{BMO^{r'}(\mathbb{Z})}. \qquad (13)$$

Hence, from (9), (10) and (13) we get for $N \in N_0$, $m \in \mathbb{Z}$

$$\frac{1}{2N+1} \sum_{k \in S_{m,N}} \left| \left[b, V_\varphi^d\right](f)(k) \right|$$
$$\leq \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} \left(n^{1/r} + n + n^{1+1/r} \right) \varphi(n) \left(M(f^r)(mn) \right)^{1/r}.$$

Finally, taking supremum over $N \in \mathbb{N}_0$ we obtain the inequality (8). ☑

Now, we are ready to prove the main result of this section. Before achieving this task, we recall that the operator M defined in (2) is bounded from $l^p(\mathbb{Z})$ into itself, for $1 < p < \infty$ (see, for example [10]).

Theorem 4.2. *Let $1 < p < q < \infty$, $1 < r < p$, and $\varphi : \mathbb{N} \to (0, \infty)$. Assume that $b \in BMO^{r'}(\mathbb{Z})$ where $\frac{1}{r} + \frac{1}{r'} = 1$, and*

$$\sum_{n=1}^{\infty} n^{1+\frac{1}{r}+\frac{1}{p}-\frac{1}{q}} \varphi(n) < \infty.$$

Then, the commutator $\left[b, V_\varphi^d\right]$ is a bounded operator from l_q^p into itself. Moreover,

$$\left\| \left[b, V_\varphi^d\right] \right\|_{l_q^p \to l_q^p} \leq C \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} n^{1+\frac{1}{r}+\frac{1}{p}-\frac{1}{q}} \varphi(n). \tag{14}$$

Proof. From Lemma 4.1 we have for every $m \in \mathbb{Z}$

$$M\left(\left[b, V_\varphi^d\right](f)\right)(m) \leq \|b\|_{BMO^{r'}(\mathbb{Z})} \sum_{n=1}^{\infty} \left(n^{1/r} + n + n^{1+1/r} \right) \varphi(n) \left(M(f^r)(mn) \right)^{1/r}.$$

Thus, using Minkowski's inequality for integrals and the boundedness of M on $l^{p/r}(\mathbb{Z})$, we obtain for every $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$

$$\begin{aligned}
&\frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{k\in S_{m,N}}\left|M\left(\left[b,V_{\varphi}^{d}\right](f)\right)(k)\right|^{p}\right)^{1/p}\\
&\leq \|b\|_{BMO^{r'}(\mathbb{Z})}\sum_{n=1}^{\infty}\left[\frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{k\in S_{m,N}}\left|M\left(f^{r}\right)(kn)\right|^{p/r}\right)^{1/p}\right.\\
&\left.\times\left(n^{1/r}+n+n^{1+1/r}\right)\varphi(n)\right]\\
&\leq \|b\|_{BMO^{r'}(\mathbb{Z})}\sum_{n=1}^{\infty}\left[\frac{1}{(2N+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in S_{mn,Nn}}\left|M\left(f^{r}\right)(l)\right|^{p/r}\right)^{1/p}\right.\\
&\left.\times\left(n^{1/r}+n+n^{1+1/r}\right)\varphi(n)\right]\\
&\leq \|b\|_{BMO^{r'}(\mathbb{Z})}\sum_{n=1}^{\infty}\left[\frac{n^{\frac{1}{p}-\frac{1}{q}}}{(2Nn+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in S_{mn,Nn}}\left|M\left(f^{r}\right)(l)\right|^{p/r}\right)^{1/p}\right.\\
&\left.\times\left(n^{1/r}+n+n^{1+1/r}\right)\varphi(n)\right]\\
&\leq C\,\|b\|_{BMO^{r'}(\mathbb{Z})}\sum_{n=1}^{\infty}\left[\frac{n^{\frac{1}{p}-\frac{1}{q}}}{(2Nn+1)^{\frac{1}{p}-\frac{1}{q}}}\left(\sum_{l\in S_{mn,Nn}}\left|f(l)\right|^{p}\right)^{1/p}\right.\\
&\left.\times\left(n^{1/r}+n+n^{1+1/r}\right)\varphi(n)\right]\\
&\leq C\,\|b\|_{BMO^{r'}(\mathbb{Z})}\,\|f\|_{l_q^p}\sum_{n=1}^{\infty}n^{1+\frac{1}{r}+\frac{1}{p}-\frac{1}{q}}\varphi(n)
\end{aligned}$$

and taking supremum over $m \in \mathbb{Z}$ and $N \in \mathbb{N}_0$ we obtain the desired estimate (14). ☑

We end this paper with the following remark.

We recall that a triplet (X, d, μ) is called a doubling measure metric space if d is a metric on $X \times X$, and μ is a Borel measure satisfying the following condition: open balls have positive and finite measure, and there is a positive constant C only depending on μ such that

$$\mu(2B) \leq C\mu(B) \tag{15}$$

for each ball B. Here, $2B$ means the ball with the same center as B but twice its radius.

It is known that in a doubling measure metric space (X, d, μ), all the spaces $BMO^s(X)$ coincide for $1 \leq s < \infty$, with equivalent seminorms (see [1]). If we consider the set $\mathbb{Z}$ with the metric inherited by $\mathbb{R}$, and we take μ as the counting measure defined on the subsets of $\mathbb{Z}$, then $\mathbb{Z}$ becomes a doubling measure metric space with constant 3 in (15).

The above implies that Lemma 4.1 and Theorem 4.2, can be stated assuming that $b \in BMO(\mathbb{Z})$, and also, estimates (8) and (14) remain the same, but now writing $\|b\|_{BMO(\mathbb{Z})}$ instead of $\|b\|_{BMO^{r'}(\mathbb{Z})}$.

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