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Normality of k-Matching Polytopes of Bipartite Graphs

Normalidad del politopo de k-emparejamientos de grafos bipartitos

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Abstract. The k-matching polytope of a graph is the convex hull of all its matchings of a given size k when they are considered as indicator vectors. In this paper, we prove that the k-matching polytope of a bipartite graph is normal, that is, every integer point in its t-dilate is the sum of t integers points of the original polytope. This generalizes the known fact that Birkhoff polytopes are normal. As a preliminary result, we prove that for bipartite graphs the k-matching polytope is equal to the fractional k-matching polytope, having thus the H-representation of the polytope. This generalizes the Birkhoff-Von Neumann Theorem which establish that every doubly stochastic matrix can be written as a convex combination of permutation matrices.

Key words and phrases. Polytopes, matchings, matching polytopes, normal polytopes, bipartite graphs.

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Resumen. El politopo de k-emparejamientos de un grafo es la envolvente convexa de todos sus emparejamientos de un tama no dado k cuando estos son considerados como vectores indicadores. En este artículo, demostramos que el politopo de k-emparejamientos de un grafo bipartito es normal, es decir, todo punto entero en su t-dilatación es la suma de t puntos enteros del politopo original. Esto generaliza el resultado conocido de que los politopos de Birkhoff son normales. Como resultado preliminar, demostramos que para grafos bipartitos el politopo de k-emparejamientos es igual al politopo de k-emparejamientos fraccional, teniendo así la H-representación del politopo. Esto generaliza el Teorema de Birkhoff-Von Neumann que establece que toda matriz doblemente estocástica puede ser escrita como una combinación convexa de matrices de permutación

Palabras y frases clave. Politopos, emparejamientos, politopos de emparejamientos, politopos normales, grafos bipartitos.

1. Introduction

Matchings are important in combinatorial optimization not only for their applications, but also because of their intriguing status from the viewpoint of computational complexity. There is a polynomial-time algorithm to find a perfect matching if one exists, or to show that one does not exist. This is the method of augmenting paths, which is one of the first non-trivial polynomial-time algorithms that we encounter in combinatorial optimization. Furthermore, although decision problems such as "does G have a perfect matching?", or "does G have a k-matching?" are in P, the counting problem "how many perfect matchings does G have?" is $\#P$-complete, even for bipartite graphs (see, for example, [5]). That means that counting matchings is just as hard as counting 3-colorings or large cliques or other objects for which the decision problem is NP-complete.

After indexing the vertices of a graph, matchings can be seen as vectors, and we can take any subset of matchings and form the corresponding convex hull to obtain a polytope. The polytopes of all matchings and of maximal matchings are well-studied but much less so when we only consider matchings of a given size k. We will call them k-matching polytopes, and they will be the focus of our paper. (There is a concept similar in name called the $\mathbf{b}$-matching polytope but it is a different object. For a graph $G = (V, E)$ and a vector $\mathbf{b} \in \mathbb{Z}^V$, the $\mathbf{b}$-matching polytope is the convex hull of all vectors $\mathbf{x} \in \mathbb{Z}^E$ with nonnegative entries such that $\sum_{e \in v} x_e \leq b_v$ for all $v \in V$. See [1] for more about these of polytopes.)

The main purpose of our paper is to prove that the k-matching polytope of a bipartite graph is normal, that is, every integer point in its t-dilate is the sum of t integer points of the original polytope, which will be done in Section 4. It is known that Birkhoff polytopes are normal which is a special case of our result. As a preliminary result, in Section 3, we will prove that the k-matching polytope is equal to the fractional k-matching polytope for bipartite graphs, thus obtaining the H-representation for this polytope (a result proved in [6] for the special case of k-assignment polytopes). These give a generalization of the Birkhoff-Von Neumann Theorem that asserts that every doubly stochastic matrix is a convex combination of permutation matrices.

The results presented in this paper were part of the doctoral thesis of the author, see [10]. He would like to thank his advisor Tristram Bogart for useful discussions on this topic.

2. Preliminaries

In this section, we recall some basic concepts and results about matchings, which have been of special interest in graph theory and combinatorial optimization.

Definition 2.1. A *matching* M of a graph $G = (V, E)$ is a collection of edges of G such that no two different edges in M are incident to a common vertex. If

$|M| = k$, we say that M is a *k-matching*, and the matching is *perfect* if every vertex of G belongs to an edge in M. The vector $\chi(M) \in \mathbb{R}^E$ defined by

$$\chi(M)_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{if } e \notin M \end{cases}$$

is called the characteristic vector of M.

Given a graph, we can construct a polytope that encodes its matching structure.

Definition 2.2. Let G be a graph. The polytope

$$M(G) := conv\{\chi(M) : M \text{ is a matching of } G\}$$

in $\mathbb{R}^E$ is called the *matching polytope* of G.

Associated to this construction is the fractional matching polytope, which is a natural linear relaxation of the matching polytope.

Definition 2.3. Let G be a graph. The polytope

$$FM(G) := \{\mathbf{x} \in \mathbb{R}^E : \sum_{e \ni v} x_e \leq 1 \ \ \forall v \in V, \ x_e \geq 0 \ \ \forall e \in E\}$$

in $\mathbb{R}^E$ is called the *fractional matching polytope* of G.

In general, the matching polytope and the fractional matching polytope are not equal, take as an example a triangle: if we assign $1/2$ to each edge, then this function is in $FM(G)$ but not in $M(G)$. Nonetheless, they are equal for bipartite graphs.

Theorem 2.4. *If G is a bipartite graph, then $M(G) = FM(G)$.*

So for bipartite graphs, we know the V-representation as well as a simple H-representation for the matching polytope. Another fundamental result for matchings in bipartite graphs is Hall's Marriage Theorem.

Theorem 2.5 (Hall's Marriage Theorem)**.** *Let G be a bipartite graph on $A \sqcup B$. Then G has a matching that covers A if and only if for every $S \subseteq A$, the set of neighbors of S (the set of vertices that are connected with at least one vertex in S), denoted by $\Gamma(S)$, satisfies $|\Gamma(S)| \geq |S|$.*

For a thorough reference on matchings (including the above results), see the book [9]. Also, we will need the following well known result about the union of two matchings.

Lemma 2.6. *Let G be a bipartite graph on $A \sqcup B$, $V_1 \subseteq A$, and $V_2 \subseteq B$. If M_1 is a matching that covers V_1 and M_2 a matching that covers V_2, then $M_1 \cup M_2$ contains a matching that covers $V_1 \cup V_2$.*

Proof. Consider the digraph D induced by $M_1 \cup M_2$ where each edge in M_1 induces a directed edge from A to B, and each edge in M_2 induces a directed edge from B to A. Every vertex in D has indegree and outdegree of at most 1, so D is composed by disjoint directed even cycles and paths starting at a vertex in $V_1 \cup V_2$ and ending at a vertex not in $V_1 \cup V_2$ (since every vertex in this set has an outgoing edge). From here, it is clear we can take a matching from $M_1 \cup M_2$ that covers $V_1 \cup V_2$. ☑

We can also construct polytopes from matchings of a given size k, which are less well-studied for k other than the maximum possible size.

Definition 2.7. Let G be a graph and $k \in \mathbb{N}$. The polytope

$$M_k(G) := conv\{\chi(M) : M \text{ is a } k\text{-matching of } G\}$$

in $\mathbb{R}^E$ is called the *k-matching polytope* of G, and the polytope

$$FM_k(G) := \{\mathbf{x} \in \mathbb{R}^E : \sum_{e \ni v} x_e \leq 1 \ \ \forall v \in V,\ x_e \geq 0 \ \ \forall e \in E,\ \sum_{e \in E} x_e = k\}$$

is called the *fractional k-matching polytope* of G.

We will see in the next section that again these two polytopes are equal for bipartite graphs. Even further, we can work with matchings up to a given size and define analogously the polytopes $M_{\leq k}(G)$ and $FM_{\leq k}(G)$ and still obtain the same equality for bipartite graphs.

We have defined the polytope $M_k(G)$ for a bipartite graph $G = (V, E)$ on $A \sqcup B$ as living in $\mathbb{R}^E$, but we can also think of it as a polytope in $\mathbb{R}^{m \times n}$, where m and n are the sizes of A and B respectively, in the following way: if $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, then a matching M of G can be represented as a 0/1 matrix of size $m \times n$ with a one in those entries (i, j) such that $\{a_i, b_j\} \in M$ and zeros elsewhere. In this case $M_k(G) = conv\{M \in \mathbb{R}^{m \times n} : M \text{ is a } k\text{-matching of } G\}$, and $FM(G)$ is the set of matrices $X \in \mathbb{R}^{m \times n}$ such that the entries in each row and column sum to at most one, $X_{ij} = 0$ when $\{a_i, b_j\} \notin E$, and all the entries of X sum to k.

Of special interest is when $G = K_{n,n}$. With the matrix representation for matchings, the n-matchings (which are the perfect matchings) of $K_{n,n}$ correspond to permutation matrices.

Definition 2.8. Let $n \in \mathbb{N}$. The polytope

$$\begin{aligned} B_n := \{N \in \mathbb{R}^{n\times n} : & \sum_{j=1}^{n} x_{ij} = 1 \ \ \forall\, 1 \le i \le n, \\ & \sum_{i=1}^{n} x_{ij} = 1 \ \ \forall\, 1 \le j \le n, \\ & x_{ij} \ge 0 \ \ \forall\, 1 \le i, j \le n\} \end{aligned}$$

is called a Birkhoff polytope. An element of B_n is called a $n\times n$ doubly stochastic matrix.

We can see that $B_n = FM_n(K_{n,n})$, and the equality $M_n(K_{n,n}) = FM_n(K_{n,n})$ is the famous Birkhoff-von Neumann Theorem which says that every doubly stochastic matrix is a convex combination of permutation matrices. Also the polytopes $M_k(K_{m,n})$ are studied in [6] under the name of k-assignment polytopes.

Using the Birkhoff polytopes as example, we transition to the concept of normality.

Definition 2.9. A polytope P is normal if for all $t \in \mathbb{N}$, every integer point (a point with integer coordinates) in $tP := \{t\mathbf{x} : \mathbf{x} \in P\}$ is equal to the sum of t integer points in P.

Normal polytopes are also known as polytopes with the *integer decomposition property* (see [3] Remark 0.1 on the use of different terminology for this concept). It is known that Birkhoff polytopes are normal, and in Section 4 we generalize that by showing that in general k-matching polytopes of bipartite graphs are normal. Normal polytopes have connections with monoid algebras [2] and toric varities [4]. A recent survey on normal polytopes is [7].

3. The H-representation of the k-matching polytope of a bipartite graph

In this section we extend a standard result on matchings in bipartite graphs, $M(G) = FM(G)$, to k-matchings. That is, we will prove that $M_k(G) = FM_k(G)$ if G is a bipartite graph. This result was known and proved in [6] for the special case of k-assignment polytopes, that is, when $G = K_{m,n}$.

Theorem 3.1. *Let $G = (V, E)$ be a bipartite graph and $k \in \mathbb{N}$. Then $M_k(G) = FM_k(G)$.*

Proof. Clearly $M_k(G) \subseteq FM_k(G)$ and $FM_k(G) \cap \mathbb{Z}^E \subseteq M_k(G)$. So it is enough to prove that every vertex of $FM_k(G)$ has integer coordinates. We do so by showing that if $\mathbf{x}$ is a non-integer point of $FM_k(G)$, then it can be written as

$\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$ where $\mathbf{x}', \mathbf{x}'' \in FM_k(G) \setminus \{\mathbf{x}\}$, and thus it cannot be a vertex of $FM_k(G)$. (We construct $\mathbf{x}'$ and $\mathbf{x}''$ by properly modifying $\mathbf{x}$; this is the same idea behind the proof $M(G) = FM(G)$, but now we need to be careful with the extra condition $\sum_{e \in E} x_e = k$.)

Let $\mathbf{x}$ be a point of $FM_k(G)$ that is not integer, and consider the subgraph $H := (V, F)$ where $F := \{e \in E : x_e \notin \{0, 1\}\}$. We divide the proof by cases.

Case 1: H has a cycle.

If H has a cycle, then it has to be an even cycle since G is bipartite. Let $e_1, \ldots, e_m$ be the edges that appear in the cycle in that precise order.

Define $\varepsilon := \min\{x_{e_1}, \ldots, x_{e_m}, 1 - x_{e_1}, \ldots, 1 - x_{e_m}\}$. Let $\mathbf{x}' = (x'_e)_{e \in E}$ and $\mathbf{x}'' = (x''_e)_{e \in E}$ where

$$x'_e = \begin{cases} x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd,} \\ x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even,} \\ x_e & \text{otherwise,} \end{cases} \tag{$*$}$$

and

$$x''_e = \begin{cases} x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd,} \\ x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even,} \\ x_e & \text{otherwise.} \end{cases} \tag{$**$}$$

Then $\mathbf{x}', \mathbf{x}'' \in FM_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$.

Case 2: H has no cycles.

Let P be a maximal path of H with consecutive vertices $v_1, v_2, \ldots, v_m, v_{m+1}$ and edges $e_1 = \{v_1, v_2\}, \ldots, e_m = \{v_m, v_{m+1}\}$. Then $x_e = 0$ if e is an edge of G incident to v_1 different from e_1. Indeed, it cannot be equal to 1 due to the inequality $\sum_{e \ni v_1} x_e \leq 1$, and it cannot be strictly between 0 and 1 since otherwise the path could be extended. A similar analysis can be done for v_{m+1}. Thus $\sum_{e \ni v_1} x_e = x_{e_1} < 1$ and $\sum_{e \ni v_{m+1}} x_e = x_{e_m} < 1$.

Furthermore, if P is of even length, take $\varepsilon := \min\{x_{e_1}, \ldots, x_{e_m}, 1 - x_{e_1}, \ldots, 1 - x_{e_m}\}$. Define $\mathbf{x}'$ and $\mathbf{x}''$ as in $(*)$ and $(**)$. Then $\mathbf{x}', \mathbf{x}'' \in FM_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$. In conclusion, if we can find a maximal path of even length in H we are done.

Subcase 2.1: H has no cycles, but it is connected.

If P is of even length, we are done, so suppose P has odd length. We analize the cases $H = P$ and $H \neq P$ separately.

Suppose $H = P$. If $\sum_{e \ni v} x_e = 1$ for every interior vertex v of H, then

$$\begin{aligned}
k &= \sum_{e \in E} x_e \\
&= \sum_{e \in H} x_e + \text{ some integer} \\
&= (x_{e_1} + x_{e_2}) + (x_{e_3} + x_{e_4}) + \cdots + (x_{e_{m-2}} + x_{e_{m-1}}) + x_{e_m} + \text{ some integer} \\
&= 1 + 1 + \cdots + 1 + x_{e_m} + \text{ some integer} \\
&= x_{e_m} + \text{ some integer}
\end{aligned}$$

which is a contradiction since x_{e_m} is not an integer. So there is an interior vertex v of H such that $\sum_{e \ni v} x_e < 1$. This vertex v divides H into two subpaths and one of them has to be of even length. Let's called this subpath P_0, and suppose without loss of generality that its consecutive edges are $e_1, \ldots, e_n$ $(n < m)$. Take $\varepsilon := \min\{x_{e_1}, \ldots, x_{e_n}, 1 - x_{e_1}, \ldots, 1 - x_{e_{n-1}}, 1 - x_{e_n} - x_{e_{n+1}}\}$. Now define $\mathbf{x}'$ and $\mathbf{x}''$ as in $(*)$ and $(**)$ with m replaced by n. Then $\mathbf{x}', \mathbf{x}'' \in FM_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$.

Now suppose $H \neq P$. Then there is an edge e_1' that goes from an interior vertex v_i of P (it cannot be an end-vertex by the maximality of P) to a vertex of H not belonging to P (since there is no cycles in H). Extend the path $e_1, \ldots, e_{i-1}, e_1'$ as much as possible to a maximal path $e_1, \ldots, e_{i-1}, e_1', \ldots, e_j'$. Then one of the maximal paths $e_1, \ldots, e_{i-1}, e_1', \ldots, e_j'$ or $e_m, \ldots, e_i, e_1', \ldots, e_j'$ is of even length, and we are done.

Subcase 2.2: H has no cycles and is not connected.

Finally, if H is disconnected, then it has two maximal paths P and P' with no vertex in common. If one of them is of even length, we are done, so suppose both of them are of odd length. Let $e_1, \ldots, e_m$ be the consecutive edges of P and $e_1', \ldots, e_t'$ the consecutive edges of P'. Define $\varepsilon := \min\{x_{e_1}, \ldots, x_{e_m}, x_{e_1'}, \ldots, x_{e_t'}, 1 - x_{e_1}, \ldots, 1 - x_{e_m}, 1 - x_{e_1'}, \ldots, 1 - x_{e_t'}\}$. Let $\mathbf{x}' = (x_e')_{e \in E}$ and $\mathbf{x}'' = (x_e'')_{e \in E}$ where

$$x_e' = \begin{cases} x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd} \\ x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even} \\ x_e - \varepsilon & \text{if } e = e_i' \text{ for some } i \in [t] \text{ odd} \\ x_e + \varepsilon & \text{if } e = e_i' \text{ for some } i \in [t] \text{ even} \\ x_e & \text{otherwise} \end{cases}$$

and

$$x_e'' = \begin{cases} x_e - \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ odd} \\ x_e + \varepsilon & \text{if } e = e_i \text{ for some } i \in [m] \text{ even} \\ x_e + \varepsilon & \text{if } e = e_i' \text{ for some } i \in [t] \text{ odd} \\ x_e - \varepsilon & \text{if } e = e_i' \text{ for some } i \in [t] \text{ even} \\ x_e & \text{otherwise} \end{cases}.$$

Then $\mathbf{x}', \mathbf{x}'' \in FM_k(G) \setminus \{\mathbf{x}\}$ and $\mathbf{x} = \frac{1}{2}(\mathbf{x}' + \mathbf{x}'')$.

☑

Observation. The same proof also works to show that $M_{\leq k}(G) = FM_{\leq k}(G)$ for bipartite graphs, where

$$M_{\leq k}(G) := conv\{\chi(M) : M \text{ is a matching of } G \text{ of size at most } k\}$$

and

$$FM_{\leq k}(G) := \{\mathbf{x} \in \mathbb{R}^E : \sum_{e \ni v} x_e \leq 1 \ \ \forall v \in V,\ x_e \geq 0 \ \ \forall e \in E,\ \sum_{e \in E} x_e \leq k\}.$$

A similar result holds for $\geq$ instead of $\leq$.

4. The normality of k-matching polytopes of bipartite graphs

The objective of this section is to prove that k-matching polytopes of bipartite graphs are normal. Little is known about these polytopes, and this opens a new research direction to see what other nice properties they have (see [8] Proposition 9.3.20 for some properties worth studying).

As explained in the preliminaries, we will think of a matching in a bipartite graph not only as a set of edges but also as a matrix. Our starting point is the following result.

Lemma 4.1. *If G is a bipartite graph such that $M_k(G)$ is normal, then so is $M_k(H)$ for every subgraph H of G.*

Proof. By adding, if necessary, vertices with no incident edges, we can suppose without loss of generality that H and G have the same vertices. If N is an integer point of $tM_k(H)$, N is also an integer point of $tM_k(G)$. Since $M_k(G)$ is normal, $N = M_1 + \cdots + M_t$ where each M_i is a k-matching of G. Since N and the M_i's have nonnegative entries, every time that N has a zero in the (r, s)-entry so do the M_i's. Thus the M_i's are k-matchings of H, and therefore $M_k(H)$ is also normal. ☑

Since every bipartite graph is a subgraph of the complete bipartite graph $K_{n,n}$ for some $n \in \mathbb{Z}^+$, it is enough to prove our main result for the polytopes $M_k(K_{n,n})$, and that is what we are going to do.

Definition 4.2. Let G be a bipartite graph with vertex sets $A = \{a_1, \cdots, a_m\}$ and $B = \{b_1, \cdots, b_n\}$. If $N \in tM_k(G)$ ($t \in \mathbb{Z}^+$) is an integer point, then the graph H induced by N is the graph with the same vertices of G and such that $\{a_i, b_j\}$ is an edge of H if and only if the entry (i, j) of N is nonzero.

For completeness we present a proof that Birkhoff polytopes $B_n = M_n(K_{n,n})$ are normal (Theorem 4.4). This is a known result in matching theory; nonetheless it is difficult to find a reference for its proof. We begin by extracting a lemma which we will use later on.

Lemma 4.3. *Let $N \in tB_n$ be a an integer point. Then the graph induced by N has a perfect matching M, and thus $N - M \in (t-1)B_n$.*

Proof. Let H be the graph induced by N, and suppose it has no perfect matching. By Hall's Marriage Theorem, there is a subset S of A such that $|\Gamma(S)| < |S|$, where $\Gamma(S)$ is the set of vertices in B that are connected in H by an edge to at least one element in S. Let $k = |S|$ and, without loss of generality, assume $S = \{a_1, \ldots, a_k\}$ and $B \smallsetminus \Gamma(S) = \{b_1, \ldots, b_m\}$ where $m = n - |\Gamma(S)|$. From $|\Gamma(S)| < |S|$, it follows that $m = n - |\Gamma(S)| > n - |S|$, so $m \geq n - k + 1$. If $b_j \notin \Gamma(S)$, then the entries $N_{1j}, \ldots, N_{kj}$ of N are zero. Thus in the upper-left corner of N, there is a submatrix of zeros of size $k \times (n-k+1)$.

Since $N \in tB_n$, the entries of N in each row and column have to sum up to t. This means that the entries of N in the upper-right $k \times (k-1)$-submatrix of N must sum up to kt, but on the other hand, the entries in the right $n \times (k-1)$ submatrix of N must sum up to $(k-1)t$. This is a contradiction. ☑

Theorem 4.4. *The Birkhoff polytope B_n is normal.*

Proof. We prove by induction on t that every integer point in tB_n is the sum of t integer points in B_n. The result clearly holds for $t = 1$. So suppose it is true for $t - 1$, and we will prove that it also holds for t.

Let N be an integer point of tB_n. By Lemma 4.3, the graph induced by N has a perfect matching M, and thus $N - M \in (t-1)B_n$ which by induction can be written as the sum of $t - 1$ integer points of B_n. Notice that M is an integer point of B_n, so N is the sum of t integer points of B_n. ☑

The plan now is to generalize both Lemma 4.3 and Theorem 4.4 to the case of the polytopes $M_k(K_{n,n})$. For that, we need the next auxiliary result.

Lemma 4.5. *Let G be a bipartite graph on $A \sqcup B$. Given $A' \subseteq A$ with $|A'| = r$, $B' \subseteq B$ with $|B'| = c$, and a nonnegative integer $k \leq r + c$, suppose:*

- *there is a k-matching M_1 of G covering A'*
- *there is a k-matching M_2 of G covering B'*
- *there is a $(r + c - k)$-matching M_3 of the subgraph induced by $A' \sqcup B'$.*

Then $M_1 \cup M_2 \cup M_3$ contains a k-matching of G covering $A' \sqcup B'$.

Proof. We prove this by induction on $r+c-k$. If $r+c-k=0$, we use Lemma 2.6: if V_1 is the set of all vertices in A that are covered by M_1, and V_2 is the set of all vertices in B that are covered by M_2, then $M_1 \cup M_2$ contains a matching M that covers $V_1 \cup V_2$. This matching M contains at least k edges since $|V_1| = |M_1| = k$, and it uses at most k edges to cover $A' \sqcup B'$ since $k = r + c$. So we can remove edges from M to obtain a k-matching covering $A' \sqcup B'$. Suppose now that $r + c - k > 0$ and that the result holds for smaller values.

Case 1: M_1 or M_2 contains an edge from A' to B'. Without loss of generality we assume the edge is in M_1. If M_1 contains the edge $e = \{a, b\}$ where $a \in A'$ and $b \in B'$, define $M_1' := M_1 \setminus \{e\}$, $M_2' := M_2 \setminus \{e'\}$ where e' is the edge of M_2 incident to b, and M_3' as the set obtained from M_3 by removing from this all the edges incident to a and b; in case that there is no edge in M_3 incident to a or b, remove any arbitrary edge of M_3. Thus M_3' has one or two elements less than M_3.

If $|M_3 \setminus M_3'| = 1$, we have

- a matching M_1' of size $k-1$ covering $A' \setminus \{a\}$
- a matching M_2' of size $k-1$ covering $B' \setminus \{b\}$
- a matching M_3' of size $r+c-k-1 = (r-1)+(c-1)-(k-1)$ in the subgraph induced by $(A' \setminus \{a\}) \sqcup (B' \setminus \{b\})$,

so by the induction hypothesis $M_1' \cup M_2' \cup M_3'$ contains a matching of size $k-1$ covering $(A' \setminus \{a\}) \sqcup (B' \setminus \{b\})$. Add e to this matching, and the result follows.

If $|M_3 \setminus M_3'| = 2$, we have two edges in M_3 of the form $e_1 = \{a, b'\}$ and $e_2 = \{a', b\}$ with $a \neq a' \in A'$ and $b \neq b' \in B$. Remove from M_1' the edge that is incident to a' and from M_2' the edge that is incident to b'. Call this new sets M_1'' and M_2''. We have then

- a matching M_1'' of size $k-2$ covering $A' \setminus \{a, a'\}$
- a matching M_2'' of size $k-2$ covering $B' \setminus \{b, b'\}$
- a matching M_3' of size $r+c-k-2 = (r-2)+(c-2)-(k-2)$ in the subgraph induced by $(A' \setminus \{a, a'\}) \sqcup (B' \setminus \{b, b'\})$,

so by the induction hypothesis $M_1'' \cup M_2'' \cup M_3'$ contains a matching of size $k-2$ covering $(A' \setminus \{a, a'\}) \sqcup (B' \setminus \{b, b'\})$. Add e_1 and e_2 to this matching, and the result follows.

Case 2: Neither M_1 nor M_2 has an edge from A' to B'. Let A'' be the vertices in A' that are covered by M_3, and define B'' similarly. Let $M_1' := \{e \in M_1 : e \text{ is incident to a vertex in } A' \setminus A''\}$, and define M_2' similarly. Notice that $|M_1'| = |A'| - |A''| = r - (r+c-k) = k-c$, and similarly, $|M_2'| = k - r$. Then

$M_1' \sqcup M_2' \sqcup M_3$ is a matching of G of size $|M_1'|+|M_2'|+|M_3| = (k-c)+(k-r)+(r+c-k) = k$ covering $A' \sqcup B'$ (no induction is required in this case). ☑

To make it easier to understand the following proofs, let us consider the elements in $M_k(K_{n,n})$ and $tM_k(K_{n,n})$ as matrices. By Theorem 3.1,

$$\begin{aligned} M_k(K_{n,n}) := \{X \in \mathbb{R}^{n\times n} : &\sum_{j=1}^{n} x_{ij} \leq 1 \ \ \forall i \in [n],\ \sum_{i=1}^{n} x_{ij} \leq 1 \ \ \forall j \in [n], \\ & x_{ij} \geq 0 \ \ \forall i,j \in [n], \\ & \sum_{i,j} x_{ij} = k\}. \end{aligned}$$

Since $X \in tM_k(K_{n,n})$ if and only if $\frac{1}{t}X \in M_k(K_{n,n})$, we have

$$\begin{aligned} tM_k(K_{n,n}) := \{X \in \mathbb{R}^{n\times n} : &\sum_{j=1}^{n} x_{ij} \leq t \ \ \forall i \in [n],\ \sum_{i=1}^{n} x_{ij} \leq t \ \ \forall j \in [n], \\ & x_{ij} \geq 0 \ \ \forall i,j \in [n], \\ & \sum_{i,j} x_{ij} = tk\}. \end{aligned}$$

Lemma 4.6. *Let $N \in tM_k(K_{n,n})$ be an integer point. Then the graph induced by N has a k-matching M such that $N - M \in (t-1)M_k(K_{n,n})$.*

Proof. The case $k = n$ is Lemma 4.3. We prove this result by induction on k. The case $k = 0$ is trivial, so suppose $0 < k < n$, and that the result is true for smaller values.

Since $k < n$, there is a row i such that the sum of its entries is less than t, since otherwise the sum of all the entries in N would be tn. Similarly there is a column j such that the sum of its entries is less than t. We add 1 to the entry (i,j) to obtain a new matrix N_1. We now repeat this process to N_1 to obtain a new matrix N_2. Repeating this process $t(n-k)$ times, we obtain a matrix $N_{t(n-k)} \in B_n$. By the normality of B_n, $N_{t(n-k)} = M_1^* + \cdots + M_t^*$ where each M_i^* is a perfect matching of $K_{n,n}$.

Think of the 1's we added to N to obtain $N_{t(n-k)}$ as colored with red, and think of each original entry of N as the sum of black 1's. Since $N_{t(n-k)} = M_1^* + \cdots + M_t^*$, each 1 in $N_{t(n-k)}$ (black or red) has to appear in one of the M_i^*'s. The total number of black 1's that appear in all the M_i^*'s is tk, so the average number of black 1's in each perfect matching is $tk/t = k$. This implies that among the M_i^*'s there is at least one, which we denote by M^*, with at least k black 1's. This implies that there is a k-matching M of H, where H is the graph induced by N. The problem is that $N - M$ does not necessarily belong to $(t-1)M_k(K_{n,n})$ since it may have rows or columns that sum up to t, so further analysis has to be done.

Since we never add red 1's in those rows and columns whose entries sum up to t, we have a black 1 in each of these rows and columns in M^*. Let r and c be the number of rows and columns of N, respectively, whose entries sum up to t. If $r+c \leq k$, then we can always take k black 1's from M^*, to form a k-matching M of H, in such a way that we take all 1's in those rows and columns whose entries sum up to t. This implies that $N - M \in (t-1)M_k(K_{n,n})$.

Let A and B be the sets which partition the vertices of $K_{n,n}$. Since $r, c \leq k$ (otherwise the sum of the entries of N would be greater than tk), we can always find a k-matching M_1 that covers

$$A' := \{\text{vertices in } A \text{ whose respective rows sum up to } t\}$$

and a k-matching M_2 that covers

$$B' := \{\text{vertices in } B \text{ whose respective columns sum up to } t\}.$$

Let's consider now the case $r + c > k$. Without loss of generality suppose that the first r rows and the first c columns of N are the ones whose entries sum up to t. Then N can be written in blocks as

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

where A_1 is a block of size $r \times c$. If a_i denotes the sum of the entries in A_i, then $a_1 + a_2 + a_3 \leq tk$. Thus $tr + tc = (a_1 + a_2) + (a_1 + a_3) \leq a_1 + tk$, and therefore $t(r + c - k) \leq a_1$.

If $r = c = k$, A_2, A_3, and A_4 would be blocks with only zeros, and $A_1 \in tB_k$. By Lemma 4.3, the graph induced by A_1 has a perfect matching P, and in this case the k-matching

$$M = \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

satisfies that $N - M \in (t-1)M_k(K_{n,n})$.

So suppose now that r or c is not equal to k, and thus $0 < r + c - k < k$.

We reduce some positive entries of A_1 to obtain a matrix C such that its entries sum up to $t(r + c - k)$, which we can do since $t(r + c - k) \leq a_1$. Then

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in tM_{r+c-k}(K_{n,n}).$$

Since $0 < r + c - k < k$, by induction, the graph induced by

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

has a matching of size $r + c - k$ of the form

$$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

From the way C is obtained, we have that the induced graph of

$$\begin{bmatrix} C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

is a subgraph of the induced graph H of N. Thus

$$\begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

is also a matching of H of size $r + c - k$. That is, we have found a matching M_3 of size $r + c - k$ in the subgraph induced by $A' \sqcup B'$. We use Lemma 4.5 to conclude that there is a matching M of H of size k covering $A' \sqcup B'$, which implies that $N - M \in (t-1)M_k(K_{n,n})$. ☑

Proposition 4.7. *The polytope $M_k(K_{n,n})$ is normal.*

Proof. We need to prove that if $N \in tM_k(K_{n,n})$, then it can be written as the sum of t integer points of $M_k(K_{n,n})$, that is, as the sum of t k-matchings of $K_{n,n}$. We prove this by induction on t.

The case $t = 1$ is clear. Suppose the result is true for $t - 1$, and let us prove it for t. By Lemma 4.6, there is a k-matching of the induced graph of N, which is also a k-matching of $K_{n,n}$, such that $N - M \in (t-1)M_k(K_{n,n})$. By the induction hypothesis, $N - M$ is the sum of $t - 1$ k-matchings of $K_{n,n}$. Thus N is the sum of t k-matchings of $K_{n,n}$. ☑

Theorem 4.8. *Let G be a bipartite graph and $k \in \mathbb{N}$. Then the polytope $M_k(G)$ is normal.*

Proof. As we remarked before, every bipartite graph can be seen as a subgraph of $K_{n,n}$ for some $n \in \mathbb{Z}^+$. Thus, our result follows from Lemma 4.1 and Proposition 4.7. ☑

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