DOI: https://doi.org/10.15446/recolma.v58n1.117432

Revista Colombiana de Matemáticas
Volumen 58(2024)1, páginas 25-46

New results of the ρ-Jafari transform and their application to linear and nonlinear generalized fractional differential equations

Nuevos resultados de la transformada de ρ-Jafari y su aplicación a ecuaciones diferenciales fraccionarias generalizadas lineales y no lineales

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Abstract. Recently Hossein Jafari introduced a new general integral transform called Jafari transform to solve higher order initial value problems and integral equations. The main objective of this paper is to modify this integral transform that we call the ρ-Jafari transform and to study its properties. Then, we present interesting results and apply them to solve linear and nonlinear generalized fractional differential equations. The results obtained confirmed that the ρ-Jafari transform acts as a powerful tool for generalized fractional problems. As a result, we assert that in the future, the modified transform can be applied to many generalized fractional differential equations that arise in applied science and engineering.

Key words and phrases. Jafari transform, generalized fractional integral, Caputo generalized fractional derivative, exact solution.

2020 Mathematics Subject Classification. 26A33, 35A22, 33E12, 44A10.

Resumen. Recientemente, Hossein Jafari introdujo una nueva transformada integral general llamada transformada de Jafari para resolver problemas de valores iniciales de orden superior y ecuaciones integrales. El principal objetivo de este trabajo es modificar esta transformada integral que llamamos transformada ρ-Jafari y estudiar sus propiedades. Luego, presentamos resultados interesantes y los aplicamos para resolver ecuaciones diferenciales fraccionarias generalizadas lineales y no lineales. Los resultados obtenidos confirmaron que la transformada ρ-Jafari actúa como una poderosa herramienta para problemas fraccionarios generalizados. Como resultado, afirmamos que en el futuro, la transformada modificada se podrá aplicar a muchas ecuaciones diferenciales fraccionarias generalizadas que surgen en las ciencias aplicadas y la ingeniería.

Palabras y frases clave. Transformada de Jafari, integral fraccionaria generalizada, derivada fraccionaria generalizada de Caputo, solución exacta.

1. Introduction

Today, fractional calculus and its applications in various branches of science and engineering is an important area in the modeling of many real-world problems [20, 21, 23, 26].

Fractional differential equations are an equations that contains fractional derivatives. These equations appear in many scientific and engineering applications such as fluid dynamics, plasma physics, hydrodynamics, solid state physics, biological population models, acoustics and some others, which are well described by differential fractional equations.

In recent years, many authors have been mainly concerned with solving fractional differential equations using various methods, for example, the Adomian decomposition method [28], the variational iteration method [22], the homotopy perturbation method [27], the homotopy analysis method [25], the differential transform method [6], the general fractional residual power series method (GFRPSM) [16] and the modified fractional Taylor series method (MFTSM) [14].

The integral transform method is one of the most popular schemes used by many researchers to get analytical solutions fractional differential equations, thus in the literature, there are various types of integral transforms such as Laplace transform [19], Sumudu transform [5], natural transform [1], Shehu transform [3] and many others.

The aim of the paper is to propose a new extension of the general integral transform that we call the ρ-Jafari transform to solve various kinds of linear and nonlinear generalized fractional differential equations.

The main advantages of the proposed method are:

(i) The ability to convert generalized fractional differential equations into a simple system of algebraic equations that can be easily solved.

(ii) This method gives the solution with less computational calculations and with high efficiency

(iii) The ρ-Jafari transform method is free from any restrictive assumptions, perturbations, discretization, or linearization.

(iv) The possibility of combining an important semi-analytic method called the Adomian decomposition method and ρ-Jafari transform method in the sense of Caputo generalized fractional derivative.

Furthermore, the novelty of this work lies in the possibility of combining an important semi-analytic method called the Adomian decomposition method and the Jafari transform in the sense of Caputo generalized fractional derivation, which is a generalization of th above integral transforms.

The outline of this paper is as follows. In the next section, we present some necessary definitions and results from fractional calculus theory. In Section 3, our main results regarding to the ρ-Jafari transform and their properties are presented. In Section 4, some linear and nonlinear generalized fractional differential equations of different types and orders are performed in order to illustrate capability and simplicity of the proposed technique. The conclusion is given in the last part, Section 5.

2. Preliminaries concepts

In this section, the preliminaries concepts of fractional calculus and some interesting properties are presented.

Definition 2.1. [10] Let the function $u : [0, \infty[\to \mathbb{R}$, the generalized fractional integral of order $\alpha, \rho > 0$ of the function u, is defined by

$$I^{\alpha,\rho}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} \frac{u(\tau)}{\tau^{1-\rho}} d\tau, t > 0. \tag{1}$$

Definition 2.2. [11] Let the function $u : [0, \infty[\to \mathbb{R}$, the generalized fractional derivative of order $\alpha, \rho > 0$ of the function u, is defined by

$$\begin{aligned} D^{\alpha,\rho}u(t) &= \gamma^n I^{n-\alpha,\rho}u(t) \\ &= \frac{\gamma^n}{\Gamma(n-\alpha)} \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha+1} \frac{u(\tau)}{\tau^{1-\rho}} d\tau, t > 0, \end{aligned} \tag{2}$$

where $\gamma = t^{1-\rho} d/dt$ and $n - 1 < \alpha \leq n$.

Definition 2.3. [8] Let the function $u : [0, \infty[\to \mathbb{R}$, the Caputo generalized fractional derivative of order $\alpha, \rho > 0$ of the function u, is defined by

$$\begin{aligned} D_C^{\alpha,\rho}u(t) &= I^{n-\alpha,\rho}\gamma^n u(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha-1} \frac{\gamma^n u(\tau)}{\tau^{1-\rho}} d\tau,\ t > 0, \end{aligned} \tag{3}$$

where $\gamma = t^{1-\rho} d/dt$ and $n - 1 < \alpha \leq n$.

For equations (1) and (3), we have the following relation

$$D_C^{\alpha,\rho} I^{\alpha,\rho}u(t) = u(t), \tag{4}$$

and

$$I^{\alpha,\rho} D_C^{\alpha,\rho}u(t) = u(t) - \sum_{k=0}^{n} \frac{D^{\alpha-k,\rho}u(0)}{\Gamma(\alpha-k+1)} \left(\frac{t^\rho}{\rho}\right)^{\alpha-k}. \tag{5}$$

Definition 2.4. [20] The Mittag-Leffler function for one parameter denoted by $E_{\alpha}(z)$, is defined as

$$E_{\alpha}(z)=\sum_{k=0}^{+\infty}\frac{z^{k}}{\Gamma(k\alpha+1)},\alpha\in\mathbb{R}^{+},z\in\mathbb{C}. \tag{6}$$

and for two parameter denoted by $E_{\alpha,\beta}(z)$, is defined as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{+\infty}\frac{z^{k}}{\Gamma(k\alpha+\beta)},\alpha,\beta\in\mathbb{R}^{+},z\in\mathbb{C}. \tag{7}$$

3. Main results

In this section, we present our main results related to the ρ-Jafari transform and their properties.

3.1. The ρ-Jafari transform

Definition 3.1. Let $u:[0,\infty[\,\to\mathbb{R}$ be an integrable function. The ρ-Jafari transform of u is given by

$$\begin{aligned}\mathbb{J}_{\rho}[u(t)]=\mathcal{J}_{\rho}(s)&=p(s)\int_{0}^{\infty}\exp\left(-q(s)\frac{t^{\rho}}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt\\&=p(s)\lim_{\delta\to\infty}\int_{0}^{\delta}\exp\left(-q(s)\frac{t^{\rho}}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt,\rho>0,\end{aligned}$$

where the integral exists for some $q(s)$ and the functions $p(s),q(s):\mathbb{R}^{+}\to\mathbb{R}$ such that $p(s)\neq0$ for all $s\in\mathbb{R}^{+}$.

Definition 3.2. A function $u:[0,\infty[\,\to\mathbb{R}$ is said to be of ρ-exponential order $\exp\left(B\frac{t^{\rho}}{\rho}\right)$ if there exist non-negative constants M,B,T such that $|u(t)|\leq M\exp\left(B\frac{t^{\rho}}{\rho}\right)$ for $t\geq T$.

Theorem 3.3. If the function $u:[0,\infty[\,\to\mathbb{R}$ is a piecewise continuous in every finite interval $0\leq t\leq A$ and is of ρ-exponential order $\exp\left(B\frac{t^{\rho}}{\rho}\right)$ for $t>A$. Then its ρ-Jafari transform exists for $q(s)>B$.

Proof. For any $A>0$, then we have

$$\begin{aligned}\mathbb{J}_{\rho}[u(t)]&=p(s)\int_{0}^{\infty}\exp\left(-q(s)\frac{t^{\rho}}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt\\&=p(s)\int_{0}^{A}\exp\left(-q(s)\frac{t^{\rho}}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt+p(s)\int_{A}^{\infty}\exp\left(-q(s)\frac{t^{\rho}}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt\\&=I_{1}+I_{2}.\end{aligned}$$

Since the function $u(t)$ is a piecewise continuous in every finite interval $0 \leq t \leq A$, then the first integral I_1 exists, and since $u(t)$ is of ρ-exponential order $\exp\left(B\frac{t^\rho}{\rho}\right)$ for $t > A$, then the second integral I_2 exists.

Indeed,

$$\begin{aligned}
\left|p(s)\int_A^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt\right| &\leq p(s)\int_A^\infty \left|\exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{u(t)}{t^{1-\rho}}\right|dt \\
&\leq p(s)\int_A^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{1}{t^{1-\rho}}\left|u(t)\right|dt \\
&\leq p(s)\int_A^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{1}{t^{1-\rho}}M\exp\left(B\frac{t^\rho}{\rho}\right)dt \\
&= Mp(s)\int_A^\infty \exp\left(-\left(q(s)-B\right)\frac{t^\rho}{\rho}\right)\frac{1}{t^{1-\rho}}dt \\
&= Mp(s)\lim_{\delta\to\infty}\left[-\frac{\exp\left(-\left(q(s)-B\right)\frac{t^\rho}{\rho}\right)}{q(s)-B}\right]_0^\delta \\
&= Mp(s)\left(\frac{1}{q(s)-B}\right) \\
&= \frac{Mp(s)}{q(s)-B}.
\end{aligned}$$

The proof is complete. ☑

Theorem 3.4. *Let $u : [0,\infty[\to \mathbb{R}$ be a function such that its ρ-Jafari transform exists. Then*

$$\mathbb{J}_\rho\left[u(t)\right] = \mathbb{J}\left[u\left((\rho t)^{\frac{1}{\rho}}\right)\right], \tag{8}$$

where $\mathbb{J}\left[u(.)\right]$ is the usual Jafari transform of $u(.)$ [7].

Proof. According to the Definiton 3.1 of ρ-Jafari transform, we have

$$\mathbb{J}_\rho\left[u(t)\right] = p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt. \tag{9}$$

Substituting $\psi = \frac{t^\rho}{\rho} \Longrightarrow d\psi = \frac{dt}{t^{1-\rho}}dt$ and $t = (\rho\psi)^{\frac{1}{\rho}}$ in equation (9), we get

$$\begin{aligned}
\mathbb{J}_\rho\left[u(t)\right] &= p(s)\int_0^\infty \exp\left(-q(s)\psi\right)u((\rho\psi)^{\frac{1}{\rho}})d\psi \\
&= \mathbb{J}\left[u\left((\rho t)^{\frac{1}{\rho}}\right)\right].
\end{aligned}$$

The proof is complete. ☑

Now, we present two important properties of the ρ-Jafari transform.

Theorem 3.5. *Let u and $v : [0,\infty[\to \mathbb{R}$ be given functions such that the ρ-Jafari transform of u and v exists for $q(s) > B_1$ and $q(s) > B_2$ respectively. Then for $\lambda, \mu \in \mathbb{R}$, the ρ-Jafari transform of $\lambda u + \mu v$ exists and*

$$\mathbb{J}_\rho\left[\lambda u(t) \pm \mu v(t)\right] = \lambda \mathbb{J}_\rho\left[u(t)\right] \pm \mu \mathbb{J}_\rho\left[v(t)\right], \tag{10}$$

for $q(s) > \max\{B_1, B_2\}$.

Proof. According to the Definiton 3.1 of ρ-Jafari transform, we get

$$\begin{aligned}
\mathbb{J}_\rho\left[\lambda u(t) \pm \mu v(t)\right] &= p(s)\int_0^\infty \exp\left(q(s)\frac{t^\rho}{\rho}\right)\frac{(\lambda u(t) \pm \mu v(t))}{t^{1-\rho}}dt \\
&= p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{\lambda u(t)}{t^{1-\rho}}dt \\
&\pm p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{\mu v(t)}{t^{1-\rho}}dt \\
&= \lambda\left(p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dtt\right) \\
&\pm \mu\left(p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{v(t)}{t^{1-\rho}}dt\right) \\
&= \lambda \mathbb{J}_\rho\left[u(t)\right] \pm \mu \mathbb{J}_\rho\left[v(t)\right].
\end{aligned}$$

The proof is complete. ☑

Theorem 3.6. *Let u and $v : [0,\infty[\to \mathbb{R}$ be two functions which are piecewise continuous in every finite interval $[0,T]$ and of ρ-exponential order . Then*

$$\mathbb{J}_\rho\left[\left(u *_\rho v\right)(t)\right] = \frac{1}{p(s)}\mathbb{J}_\rho\left[u(t)\right]\mathbb{J}_\rho\left[v(t)\right], \tag{11}$$

where $u *_\rho v$ is the ρ-convolution integral defined by

$$\begin{aligned}
\left(u *_\rho v\right)(t) &= \int_0^t u((t^\rho - \tau^\rho)^{\frac{1}{\rho}})\frac{v(\tau)}{\tau^{1-\rho}}d\tau \\
&= \int_0^t v((t^\rho - \xi^\rho)^{\frac{1}{\rho}})\frac{u(\xi)}{\xi^{1-\rho}}d\xi \\
&= \left(v *_\rho u\right)(t).
\end{aligned}$$

Proof. According to the Definiton 3.1 of ρ-Jafari transform , we get

$$
\begin{aligned}
\mathbb{J}_\rho\left[u(t)\right]\mathbb{J}_\rho\left[v(t)\right] &= \left(p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt\right)\\
&\quad\times\left(p(s)\int_0^\infty \exp\left(-q(s)\frac{\xi^\rho}{\rho}\right)\frac{v(\xi)}{\xi^{1-\rho}}d\xi\right)\\
&= p(s)^2\int_0^\infty\int_0^\infty \exp\left(-q(s)\left(\frac{t^\rho+\xi^\rho}{\rho}\right)\right)\frac{u(t)}{t^{1-\rho}}\frac{v(\xi)}{\xi^{1-\rho}}dtd\xi.
\end{aligned}
\tag{12}
$$

Substituting $\tau^\rho = t^\rho + \xi^\rho$ and $t^{\rho-1}dt = \tau^{\rho-1}dt$ in equation (12), we get

$$
\mathbb{J}_\rho\left[u(t)\right]\mathbb{J}_\rho\left[v(t)\right] = p(s)^2\int_0^\infty\int_\xi^\infty \exp\left(-q(s)\frac{\tau^\rho}{\rho}\right)\frac{u((\tau^\rho-\xi^\rho)^{\frac{1}{\rho}})}{\tau^{1-\rho}}\frac{v(\xi)}{\xi^{1-\rho}}d\tau d\xi.
\tag{13}
$$

Changing the order of integration, i.e., equation (13) becomes

$$
\begin{aligned}
\mathbb{J}_\rho\left[u(t)\right]\mathbb{J}_\rho\left[v(t)\right] &= p(s)^2\int_0^\infty\int_0^\tau \exp\left(-q(s)\frac{\tau^\rho}{\rho}\right)\frac{u((\tau^\rho-\xi^\rho)^{\frac{1}{\rho}})}{\tau^{1-\rho}}\frac{v(\xi)}{\xi^{1-\rho}}d\tau d\xi\\
&= p(s)\left(p(s)\int_0^\infty \exp\left(-q(s)\frac{\tau^\rho}{\rho}\right)\left(\int_0^\tau u((\tau^\rho-\xi^\rho)^{\frac{1}{\rho}})\frac{v(\xi)}{\xi^{1-\rho}}d\xi\right)\frac{d\tau}{\tau^{1-\rho}}\right)\\
&= p(s)\left(p(s)\int_0^\infty \exp\left(-q(s)\frac{\tau^\rho}{\rho}\right)\left(u *_\rho v\right)(\tau)\frac{d\tau}{\tau^{1-\rho}}\right)\\
&= p(s)\left(p(s)\int_0^\infty \exp\left(-q(s)\frac{\tau^\rho}{\rho}\right)\left(u *_\rho v\right)(\tau)\frac{d\tau}{\tau^{1-\rho}}\right)\\
&= p(s)\mathbb{J}_\rho\left[\left(u *_\rho v\right)(t)\right].
\end{aligned}
$$

The proof is complete. ☑

Now, we give some of the ρ-Jafari transforms of elementary functions.

Theorem 3.7. *Let a, b and $c \in \mathbb{R}$ and $\rho > 0$, then*

$$1)\ \mathbb{J}_\rho\left[1\right] = \frac{p(s)}{q(s)}.$$

$$2)\ \mathbb{J}_\rho\left[t^c\right] = \frac{\rho^{\frac{c}{\rho}}p(s)}{q^{\frac{c}{\rho}+1}(s)}\Gamma\left(\frac{c}{\rho}+1\right).$$

$$3)\ \mathbb{J}_\rho\left[\frac{t^{n\rho}}{\rho^n}\right] = \frac{p(s)}{q^{n+1}(s)}\Gamma\left(n+1\right).$$

$$4)\ \mathbb{J}_\rho\left[\exp\left(a\frac{t^\rho}{\rho}\right)\right] = \frac{p(s)}{q(s)-a}, q(s) > a.$$

$$5)\ \mathbb{J}_\rho\left[\sin\left(a\frac{t^\rho}{\rho}\right)\right]=\frac{ap(s)}{q^2(s)+a^2}.$$

$$6)\ \mathbb{J}_\rho\left[\sinh\left(a\frac{t^\rho}{\rho}\right)\right]=\frac{ap(s)}{q^2(s)-a^2}.$$

$$7)\ \mathbb{J}_\rho\left[\cos\left(a\frac{t^\rho}{\rho}\right)\right]=\frac{q(s)p(s)}{q(s)^2+a^2}.$$

$$8)\ \mathbb{J}_\rho\left[\cosh\left(a\frac{t^\rho}{\rho}\right)\right]=\frac{q(s)p(s)}{q^2(s)-a^2}.$$

Proof. From Theorem 3.4, we have

1)

$$\mathbb{J}_\rho[1]=\mathbb{J}[1]=\frac{p(s)}{q(s)}.$$

2)

$$\begin{aligned}\mathbb{J}_\rho[t^c]&=\mathbb{J}\left[(\rho t)^{\frac{c}{\rho}}\right]=\rho^{\frac{c}{\rho}}\mathbb{J}\left[t^{\frac{c}{\rho}}\right]\\&=\frac{\rho^{\frac{c}{\rho}}p(s)}{q^{\frac{c}{\rho}+1}(s)}\Gamma\left(\frac{c}{\rho}+1\right).\end{aligned}$$

3) If we take $c=n\rho$, then we have

$$\mathbb{J}_\rho[t^{n\rho}]=\mathbb{J}\left[(\rho t)^{\frac{n\rho}{\rho}}\right]=\rho^n\mathbb{J}[t^n].$$

So,

$$\mathbb{J}_\rho\left[\frac{t^{n\rho}}{\rho^n}\right]=\mathbb{J}[t^n]=\frac{p(s)}{q^{n+1}(s)}\Gamma(n+1).$$

4)

$$\begin{aligned}\mathbb{J}_\rho\left[\exp\left(a\frac{t^\rho}{\rho}\right)\right]&=\mathbb{J}\left[\exp\left(a\frac{\left((\rho t)^{\frac{1}{\rho}}\right)^\rho}{\rho}\right)\right]\\&=\mathbb{J}[\exp(at)]\\&=\frac{p(s)}{q(s)-a},q(s)>a.\end{aligned}$$

5)

$$\begin{aligned}\mathbb{J}_\rho\left[\sin\left(a\frac{t^\rho}{\rho}\right)\right] &= \mathbb{J}\left[\sin\left(a\frac{\left((\rho t)^{\frac{1}{\rho}}\right)^\rho}{\rho}\right)\right]\\ &= \mathbb{J}\left[\sin\left(at\right)\right]\\ &= \frac{ap(s)}{q^2(s)+a^2}.\end{aligned}$$

6)

$$\begin{aligned}\mathbb{J}_\rho\left[\sinh\left(a\frac{t^\rho}{\rho}\right)\right] &= \mathbb{J}\left[\sinh\left(a\frac{\left((\rho t)^{\frac{1}{\rho}}\right)^\rho}{\rho}\right)\right]\\ &= \mathbb{J}\left[\sinh\left(at\right)\right]\\ &= \frac{ap(s)}{q(s)^2-a^2}.\end{aligned}$$

7)

$$\begin{aligned}\mathbb{J}_\rho\left[\cos\left(a\frac{t^\rho}{\rho}\right)\right] &= \mathbb{J}\left[\cos\left(a\frac{\left((\rho t)^{\frac{1}{\rho}}\right)^\rho}{\rho}\right)\right]\\ &= \mathbb{J}\left[\cos\left(at\right)\right]\\ &= \frac{q(s)p(s)}{q^2(s)+a^2}.\end{aligned}$$

8)

$$\begin{aligned}\mathbb{J}_\rho\left[\cosh\left(a\frac{t^\rho}{\rho}\right)\right] &= \mathbb{J}\left[\cosh\left(a\frac{\left((\rho t)^{\frac{1}{\rho}}\right)^\rho}{\rho}\right)\right]\\ &= \mathbb{J}\left[\cosh\left(at\right)\right]\\ &= \frac{q(s)p(s)}{q^2(s)-a^2}.\end{aligned}$$

The proof is complete. ☑

Now, we present the ρ-Jafari transforms of the derivatives.

Theorem 3.8. *Let $u : [0,\infty[\to \mathbb{R}$ be a function which is continuous and is of ρ-exponential order $\exp\left(B\frac{t^\rho}{\rho}\right)$ such that $\gamma u(t)$ is piecewise continuous in every finite interval $[0,T]$. Then ρ-Jafari transform of $\gamma u(t)$ exists for $q(s) > B$ and*

$$\mathbb{J}_\rho\left[\gamma u(t)\right] = q(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)u(0). \tag{14}$$

Proof. Using the definition of ρ-Jafari transform 3.1, we get

$$\begin{aligned}\mathbb{J}_\rho\left[\gamma u(t)\right] &= p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{\gamma u(t)}{t^{1-\rho}}dt\\ &= p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)t^{1-\rho}\frac{du(t)}{dt}\frac{1}{t^{1-\rho}}dt\\ &= p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)u'(t)dt.\end{aligned}$$

Now integrating by parts gives

$$\begin{aligned}\mathbb{J}_\rho\left[\gamma u(t)\right] &= p(s)\left(\lim_{\sigma\to\infty}\left[\exp\left(-q(s)\frac{t^\rho}{\rho}\right)u(t)\right]_0^\sigma\right.\\ &\quad\left.+q(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\frac{u(t)}{t^{1-\rho}}dt\right)\\ &= -p(s)u(0)+q(s)\mathbb{J}_\rho\left[u(t)\right]\\ &= q(s)\mathbb{J}_\rho\left[u(t)\right]-p(s)u(0).\end{aligned}$$

The proof is complete. ☑

Theorem 3.9. *Let* $u\in C_\gamma^{n-1}[0,\infty)$ *such that* $\gamma^i u, i=0,1,2,...,n-1$ *are of* ρ*-exponential order* $\exp\left(B\frac{t^\rho}{\rho}\right)$. *Let* $\gamma^n u$ *be a piecewise continuous function in every finite interval* $[0,T]$. *Then, the* ρ*-Jafari transform of* $\gamma^n u(t)$ *exists for* $q(s)>B$ *and*

$$\mathbb{J}_\rho\left[\gamma^n u(t)\right] = q^n(s)\mathbb{J}_\rho\left[u(t)\right]-p(s)\sum_{k=0}^{n-1}q^{n-1-k}(s)\gamma^k u(0). \tag{15}$$

Proof. The proof can be done by mathematical induction on n and with the help of Theorem 3.8. ☑

3.2. The ρ-Jafari transforms of the generalized fractional integrals and derivatives

In the following theorems, we present new results related to the ρ-Jafari transforms of generalized fractional integrals and derivatives.

Theorem 3.10. *Let the function* $u:[0,\infty[\,\to\mathbb{R}$ *be a piecewise continuous in every finite interval* $[0,T]$ *and is of* ρ*-exponential order* $\exp\left(B\frac{t^\rho}{\rho}\right)$. *Then the* ρ*-Jafari transform of generalized fractional integral of order* $\alpha,\rho>0$, *is given by*

$$\mathbb{J}_\rho\left[I^{\alpha,\rho}u(t)\right] = \frac{1}{q^\alpha(s)}\mathbb{J}_\rho\left[u(t)\right], q(s)>B. \tag{16}$$

Proof. Applying the ρ-Jafari transform to both sides of equation (1), we get

$$\begin{aligned}
\mathbb{J}_\rho\left[I^{\alpha,\rho}u(t)\right] &= \mathbb{J}_\rho\left[\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\frac{u(\tau)}{\tau^{1-\rho}}d\tau\right]\\
&= p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\left[\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\alpha-1}\frac{u(\tau)}{\tau^{1-\rho}}d\tau\right]\frac{dt}{t^{1-\rho}}\\
&= p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t\left((t^\rho-\tau^\rho)^{\frac{1}{\rho}}\right)^{\rho(\alpha-1)}\frac{u(\tau)}{\tau^{1-\rho}}d\tau\right]\frac{dt}{t^{1-\rho}}\\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}p(s)\int_0^\infty \exp\left(-q(s)\frac{t^\rho}{\rho}\right)\left(t^{\rho(\alpha-1)} *_\rho u(t)\right)\frac{dt}{t^{1-\rho}}\\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\mathbb{J}_\rho\left[t^{\rho(\alpha-1)} *_\rho u(t)\right].
\end{aligned}$$

Using Theorems 3.6 and 3.7, we obtain

$$\begin{aligned}
\mathbb{J}_\rho\left[I^{\alpha,\rho}u(t)\right] &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{1}{p(s)}\mathbb{J}_\rho\left[t^{\rho(\alpha-1)}\right]\mathbb{J}_\rho\left[u(t)\right]\\
&=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\frac{1}{p(s)}\frac{\rho^{\frac{\rho(\alpha-1)}{\rho}}p(s)}{q^{\frac{\rho(\alpha-1)}{\rho}+1}(s)}\Gamma\left(\frac{\rho\left(\alpha-1\right)}{\rho}+1\right)\mathbb{J}_\rho\left[u(t)\right]\\
&= \frac{1}{q^\alpha(s)}\mathbb{J}_\rho\left[u(t)\right].
\end{aligned}$$

The proof is complete. ☑

Theorem 3.11. *Let* $u \in AC_\gamma^n\left[0,T\right], T>0$ *and* $I^{n-\alpha-k,\rho}u, k=0,1,2,...n-1$ *are of* ρ*-exponential order* $\exp\left(B\frac{t^\rho}{\rho}\right)$. *Then the* ρ*-Jafari transform of generalized fractional derivative of order* $\alpha,\rho>0$, *is given by*

$$\mathbb{J}_\rho\left[D^{\alpha,\rho}u(t)\right] = q^\alpha(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1}q^{n-1-k}(s)I^{n-\alpha-k,\rho}u(0), q(s)>B. \tag{17}$$

Proof. Applying the ρ-Jafari transform to both sides of equation (2), we get

$$\mathbb{J}_\rho\left[D^{\alpha,\rho}u(t)\right] = \mathbb{J}_\rho\left[\gamma^n I^{n-\alpha,\rho}u(t)\right].$$

Using Theorem 3.9, we get

$$\begin{aligned}
\mathbb{J}_\rho\left[D^{\alpha,\rho}u(t)\right] &= \mathbb{J}_\rho\left[\gamma^n I^{n-\alpha,\rho}u(t)\right]\\
&= q^n(s)\mathbb{J}_\rho\left[I^{n-\alpha,\rho}u(t)\right] - p(s)\sum_{k=0}^{n-1}q^{n-1-k}(s)\gamma^k I^{n-\alpha,\rho}u(0).
\end{aligned}$$

By Theorem 3.10, we have

$$\begin{aligned}\mathbb{J}_\rho\left[D^{\alpha,\rho}u(t)\right] &= q^n(s)\mathbb{J}_\rho\left[I^{n-\alpha,\rho}u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)\gamma^k I^{n-\alpha,\rho}u(0)\\ &= q^n(s)\frac{1}{q^{n-\alpha}(s)}\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)\gamma^k I^{n-\alpha,\rho}u(0)\\ &= q^\alpha(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)\gamma^k I^{n-\alpha,\rho}u(0).\end{aligned}$$

From Theorem 2.5 in [8], we obtain

$$\begin{aligned}\mathbb{J}_\rho\left[D^{\alpha,\rho}u(t)\right] &= q^\alpha(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)\gamma^k I^{n-\alpha,\rho}u(0)\\ &= q^\alpha(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s) I^{n-\alpha-k,\rho}u(0).\end{aligned}$$

The proof is complete. ☑

Theorem 3.12. *Let* $u \in AC_\gamma^n\left[0,T\right], T > 0$ *and* $\gamma^k u, k = 0,1,2,...n$ *are of* ρ*-exponential order* $\exp\left(B\frac{t^\rho}{\rho}\right)$. *Then the* ρ*-Jafari transform of Caputo generalized fractional derivative of order* $\alpha, \rho > 0$, *is given by*

$$\mathbb{J}_\rho\left[D_C^{\alpha,\rho}u(t)\right] = q^\alpha(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{\alpha-1-k}(s)\gamma^k u(0), q(s) > B. \tag{18}$$

Proof. Applying the ρ-Jafari transform to both sides of equation (3), we get

$$\mathbb{J}_\rho\left[D_C^{\alpha,\rho}u(t)\right] = \mathbb{J}_\rho\left[I^{n-\alpha,\rho}\gamma^n u(t)\right].$$

Using Theorem 3.10, we get

$$\begin{aligned}\mathbb{J}_\rho\left[D_C^{\alpha,\rho}u(t)\right] &= \mathbb{J}_\rho\left[I^{n-\alpha,\rho}\gamma^n u(t)\right]\\ &= \frac{1}{q^{n-\alpha}(s)}\mathbb{J}_\rho\left[\gamma^n u(t)\right].\end{aligned}$$

From Theorem 3.9, we obtain

$$\begin{aligned}\mathbb{J}_\rho\left[D_C^{\alpha,\rho}u(t)\right] &= \frac{1}{q^{n-\alpha}(s)}\mathbb{J}_\rho\left[\gamma^n u(t)\right]\\ &= \frac{1}{q^{n-\alpha}(s)}\left(q^n(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{n-1-k}(s)\gamma^k u(0)\right).\\ &= q^\alpha(s)\mathbb{J}_\rho\left[u(t)\right] - p(s)\sum_{k=0}^{n-1} q^{\alpha-1-k}(s)\gamma^k u(0).\end{aligned}$$

The proof is complete. ☑

Corollary 3.13. *Taking $\rho = 1$ in equation (18), we get the following results*

- *The Laplace transform of the Caputo fractional derivative [12], if $p(s) = 1$ and $q(s) = s$*

$$\mathbb{L}\left[D_C^{\alpha} u(t)\right] = s^{\alpha} \mathbb{L}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{\alpha-1-k} u^{(k)}(0),$$

where $\mathbb{L}\left[u(t)\right]$ is the usual Laplace transform of $u(t)$.

- *The Aboodh transform of the Caputo fractional derivative [18], if $p(s) = \frac{1}{s}$ and $q(s) = s$*

$$\begin{aligned}\mathbb{A}\left[D_C^{\alpha} u(t)\right] &= s^{\alpha} \mathbb{A}\left[u(t)\right] - \frac{1}{s} \sum_{k=0}^{n-1} s^{\alpha-1-k} u^{(k)}(0)\\ &= s^{\alpha} \mathbb{A}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{\alpha-2-k} u^{(k)}(0),\end{aligned}$$

where $\mathbb{A}\left[u(t)\right]$ is the usual Aboodh transform of $u(t)$.

- *The Elzaki transform of the Caputo fractional derivative [13], if $p(s) = s$ and $q(s) = \frac{1}{s}$*

$$\begin{aligned}\mathbb{E}\left[D_C^{\alpha} u(t)\right] &= \frac{1}{s^{\alpha}} \mathbb{E}\left[u(t)\right] - s \sum_{k=0}^{n-1} \frac{1}{s^{\alpha-1-k}} u^{(k)}(0)\\ &= \frac{1}{s^{\alpha}} \mathbb{E}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{2-\alpha+k} u^{(k)}(0),\end{aligned}$$

where $\mathbb{E}\left[u(t)\right]$ is the usual Elzaki transform of $u(t)$.

- *The Sumudu transform of the Caputo fractional derivative [9], if $p(s) = q(s) = \frac{1}{s}$*

$$\begin{aligned}\mathbb{S}\left[D_C^{\alpha} u(t)\right] &= \frac{1}{s^{\alpha}} \mathbb{S}\left[u(t)\right] - \frac{1}{s} \sum_{k=0}^{n-1} \frac{1}{s^{\alpha-1-k}} u^{(k)}(0)\\ &= s^{-\alpha}\left[\mathbb{S}\left[u(t)\right] - \sum_{k=0}^{n-1} s^{k} u^{(k)}(0)\right],\end{aligned}$$

where $\mathbb{S}\left[u(t)\right]$ is the usual Sumudu transform of $u(t)$.

- *The natural transform of the Caputo fractional derivative [24], if* $p(s)=\frac{1}{v}$ *and* $q(s)=\frac{s}{v}$

$$\begin{aligned}\mathbb{N}^{+}\left[D_{C}^{\alpha}u(t)\right] &= \left(\frac{s}{v}\right)^{\alpha}\mathbb{N}^{+}\left[u(t)\right]-\frac{1}{v}\sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\alpha-1-k}u^{(k)}(0).\\ &= \left(\frac{s}{v}\right)^{\alpha}\mathbb{N}^{+}\left[u(t)\right]-\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{v^{\alpha-k}}u^{(k)}(0),\end{aligned}$$

  where $\mathbb{N}^{+}\left[u(t)\right]$ *is the usual natural transform of* $u(t)$.

- *The Shehu transform of the Caputo fractional derivative [15], if* $p(s)=1$ *and* $q(s)=\frac{s}{v}$

$$\mathbb{H}\left[D_{C}^{\alpha}u(t)\right]=\left(\frac{s}{v}\right)^{\alpha}\mathbb{H}\left[u(t)\right]-\sum_{k=0}^{n-1}\left(\frac{s}{v}\right)^{\alpha-1-k}u^{(k)}(0),$$

  where $\mathbb{H}\left[u(t)\right]$ *is the usual Shehu transform of* $u(t)$.

Now, we present the ρ-Jafari transform transform of Mittag-Leffler functions.

Theorem 3.14. *Let* $\alpha,\beta,\rho>0,\lambda\in\mathbb{R}$, *and* $|\lambda|<\left|q^{\alpha}(s)\right|$, *then*

$$\mathbb{J}_{\rho}\left[E_{\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right]=\frac{p(s)q^{\alpha-1}(s)}{q^{\alpha}(s)-\lambda}, \tag{19}$$

and

$$\mathbb{J}_{\rho}\left[\left(\frac{t^{\rho}}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right]=\frac{p(s)q^{\alpha-\beta}(s)}{q^{\alpha}(s)-\lambda}. \tag{20}$$

Proof. From equation (6) and Theorem 3.5, we have

$$\begin{aligned}\mathbb{J}_{\rho}\left[E_{\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right] &= \mathbb{J}_{\rho}\left[\sum_{k=0}^{+\infty}\frac{\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)^{k}}{\Gamma(k\alpha+1)}\right]\\ &= \sum_{k=0}^{+\infty}\frac{\lambda^{k}}{\rho^{k\alpha}\Gamma(k\alpha+1)}\mathbb{J}_{\rho}\left[t^{\rho k\alpha}\right].\end{aligned}$$

According to Theorem 3.7, we get

$$\begin{aligned}
\mathbb{J}_\rho\left[E_\alpha\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right] &= \sum_{k=0}^{+\infty}\frac{\lambda^k}{\rho^{k\alpha}\Gamma(k\alpha+1)}\mathbb{J}_\rho\left[t^{\rho k\alpha}\right] \\
&= \sum_{k=0}^{+\infty}\frac{\lambda^k}{\rho^{k\alpha}\Gamma(k\alpha+1)}\frac{\rho^{k\alpha}p(s)}{q^{k\alpha+1}(s)}\Gamma\left(k\alpha+1\right) \\
&= \frac{p(s)}{q(s)}\sum_{k=0}^{+\infty}\left(\frac{\lambda}{q^\alpha(s)}\right)^k \\
&= \frac{p(s)}{q(s)}\frac{1}{1-\frac{\lambda}{q^\alpha(s)}}, \left|\frac{\lambda}{q^\alpha(s)}\right| < 1 \\
&= \frac{p(s)q^{\alpha-1}(s)}{q^\alpha(s)-\lambda}, |\lambda| < \left|q^\alpha(s)\right|.
\end{aligned}$$

From equation (7) and Theorem 3.5, we have

$$\begin{aligned}
\mathbb{J}_\rho\left[\left(\frac{t^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right] &= \mathbb{J}_\rho\left[\left(\frac{t^\rho}{\rho}\right)^{\beta-1}\sum_{k=0}^{+\infty}\frac{\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)^k}{\Gamma(k\alpha+\beta)}\right] \\
&= \sum_{k=0}^{+\infty}\frac{\lambda^k}{\rho^{k\alpha+\beta-1}\Gamma(k\alpha+\beta)}\mathbb{J}_\rho\left[t^{(k\alpha+\beta-1)\rho}\right].
\end{aligned}$$

According to Theorem 3.7, we get

$$\begin{aligned}
\mathbb{J}_\rho\left[\left(\frac{t^\rho}{\rho}\right)^{\beta-1}E_{\alpha,\beta}\left(\lambda\left(\frac{t^\rho}{\rho}\right)^\alpha\right)\right] &= \sum_{k=0}^{+\infty}\frac{\lambda^k}{\rho^{k\alpha+\beta-1}\Gamma(k\alpha+\beta)}\mathbb{J}_\rho\left[t^{(k\alpha+\beta-1)\rho}\right] \\
&= \sum_{k=0}^{+\infty}\frac{\lambda^k}{\rho^{k\alpha+\beta-1}\Gamma(k\alpha+\beta)}\frac{\rho^{k\alpha+\beta-1}p(s)}{q^{k\alpha+\beta}(s)}\Gamma\left(k\alpha+\beta\right) \\
&= \frac{p(s)}{q^\beta(s)}\sum_{k=0}^{+\infty}\left(\frac{\lambda}{q^\alpha(s)}\right)^k \\
&= \frac{p(s)}{q^\beta(s)}\frac{1}{1-\frac{\lambda}{q^\alpha(s)}}, \left|\frac{\lambda}{q^\alpha(s)}\right| < 1 \\
&= \frac{p(s)q^{\alpha-\beta}(s)}{q^\alpha(s)-\lambda}, |\lambda| < \left|q^\alpha(s)\right|.
\end{aligned}$$

The proof is complete. ☑

4. Applications

In this section, we examine the validity of the ρ-Jafari transform to some linear and nonlinear generalized fractional differential equations.

Example 4.1. Consider the generalized fractional differential equation

$$D^{\alpha,\rho}u(t)-\lambda u(t)=f(t),t>0,0<\alpha\leq 1, \tag{21}$$

with the initial condition

$$I^{1-\alpha,\rho}u(0)=b, \tag{22}$$

where λ and b are constants and $D^{\alpha,\rho}$ denotes the generalized fractional derivative of order $\alpha,\rho>0$ of the function $u(t)$.

Applying the ρ-Jafari transform on both sides of equation (21) and using Theorem 3.11, we get

$$q^{\alpha}(s)\mathbb{J}_{\rho}\left[u(t)\right]-p(s)I^{1-\alpha,\rho}u(0)-\lambda\mathbb{J}_{\rho}\left[u(t)\right]=\mathbb{J}_{\rho}\left[f(t)\right]. \tag{23}$$

Substituting equation (22) into equation (23), we get

$$\left(q^{\alpha}(s)-\lambda\right)\mathbb{J}_{\rho}\left[u(t)\right]-p(s)b=\mathbb{J}_{\rho}\left[f(t)\right].$$

So

$$\mathbb{J}_{\rho}\left[u(t)\right]=\frac{p(s)}{q^{\alpha}(s)-\lambda}b+\frac{1}{q^{\alpha}(s)-\lambda}\mathbb{J}_{\rho}\left[f(t)\right].$$

Using Theorems 3.5, 3.6 and 3.14, we get

$$\begin{aligned}\mathbb{J}_{\rho}\left[u(t)\right]&=b\mathbb{J}_{\rho}\left[\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right]+\mathbb{J}_{\rho}\left[\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)*f(t)\right]\\&=\mathbb{J}_{\rho}\left[b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)+\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)*f(t)\right].\end{aligned}$$

Therefore, we have

$$u(t)=b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)+\int_{0}^{t}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\alpha}\right)\frac{f(\tau)}{\tau^{1-\rho}}d\tau. \tag{24}$$

When $\rho=1$ in equation (24), we get

$$u(t)=bt^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)+\int_{0}^{t}\left(t-\tau\right)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda\left(t-\tau\right)^{\alpha}\right)f(\tau)d\tau.$$

This is the exact solution of equations (21)-(22) which is perfectly consistent with other existing methods [17, 24].

Example 4.2. Consider the Caputo generalized fractional differential equation

$$D_C^{\alpha,\rho}u(t) + bu(t) = 0, t > 0, 1 < \alpha \leq 2, \tag{25}$$

with the initial conditions

$$u(0) = c_0, \gamma u(0) = c_1, \tag{26}$$

where b, c_0 and c_1 are constants and $D_C^{\alpha,\rho}$ denotes the Caputo generalized fractional derivative of order $\alpha, \rho > 0$ of the function $u(t)$.

Applying the ρ-Jafari transform on both sides of equation (25) and using Theorem 3.12, we get

$$q^{\alpha}(s)\mathbb{J}_{\rho}\left[u(t)\right] - p(s)\left(q^{\alpha-1}(s)u(0) + q^{\alpha-2}(s)\gamma u(0)\right) + b\mathbb{J}_{\rho}\left[u(t)\right] = 0. \tag{27}$$

Substituting equation (26) into equation (27), we get

$$\left(q^{\alpha}(s) + b\right)\mathbb{J}_{\rho}\left[u(t)\right] - p(s)\left(c_0 q^{\alpha-1}(s) + c_1 q^{\alpha-2}(s)\right) = 0.$$

So

$$\mathbb{J}_{\rho}\left[u(t)\right] = c_0\frac{p(s)q^{\alpha-1}(s)}{q^{\alpha}(s)+b} + c_1\frac{p(s)q^{\alpha-2}(s)}{q^{\alpha}(s)+b}.$$

Using Theorems 3.5 and 3.14, we get

$$\begin{aligned}\mathbb{J}_{\rho}\left[u(t)\right] &= c_0\mathbb{J}_{\rho}\left[E_{\alpha}\left(-b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right] + c_1\mathbb{J}_{\rho}\left[\frac{t^{\rho}}{\rho}E_{\alpha,2}\left(-b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right]\\ &= \mathbb{J}_{\rho}\left[c_0E_{\alpha}\left(-b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) + c_1\frac{t^{\rho}}{\rho}E_{\alpha,2}\left(-b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right)\right].\end{aligned}$$

Therefore, we have

$$u(t) = c_0E_{\alpha}\left(-b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right) + c_1\frac{t^{\rho}}{\rho}E_{\alpha,2}\left(-b\left(\frac{t^{\rho}}{\rho}\right)^{\alpha}\right). \tag{28}$$

When $\rho = 1$ in equation (28), we get

$$u(t) = c_0E_{\alpha}\left(-bt^{\alpha}\right) + c_1t^{\alpha}E_{\alpha,2}\left(-bt^{\alpha}\right).$$

This is the exact solution of equations (25)-(26) which is perfectly consistent with other existing methods [17, 24].

Example 4.3. Consider the non-linear Caputo generalized fractional differential equation

$$D_C^{\alpha,\rho}u(t) = u^2(t) + 1, t > 0, 0 < \alpha \leq 1, \tag{29}$$

with the initial condition

$$u(0) = 0. \tag{30}$$

where $D_C^{\alpha,\rho}$ denotes the Caputo generalized fractional derivative of order $\alpha, \rho > 0$ of the function $u(t)$.

Applying the ρ-Jafari transform on both sides of equation (29) and using Theorems 3.7 and 3.12, we get

$$\mathbb{J}_\rho\left[u(t)\right] = \frac{p(s)}{q^{\alpha+1}(s)} + \frac{1}{q^\alpha(s)}\mathbb{J}_\rho\left[u^2(t)\right]. \tag{31}$$

Taking the inverse ρ-Jafari transform of both sides of equation (31), we obtain

$$u(t) = \frac{t^{\alpha\rho}}{\rho^\alpha\Gamma\left(\alpha+1\right)} + \mathbb{J}_\rho^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}_\rho\left[u^2(t)\right]\right). \tag{32}$$

Now, by using Adomian decomposition method, then the equation (32) can be rewritten as

$$\sum_{n=0}^{\infty} u(t) = \frac{t^{\alpha\rho}}{\rho^\alpha\Gamma\left(\alpha+1\right)} + \mathbb{J}_\rho^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}_\rho\left[\sum_{n=0}^{\infty} A_n\right]\right).$$

where A_n are the Adomian polynomials of the nonlinear term $u^2(t)$ and it can be formed as

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, n = 0, 1, 2, ...$$

The first few components of A_n polynomials are given by

$$\begin{aligned}
A_0 &= u_0^2, \\
A_1 &= 2u_0u_1, \\
A_2 &= 2u_0u_2 + u_1^2, \\
&\vdots
\end{aligned}$$

We define the following recursive formula

$$\begin{aligned}
u_0(t) &= \frac{t^{\alpha\rho}}{\rho^\alpha\Gamma\left(\alpha+1\right)}, \\
u_{n+1}(t) &= \mathbb{J}_\rho^{-1}\left(\frac{1}{q^\alpha(s)}\mathbb{J}_\rho\left[A_n\right]\right), n \geq 0.
\end{aligned}$$

This gives

$$u_0(t)=\frac{1}{\Gamma(\alpha+1)}\frac{t^{\alpha\rho}}{\rho^{\alpha}},$$

$$u_1(t)=\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\frac{t^{3\alpha\rho}}{\rho^{3\alpha}},$$

$$u_2(t)=\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\frac{t^{5\alpha\rho}}{\rho^{5\alpha}},$$

$$u_3(t)=\left(\frac{4\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)+\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}\right)\frac{t^{7\alpha\rho}}{\rho^{7\alpha}},$$

$$\vdots$$

Therefore, the series solution is given by

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\alpha+1)}\frac{t^{\alpha\rho}}{\rho^{\alpha}}+\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\frac{t^{3\alpha\rho}}{\rho^{3\alpha}}+\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{\Gamma^3(\alpha+1)\Gamma(3\alpha+1)\Gamma(5\alpha+1)}\frac{t^{5\alpha\rho}}{\rho^{5\alpha}}\\&+\left(\frac{4\Gamma(2\alpha+1)\Gamma(3\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)+\Gamma^2(2\alpha+1)\Gamma(5\alpha+1)\Gamma(6\alpha+1)}{\Gamma^4(\alpha+1)\Gamma^2(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)}\right)\frac{t^{7\alpha\rho}}{\rho^{7\alpha}}+...\end{aligned}$$

Taking $\alpha=1$ in the above equation, we get

$$\begin{aligned}u(t)&=\frac{t^{\rho}}{\rho^{\alpha}}+\frac{1}{3}\frac{t^{3\rho}}{\rho^{3}}+\frac{2}{15}\frac{t^{5\rho}}{\rho^{5}}+\frac{17}{315}\frac{t^{7\rho}}{\rho^{7}}+\cdots\\&=\tanh\left(\frac{t^{\rho}}{\rho^{\alpha}}\right).\end{aligned}$$

This is the exact solution of equations (29)-(30) which is perfectly consistent with other existing methods [2, 4].

5. Conclusions

In this paper, we have defined an interesting type of general integral transform modification of the generalized fractional integrals and derivatives. This modification called the ρ-Jafari transform. Moreover, we studied some important properties of the the ρ-Jafari transform. As an application and justification of our results, we illustrate some applications. The use of ρ-Jafari transform gives more advantages in fractional calculus, especially in generalized fractional differential equations to describe the systems under study.

Acknowledgements. The author would like to thank Professor Francisco José Marcellán (Managing Editor) and referees for their useful technical comments and valuable suggestions to improve the readability of the paper, which led to a significant improvement of the paper.

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(Recibido en marzo de 2023. Aceptado en septiembre de 2024)

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