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Independence numbers of some double vertex graphs and pair graphs

Números de independencia de algunas gráficas de vértice doble y gráficas de pares

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Abstract. The combinatorial properties of the double vertex graph of a graph have been widely studied since the 90's. However only very few results are know about the independence number of such graphs. In this paper we obtain the independence numbers of the double vertex graphs of fan graphs and wheel graphs. Also we obtain the independence numbers of the pair graphs, that is a kind of generalization of the double vertex graphs, of some families of graphs.

Key words and phrases. Double vertex graphs, pair graphs, independence number.

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Resumen. Las propiedades combinatorias del grafo de vértice doble de un grafo han sido ampliamente estudiadas desde los años 90. Sin embargo, sólo se conocen pocos resultados sobre el número de independencia de dichos grafos. En este artículo obtenemos los números de independencia de los grafos de vértice doble de las los grafos abanico y rueda. También obtenemos los números de independencia de los grafos de pares, que es un tipo de generalización de los grafos de vértice doble, de algunas familias de grafos.

Palabras y frases clave. Grafos de vértice doble, grafos de pares, número de independencia.

1. Introduction

Let G be a graph of order n. The double vertex graph $G^{(2)}$ of G is defined as the graph with vertex set all 2-subsets of $V(G)$, where two vertices are adjacent in $G^{(2)}$ whenever their symmetric difference is an edge of G. The double vertex graph of G is also denoted as $F_2(G)$. This concept, and its generalization called

k-token graphs, has been redefined several times and with different names. The double vertex graphs were defined and widely studied by Alavi et al. [1, 2, 3], but we can find them earlier in a thesis of G. Johns [17], with the name of the 2-subgraph graph of G. T. Rudolph [21] redefined the double vertex graphs with the name of symmetric powers of graphs and used this graphs to studied the graph isomorphism problem and to study some problems in quantum mechanics and has motivated several works of different authors, see, e.g., [6, 7, 8, 14] and the references therein. Later, R. Fabila-Monroy, et. al. [13] reintroduce this concept but now with the name of token graphs, where the double vertex graphs are precisely the 2-token graphs, and studied several combinatorial properties of this graphs such as: connectivity, diameter, cliques, chromatic number and Hamiltonian paths. After this work, there are a lot of results about different combinatorial parameters of token graphs, see for example [9, 10, 4, 12, 15, 19, 20].

In H. de Alba, et. al. [5] began the study of the independence number of k-token graphs and in particular for the double vertex graphs of some special graphs such as: paths, cycles, complete bipartite graphs, star graphs, etc. A subset I of vertices of G is an *independent set* if no two vertices in I are adjacent. The *independence number* $\alpha(G)$ of G is the number of vertices in a largest independent set in G. it is know that to determine the independence number is an NP-hard [18] problem in its generality.

In this work we obtain the independence number of the double vertex graphs of fan graphs and wheel graphs. The *fan graph* $F_{m,1}$ is defined as the join graph $P_m + K_1$, where P_m denote the path graph of order n and K_1 the complete graph of order 1, and wheel graph $W_{m,1}$ is defined as the joint graph $C_m + K_1$, where C_m denote the cycle of order m. Our main results about independence number of double vertex graphs are the following:

Theorem 1.1. *Let $m \geq 2$ be an integer. Then*

$$\alpha\left(F_{m,1}^{(2)}\right) = \left\lfloor \frac{m^2}{4} \right\rfloor.$$

Theorem 1.2. *Let $m \geq 4$ be and integer. Then*

$$\alpha\left(W_{m,1}^{(2)}\right) = \left\lfloor \frac{m}{2} \left\lfloor \frac{m}{2} \right\rfloor \right\rfloor.$$

1.1. Pair graph of graphs

Let G be a graph of order $n \geq 2$. The *pair graph* $C(G)$ of G is the graph whose vertex set consists of all 2-multisets of $V(G)$ where tho vertices $\{x, y\}$ and $\{u, v\}$ are adjacent if and only if $\{x, y\} \cap \{u, v\} = \{a\}$ and if $x = u = a$, then y and v are adjacent in G. The pair graphs, also called *complete double vertex graph*, of a graph, were implicitly introduced by Chartrand et al. [11] and defined explicitly by Jacob et. al. [16], were the first combinatorial properties were studied. The pair graphs are a generalization of the double vertex graphs and G is always isomorphic to a subgraph of $C(G)$.

For the case of the independence number of complete double vertex graphs we have the following results.

Theorem 1.3. *If $m \geq 3$ is an integer, then*

$$\alpha(C(P_m)) = \left\lfloor \frac{(m+1)^2}{4} \right\rfloor.$$

Theorem 1.4. *Let $m \geq 1$ be an integer. Then*

$$\alpha(C(F_{m,1})) = \alpha(C(P_m)) + 1$$

Notice that the series $\{\lfloor (m+1)^2/4 \rfloor + 1\}_{m \geq 1}$ coincides with A033638$(m+1)$ in OEIS.

Theorem 1.5. *Let $m \geq 3$ be an integer. Then*

$$\alpha(C(C_m)) = \begin{cases} k(k+1) + \lfloor (k+1)/2 \rfloor & m = 2k+1 \\ k(k+1) & m = 2k \end{cases}$$

Theorem 1.6. *Let $m \geq 3$ be an integer. Then*

$$\alpha(C(W_{m,1})) = \alpha(C(C_m)) + 1.$$

In the rest of the papers we prove all these results in different sections.

2. Preliminary results

In the proofs of some of our results, we use the following known facts.

Lemma 2.1. *If H is an induced subgraph of G, then $\alpha(H) \leq \alpha(G)$.*

Let $G \Box H$ denote the cartesian product of graphs G and H.

Proposition 2.2. *Let r and s be positive integers. Then*

$$\alpha(P_r \Box P_s) = \left\lceil \frac{r}{2} \right\rceil \left\lceil \frac{s}{2} \right\rceil + \left(r - \left\lceil \frac{r}{2} \right\rceil \right) \left(s - \left\lceil \frac{s}{2} \right\rceil \right)$$

The *disjoint union* of G_1 and G_2 is the graph $G_1 \cup G_2$ whose vertex and edge sets are the disjoint union of the corresponding sets. It is easy to prove that the disjoint union of graphs is associative and commutative. Therefore, if G is a graph whose components are $G_1, ..., G_m$, then $G = \cup_{i=1}^{m} G_i$.

Proposition 2.3. *If $G = \cup_{i=1}^{k} G_i$, where G_i is a component of G with $|G_i| \geq 2$, for every i, then*

$$G^{(2)} = \bigcup_{i=1}^{k} G_i^{(2)} \bigcup_{j>i\geq 1}^{k} G_{ij},$$

where $G_{ij} \simeq G_j \square G_i$.

The following proposition appears in the proof of Lema 12 in [10].

Proposition 2.4. *Let X be a subset of $V(G)$ and $G' = G - X$. Then $F_k(G')$ is isomorphic to the graph obtained from $F_k(G)$ by deleting all vertices in $F_k(G)$ such that have al least one element of X.*

In [5] was proved that $\alpha(P_m^{(2)}) = \lfloor m^2/4 \rfloor$, $m \geq 2$. This is sequence $A002620(n)$ in The On-Line Encyclopedia of Integer Sequences (OEIS) [22] and has the following properties that will be useful.

Proposition 2.5. *Let $a(n) = A002620(n)$, $n \geq 0$.*

(1) $a(n) = \lfloor n/2 \rfloor \lceil n/2 \rceil = \lfloor n^2/4 \rfloor$.

(2) $a(n) = a(n-1) + \lfloor n/2 \rfloor = a(n-1) + \lceil (n-1)/2 \rceil$, $n > 0$, $a(0) = a(1) = 0$.

(3) $a(n) = a(n-2) + n - 1$, $a(0) = 1, a(1) = 0$, $n \geq 2$.

If $|G| = 1$, then we use the convention that $G^{(2)} = \varnothing$ and that $\alpha(G^{(2)}) = 0$.

3. Proof of Theorem 1.1

For the graph $F_{m,1}$, we consider that $V(P_m) = \{1, \dots, m\}$, $E(P_m) = \{\{i, i+1\} : 1 \leq i \leq m-1\}$ and $V(K_1) = \{m+1\}$. We will use T_k to denote $P_k^{(2)}$.

If A_m is the set of all 2-subsets of $V(P_m)$ and $B = \{\{a, m+1\} : a \in V(P_m)\}$, then $\{A_m, B\}$ is a partition of $V(F_{m,1}^{(2)})$. Notice that the subgraph of $F_{m,1}^{(2)}$ induced by A_m is isomorphic to T_m and that the subgraph induced by B is isomorphic to P_m. Sometimes we use T_m and B as sets of vertices or as induced subgraphs of $F_{m,1}^{(2)}$ if it is clear from the context. For $q \in \{1, \dots, m\}$, we define the following subset of vertices of $F_{m,1}^{(2)}$

$$R_q = \{\{q, i\} : i \in \{1, \dots, m\} - \{q\}\}.$$

In fact, $R_q \subseteq T_m$, for every $q \in \{1, \dots, m\}$ (see Figure 1 as example).

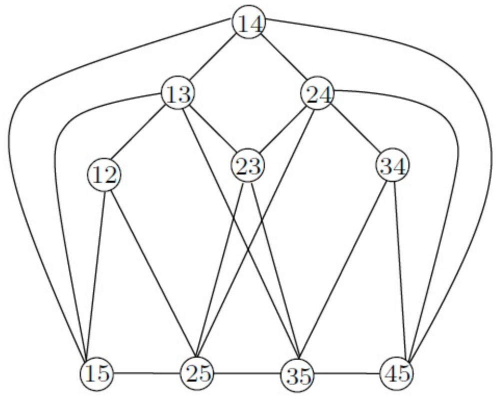

FIGURE 1. Double vertex graph of $F_{4,1}$. $B = \{\{1,5\},\{2,5\},\{3,5\},\{4,5\}\}, T_4 = V(F_{4,1}^{(2)}) - B$ and $R_2 = \{\{1,2\},\{2,3\},\{2,4\}\}$.

We need some propositions about the independence number of $T_m - R_i$ and $T_m - R_i \cup R_j$ (see Figure 2 for some examples of these graphs).

Proposition 3.1. *Let $m \geq 4$ be an integer. Then $\alpha(T_m - R_i) = \alpha(T_{m-1})$, for all $i \in \{1,\dots,m\}$.*

Proof. By Proposition 2.4, $T_m - R_i$ is isomorphic to the double vertex graph of $P_m - i$, for every $i \in \{1,\dots,m\}$. Notice that $P_m - i$ consists of two components, one isomorphic to P_{i-1} and the other isomorphic to P_{m-i}.

By Proposition 2.3 it follows that $(P_m - i)^{(2)}$ has three components. One isomorphic to T_{m-i}, another isomorphic to T_{i-1} and the last one isomorphic to $P_{m-i} \Box P_{i-1}$. Therefore

$$\begin{aligned}\alpha(T_m - R_i) &= \alpha(T_{m-i}) + \alpha(T_{i-1}) + \alpha(P_{m-i} \Box P_{i-1})\\ &= \lfloor (m-i)^2/4 \rfloor + \lfloor (i-1)^2/4 \rfloor + \\ &\quad \left\lceil \frac{m-i}{2} \right\rceil \left\lceil \frac{i-1}{2} \right\rceil + \left(m - i - \left\lceil \frac{m-i}{2} \right\rceil \right) \left(i - 1 - \left\lceil \frac{i-1}{2} \right\rceil \right)\\ &= \left\lfloor \frac{(m-1)^2}{4} \right\rfloor.\end{aligned}$$

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Proposition 3.2. *Let $m \geq 4$ be an integer. Let $S \subset \{1,\dots,m\}$ such that in S there does not exist consecutive integers and $|S| \geq 2$. Then $\alpha(T_m - \cup_{i \in S} R_i) < \alpha(T_{m-1})$.*

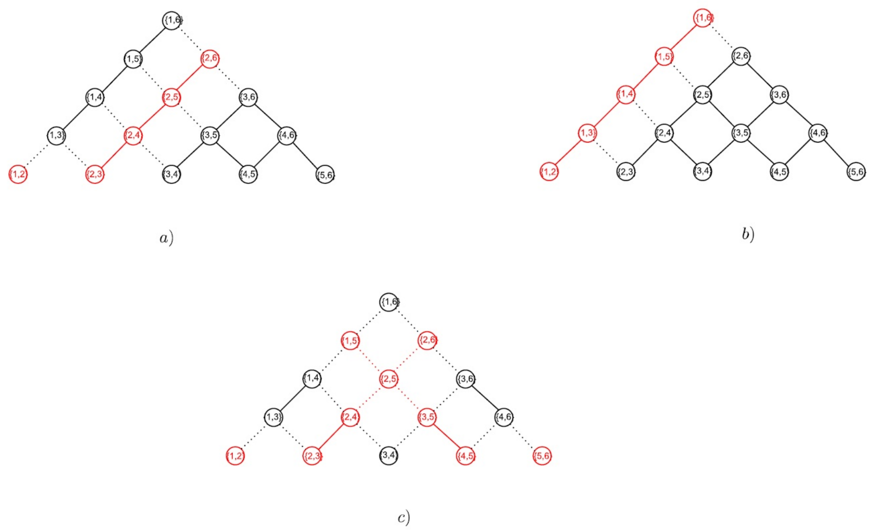

FIGURE 2. a) Graph $T_8 - R_2$, b) graph $T_6 - R_1$, c) graph $T_6 - (R_2 \cup R_5)$.

Proof. In the view of Lemma 2.1, it is enough to prove the case when $|S| = 2$. If for some x in S, $x \in \{1, m\}$ then $T_m - R_x = T_{m-1}$ and by Proposition 3.1 we have that $\alpha(T_m - R_x \cup R_y) = \alpha(T_{m-1} - R_y) = \alpha(T_{m-2}) < \alpha(T_{m-1})$. Now, let $S = \{i, j\}$, with $1 < i < j < m$. The proof of this case is by contradiction. Suppose that $\alpha(T_m - R_i \cup R_j) = \alpha(T_{m-1})$. Let I be an independent set in $T_m - R_i \cup R_j$ of cardinality $\alpha(T_{m-1})$. Let $X = \{j-1, j\}$ and $Z = \{j, j+1\}$. As the vertices X and Z belong to R_j, then both vertices does not belong to I. The open neighborhood of $\{X, Z\}$ is a subset of $\{\{j-2, j\}, \{j-1, j+1\}, \{j, j+2\}\}$. Of this vertices, only $\{j-1, j+1\}$ could be in I. Therefore, the set $I' = (I - \{\{j-1, j+1\}\}) \cup \{X, Z\}$ is and independent set in $T_m - R_i$ (because $i \notin \{j-1, j+1\}$) of cardinality greater than $\alpha(T_{m-1})$, a contradiction. ☑

It is clear that $\alpha(F^{(2)}_{1,1}) = 1$. We are ready to prove our result.

Theorem 3.3. 1.1 *Let $m \geq 2$ be an integer. Then*

$$\alpha(F^{(2)}_{m,1}) = \lfloor m^2/4 \rfloor.$$

Proof. If $m = 2$, then $F_{2,1} \simeq K_3$ and $K^{(2)}_3 \simeq K_3$. Therefore, $\alpha(F^{(2)}_{2,1}) = 1$ as desired. The case $m = 3$ can be checked by hand or by computer. We suppose that $m \geq 4$. As T_m is isomorphic to $P^{(2)}_m$ it follows from Lemma 2.1 that $\alpha(P^{(2)}_m) \leq \alpha(F^{(2)}_{m,1})$. We will show that $\alpha(F^{(2)}_{m,1}) \leq \alpha(P^{(2)}_m)$. We use the fact that the subgraph of $F_{m,1}^{(2)}$ induced by the vertex set B is isomorphic to P_m and $\alpha(P_m) = \lceil m/2 \rceil$.

Let I be an independent set in $F_{m,1}^{(2)}$. We have the following cases:

Case 1. If $I \subseteq T_m$, then $|I| \leq \alpha(P_m^{(2)})$.

Case 2. If $I \subseteq B$, then $|I| \leq \lceil m/2 \rceil \leq \lfloor m^2/4 \rfloor$.

Case 3. $I = I^{'} \cup I^{''}$, where $I^{'}$, $I^{''}$ are both non-empty and $I^{'} \subset T_m, I^{''} \subset B$, respectively.

Let $S = \{i\colon \{i, m+1\} \in I''\}$. As I'' is an independent set of P_m, there do not exist any two consecutive integers in S. We claim that $\cup_{i\in S} R_i \cap I' = \varnothing$. Indeed, suppose that $\{x, y\} \in \cup_{i\in S} R_i \cap I'$. Then $\{x, y\} \in R_i$, for some $i \in S$. Without lost of generality we can suppose that $\{x, y\} = \{i, y\}$, with $y \in \{1, \ldots, m\} - \{i\}$. By definition of S, vertex $\{i, m+1\}$ belong to I''. Now $\{i, y\} \triangle \{i, m+1\} = \{y, m+1\}$ that is an edge in $F_{m,1}$, and hence $\{i, y\} \sim \{i, m+1\}$ in $F_{m,1}^{(2)}$. But this is a contradiction, since I is an independent set. This shows that I' is a subset of $T_m - \cup_{i\in S} R_i$ and hence

$$|I'| \leq \alpha(T_m - \cup_{i\in S} R_i). \tag{1}$$

Now, if $|I''| = 1$, then $|S| = 1$ and by Propositions 3.1

$$|I| \leq \alpha(T_{m-1}) + 1 \leq \alpha(T_{m-1}) + \lfloor m/2 \rfloor = \lfloor m^2/4 \rfloor,$$

where the last inequality follows from Proposition 2.5(2). Finally, we consider that $2 \leq |I^{''}| \leq \lceil m/2 \rceil$. As $|I''| \geq 2$, then $|S| \geq 2$. By Proposition 3.2 and Equation (1) we have that

$$\begin{aligned} |\ I\ | &\leq \alpha(T_m - U_{i\in S} R_i) + |I''| \\ &< \alpha(T_{m-1}) + \lceil m/2 \rceil \\ &\leq \alpha(T_{m-1}) + \lceil m/2 \rceil - 1 \\ &\leq \alpha(T_{m-1}) + \lfloor m/2 \rfloor \\ &\leq \lfloor m^2/4 \rfloor. \end{aligned}$$

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4. Proof of Theorem 1.2

For the wheel graph $W_{m,1}$ we consider that $V(C_m) = \{1, \ldots, m\}$, $E(C_m) = \{\{i, i+1\}\colon 1 \leq i \leq m-1\} \cup \{\{1, m\}\}$ and $V(K_1) = \{m+1\}$. Let H_m denotes the subgraph of $W_{m,1}^{(2)}$ induced by all 2-subsets of $V(C_m)$. Let D denotes the subgraph of $W_{m,1}^{(2)}$ induced by the vertex set $\{\{i, m+1\}\colon i \in \{1, \ldots, m\}\}$. The graph H_m is isomorphic to $C_m^{(2)}$ and D is isomorphic to C_m. We also use H_m

and D as vertex sets. It is well-known that $\alpha(C_m) = \lfloor m/2 \rfloor$ and in [5] was proved that $\alpha(C_m^{(2)}) = \lfloor m\lfloor m/2\rfloor/2 \rfloor$, $m \geq 3$.

It can be checked by computer that $\alpha(W_{3,1}^{(2)}) = 2$. We now prove our main result in this section.

Theorem 4.1. 1.2 *Let $m \geq 4$ be and integer. Then*

$$\alpha(W_{m,1}^{(2)}) = \alpha(C_m^{(2)}).$$

Proof. As H_m is isomorphic to $C_m^{(2)}$ and it is an induced subgraph of $W_{m,1}^{(2)}$ we have that $\alpha(C_m^{(2)}) \leq \alpha(W_{m,1}^{(2)})$. We will show that $\alpha(W_{m,1}^{(2)}) \leq \alpha(C_m^{(2)})$.

Let I be an independent set in $W_{m,1}^{(2)}$. If $I \subset H_m$, then $|I| \leq \alpha(C_m^{(2)})$. If $I \subset D$, then

$$|I| \leq \lfloor m/2 \rfloor \leq \lfloor m\lfloor m/2\rfloor/2 \rfloor = \alpha(C_m^{(2)}).$$

Now suppose that $I = I' \cup I''$, where $I^{'}$, $I^{''}$ are both non-empty and $I' \subset H_m$, $I'' \subset D$. As D is isomorphic to C_m then $|I''| \leq \lfloor m/2 \rfloor$. For $q \in \{1, \ldots, m\}$, let R_q be defined as in previous section. Let

$$U = \{i \in \{1, \ldots, m\} \colon \{i, m+1\} \in I''\}.$$

It is easy to show that $\cup_{i\in U} R_i \cap I' = \varnothing$. Therefore we have $|I'| \leq \alpha(H_m - \cup_{i\in U} R_i)$.

By Proposition 2.4, the graph $H_m - R_q$ is isomorphic to the double vertex graph of $C_m - q$, for every $q \in \{1, \ldots, m\}$. But as $C_m - q$ is isomorphic to P_{m-1} then $H_m - R_q \simeq T_{m-1}$. We like to bound $|I|$ using that $|I'| \leq \alpha(H_m - \cup_{i\in U} R_i)$. First, consider that $U = \{x\}$, for some $x \in \{1, \ldots, m\}$, that is $I'' = \{\{x, m+1\}\}$. We have that

$$|I| \leq \alpha(H_m - R_x) + 1 = \alpha(T_{m-1}) + 1 = \lfloor (m-1)^2/4 \rfloor + 1 \leq \lfloor m\lfloor m/2\rfloor/2 \rfloor,$$

where the last inequality holds because $m \geq 4$.

Now, suppose that $|U| \geq 2$. First note that if $q \in V(C_m)$, then the double vertex graph of $W_{m,1} - q$ is isomorphic to the double vertex graph of $F_{m-1,1}$. Therefore, by Proposition 2.4, it follows that

$$W_{m,1}^{(2)} - (R_q \cup \{\{q, m+1\}\} \simeq (W_{m,1} - q)^{(2)} \simeq F_{m-1,1}^{(2)}. \tag{2}$$

We would like to obtain $\alpha(H_m - \cup_{i\in U} R_i)$. By Equation (2), after we delete one set R_q and the vertex $\{q, m+1\}$ from $W_{m,1}^{(2)}$, for $q \in U$, we obtain an isomorphic copy of $F_{m-1,1}^{(2)}$, which in turn contains isomorphic copies of the remaining sets R_i, for $i \in U - \{q\}$. Then we are in Case (3) of the proof of Theorem 1.1

for $F^{(2)}_{m-1,1}$ and $S = U - \{q\}$ (with the corresponding relabeling given by the isomorphism between $(W_{m,1} - q)^{(2)}$ and $F_{m-1,1}$). Therefore, for any $x \in U$

$$H_m - \cup_{i\in U} R_i = (H_m - R_x) - \cup_{i\in U-\{x\}} R_i \simeq T_{m-1} - \cup_{i\in S} R_i,$$

Using Proposition 3.2 for $F^{(2)}_{m-1,1}$ and S we have that

$$\alpha(H_m - \cup_{i\in U} R_i) = \alpha(T_{m-1} - \cup_{i\in S} R_i) \leq \alpha(T_{m-2}).$$

And hence

$$\begin{aligned} |I| &\leq \alpha(H_m - \cup_{i\in U} R_i) + |I''| \\ &\leq \alpha(T_{m-2}) + \lfloor m/2 \rfloor \\ &\leq \lfloor (m-2)^2/4 \rfloor + \lfloor m/2 \rfloor \\ &\leq \lfloor m \lfloor m/2 \rfloor /2 \rfloor \\ &= \alpha(C^{(2)}_m). \end{aligned}$$

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5. Proof of Theorem 1.3

The proof of Theorem 1.3 follows directly from the following result.

Theorem 5.1. *For any non negative integer $n \geq 3$ we have*

$$C(P_n) \simeq P^{(2)}_{n+1}$$

Proof. Without loss of generality we can suppose that for $\{a, b\} \in V(C(P_n))$, $a \leq b$, and for $\{a, b\}$ in $P^{(2)}_m$, $a < b$. Let $\phi\colon C(P_n) \to P^{(2)}_{n+1}$ be the function defined by $\phi(\{i, j\}) = \{i, j + 1\}$. It is an exercise to show that this function is a graph isomorphism between $C(P_n)$ and $P^{(2)}_{n+1}$. ☑

6. Proof of Theorem 1.4

The vertex set of $C(F_{m,1})$ can be partitioned in $\{T_{m+1}, B\}$ where T_{m+1} (as induced graph) is isomorphic to $C(P_m)$ (that is isomorphic to $P^{(2)}_{m+1}$ by Theorem 5.1) and

$$B = \{\{i, m + 1\} : 1 \leq i \leq m + 1\}.$$

Notice that the subgraph of $C(F_{m,1})$ induced by B is isomorphic to $F_{m,1}$.

We define the following subset of vertices of $C(F_{m,1})$.

$$R_i = \{\{i, j\}\colon j \in \{1, \ldots, m\}\},$$

The following proposition will be useful in the proof of Theorem 1.4.

Proposition 6.1. *For $m \geq 2$, we have that $\alpha(T_{m+1} - R_i) \leq \lfloor m^2/4 \rfloor + 1$, for any $i \in \{1, \ldots, m\}$.*

Proof. Notice that for $i \in \{1, \ldots, m\}$, the graph $T_{m+1} - R_i$ is isomorphic to the graph $C(P_m - i)$. We have several cases.

Case 1. If $i \in \{1, m\}$, then the graph $T_{m+1} - R_i$ is isomorphic to $C(P_{m-1})$, that it is isomorphic to $P_m^{(2)}$ (by Theorem 5.1), and hence $\alpha(T_{m+1} - R_i) = \lfloor m^2/4 \rfloor$.

Case 2. If $i \in \{2, m-1\}$, then $T_{m+1} - R_i$ consists of three components as follows. One component that is isomorphic to K_1. Such component K_1 is either the vertex $\{1, 1\}$, or the vertex $\{m, m\}$ if $i = 2$ or $i = m-1$, respectively. Another component consists either, in the subgraph generated by the vertices $R_1 - \{\{1, 2\}, \{1, 1\}\}$, when $i = 2$, or $R_m - \{\{m-1, m\}, \{m, m\}\}$ when $i = m-1$. This component is isomorphic to P_{m-2}. The last component is isomorphic to T_{m-1}: when $i = 2$, T_{m-1} will be the subgraph generated by the set of vertices $T_{m+1} - (R_1 \cup R_2)$, and when $i = m - 1$, T_{m-1} will be the subgraph generated by the set of vertices $T_{m+1} - (R_{m-1} \cup R_m)$. Then

$$\begin{aligned}
\alpha(T_{m+1} - R_i) &= \alpha(T_{m-1}) + \alpha(P_{m-2}) + 1 \\
&= \lfloor (m-1)^2/4 \rfloor + \lceil (m-2)/2 \rceil + 1 \\
&\leq \lfloor (m-1)^2/4 \rfloor + \lceil (m-1)/2 \rceil + 1 \\
&\leq \lfloor m^2/4 \rfloor + 1,
\end{aligned}$$

where, for the last inequality, we use the fact that $\lceil (m-2)/2 \rceil \leq \lceil (m-1)/2 \rceil$ and part 2 of Proposition 2.5.

Case 3. If $i \in \{1, \ldots, m\} - \{1, 2, m-1, m\}$, then $T_{m+1} - R_i$ consists of three components that came from the double vertex graph of $P_m - i$. The first component is isomorphic to $C(P_m - \{1, \ldots, i\})$, that in fact is isomorphic to T_{m-i+1}. The other component is isomorphic to $C(P_m - \{i, \ldots, m\})$ that is isomorphic to $C(P_{i-1})$, which in turn is isomorphic to T_i. The last component of $T_{m+1} - R_i$ is the subgraph of T_{m+1} induced for the set of vertices of the form $\{a, b\}$ with $a \in \{1, \ldots, i-1\}$ and $b \in \{i+1, \ldots, m\}$. This last component is isomorphic to the grid graph $P_{m-i} \times P_{i-1}$ and hence

$$\alpha(T_{m+1} - R_i) = \alpha(T_{m-i+1}) + \alpha(T_i) + \alpha(P_{m-i} \times P_{i-1}).$$

Therefore

$$\begin{aligned}
\alpha(T_{m+1} - R_i) &= \lfloor (m-i+1)^2/4 \rfloor + \lfloor i^2/4 \rfloor + \\
&\quad \left\lceil \frac{m-i}{2} \right\rceil \left\lceil \frac{i-1}{2} \right\rceil + \left(m - i - \left\lceil \frac{m-i}{2} \right\rceil \right) \left(i - 1 - \left\lceil \frac{i-1}{2} \right\rceil \right) \\
&\leq \left\lfloor \frac{m^2}{4} \right\rfloor + 1.
\end{aligned}$$

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Corollary 6.2. $\alpha(T_{m+1} - \cup_{i\in S} R_i) \leq \lfloor m^2/4 \rfloor + 1$, *for every* $S \subseteq \{1, \dots, m\}$, $S \neq \varnothing$.

Proof. For every $j \in S$, $T_{m+1} - \cup_{i\in S} R_i$ is an induced subgraph of $T_{m+1} - R_j$ and the result follows by Lemma 2.1 and by Proposition 6.1. ☑

In Figure 3 we show graph $T_{9+1} - R_2 \cup R_5$.

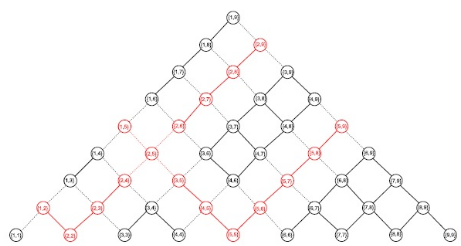

FIGURE 3. Graph $T_{10} - R_2 \cup R_5$

Theorem 6.3. 1.4 *Let* $m \geq 3$ *be an integer. Then*

$$\alpha(C(F_{m,1})) = \alpha(C(P_m)) + 1$$

Proof. Let I be an independent set in T_{m+1} of cardinality $|I| = \alpha(C(P_m))$. As $I \cup \{\{m+1, m+1\}\}$ is an independent set in $C(F_{m,1})$ then $\alpha(C(F_{m,1})) \geq \alpha(C(P_m)) + 1$.

We will show that $\alpha(C(F_{m,1})) \leq \alpha(C(P_m)) + 1$. The case $m = 3$ is easy, so that, we suppose that $m \geq 4$. Let I be an independent set in $C(F_{m,1})$. We have several cases.

Case 1. If $I \subseteq T_{m+1}$, then $|I| \leq \alpha(C(P_m))$.

Case 2. If $I \subseteq B$, then $|I| \leq \lfloor (m+1)/2 \rfloor \leq \lfloor (m+1)^2/4 \rfloor$, because $B \simeq F_{1,m}$ and $m \geq 4$.

Case 3. If $I = I^{'} \cup I^{''}$, such that $I^{'} \subset T_{m+1}$, $I^{''} \subset B$, with $I^{'}$ e $I^{''}$ non empty sets. Let $S = \{i \colon \{i, m+1\} \in I''\}$. In a similar way that in the proof of Case 3 of Theorem 1.1 we can show that $\cup_{i\in S} R_i \cap I' = \varnothing$. This shows that I' is a subset of $T_{m+1} - \cup_{i\in S} R_i$. By Corollary 6.2 we have that $|I'| \leq \lfloor m^2/4 \rfloor + 1$ and as $B \simeq C_{m+1}$ then $|I''| \leq \lfloor (m+1)/2 \rfloor$. Therefore

$$\begin{aligned}|I| &\leq \lfloor m^2/4 \rfloor + \lfloor (m+1)/2 \rfloor + 1\\ &= \lfloor (m+1)^2/4 \rfloor + 1\\ &= \alpha(C(P_m)) + 1,\end{aligned}$$

where we are using part (2) of Proposition 2.5. ☑

7. Proof of Theorem 1.5

We prove Theorem 1.5 by mean of Propositions 7.3 and 7.5. For $q = 1, \ldots, m$, let

$$L_q := \{\{j, m-(q-j)\} \colon 1 \leq j \leq q\}.$$

It is clear that $|L_q| = q$, for every q, and that $\{L_1 \ldots, L_q\}$ is a partition of $V(C(C_m))$. The following proposition shows that most of the sets L_q are independent sets in $C(C_m)$.

Proposition 7.1. *If L_q is not an independent set in $C(C_m)$, then $m = 2q-1$, where $2 \leq q \leq m-1$.*

Proof. Clearly $L_1 = \{\{1, m\}\}$ and $L_m = \{\{1,1\}, \{2,2\}, \ldots, \{m,m\}\}$ are independent sets and hence $2 \leq q \leq m-1$. As L_q is not an independent set and $q \geq 2$, then there exist two adjacent vertices in L_q, say $\{i, m-(q-i)\}$ and $\{j, m-(q-j)\}$. Notice that $i \neq j$ and $i \neq m-(q-i)$. Therefore $i = m-(q-j)$ and $|m-(q-i)-j| \in \{1, m-1\}$. From these equations we obtain that $|m-(q-i)-j| = |2(m-q)|$, which implies that $m = 2q-1$. ☑

Let G be a graph and let A and B subsets of $V(G)$. We say that A and B are linked in G, and is denoted by $A \approx B$, if G has an edge ab such that $a \in A$ and $b \in B$.

Proposition 7.2. *Let $m \geq 4$. The subsets L_i of $V(C(C_m))$ previously defined are linked as follows:*

(1) $L_i \approx L_{i+1}$, for $i \in \{1, \ldots, m-1\}$.

(2) $L_i \approx L_{m-i+1}$, for $i \in \{1, \ldots, m-1\}$.

(3) All the links between the elements in $\{L_1 \ldots, L_m\}$ are given by (1) and (2).

Proof. (1) For $1 \leq i \leq m = 1$, the sets L_i and L_{i+1} are linked because $\{i, m\} \in L_i, \{i+1, m\} \in L_{i+1}$ and $[\{i, m\}, \{i+1, m\}]$ is an edge in $C(C_m)$.
(2) By definition of L_q it follows that $\{\{i, m\}$ and $\{1, m-(i-1)\}$ belongs to L_i, and the vertices $\{1, i\}$ and $\{m-(i-1), m\}$ belongs to $L_{m-(i-1)}$. As

$[\{\{i, m\}, \{1, i\}]$ and $[\{1, m-(i-1)\}, \{m-(i-1), m\}]$ are edges in $C(C_m)$ we obtain that $L_i \approx L_{m-i+1}$.
(3) If L_i is linked with a set L_j, with $|i-j| \neq 1$, then, by the construction of $C(C_m)$, the unique possible vertices in L_i that could be adjacent with vertices in L_j are $\{1, m-i+1\}$ and $\{i, m\}$. But this would implies that $j = m-i+1$ and we are in Case 2. ☑

Proposition 7.3. *Let $k \geq 2$ be an integer. Then*

$$\alpha(C(C_{2k})) = k(k+1).$$

Proof. By Propositions 7.1 and 7.2 we have that

$$I = L_2 \cup L_4 \cup \cdots \cup L_{m-2} \cup L_m$$

is an independent set in $C(C_m))$. Now

$$\begin{aligned} |I| &= |L_2| + |L_4| + \cdots + |L_{2k-2}| + |L_{2k}| \\ &= 2 + 4 + \cdots + 2k - 2 + 2k \\ &= k(k+1). \end{aligned}$$

Therefore $\alpha(C(C_{2k})) \geq k(k+1)$.

Now, as $C(P_m)$ is a subgraph of $C(C_m)$ with $V(C(P_m)) = V(C(C_m))$ and $E(C(P_m)) \subset E(C(C_m))$, then $\alpha(C(C_m)) \leq \alpha(C(P_m))$. Using Theorem 5.1 we obtain

$$\begin{aligned} \alpha(C(C_{2k})) &\leq \alpha(C(P_{2k})) \\ &= \alpha(P_{2k+1}^{(2)})) \\ &= \lfloor (2k+1)^2/4 \rfloor \\ &= k(k+1). \end{aligned}$$

☑

The following proposition will be useful.

Proposition 7.4. *Let $n = 2k+1$ be an odd positive integer. Let I be an independent set of $C(C_n)$. If the vertex $\{1, n\}$ belongs to I, then there exist an independent set I' such that $\{1, n\} \notin I'$ and $|I'| \geq |I|$.*

Proof. Let $I = I_1 \cup I_2$, where $I_2 = L_n \cap I$ and $I_1 = I - I_2$. If $\{1, n\} \in I$, then the vertices $\{1, 1\}$ and $\{n, n\}$ does not belongs to I. Let $m = |L_{n-1} \cap I|$. Then, there are at least m vertices in $L_n - \{\{1, 1\}, \{n, n\}\}$ such that does not belongs to I. That is, $|L_n \cap I| \leq n - 2 - m$, and hence $|I| \leq |I_1| + n - 2 - m$. We construct $I' = I_1' \cup I_2'$ as follows, $I_1' = I_1 - (L_{n-1} \cap I) \cup \{\{1, n\}\}$ and $I_2' = L_n$. Clearly I' is an independent set. Therefore

$$|I'| = |I_1'| + |I_2'| = |I_1| - m - 1 + n \geq |I|.$$

☑

If x is a vertex in $V(C(C_n))$ of the form $\{1, j\}$ or $\{j, n\}$, for some j, then x is called an *extreme vertex* in $C(C_n)$. Let $\{i, j\} \in V(C(C_n))$, with $i < j$. We say that $x \in \{\{i, j+1\}, \{i+1, j\}\}$ (resp. $x \in \{\{i-1, j\}, \{i, j-1\}\}$) is a *right neighbor* of $\{i, j\}$ (resp. *left neighbor*) if x is adjacent to $\{i, j\}$ in $C(C_n)$.

Proposition 7.5. *Let $k \geq 1$ be an integer. Then*

$$\alpha(C(C_{2k+1})) = k^2 + k + \lfloor (k+1)/2 \rfloor.$$

***Proof*.** The case $k = 1$ is easy so we assume that $k \geq 2$.

Case k odd. Let

$$L = L_2 \cup L_4 \cup \cdots \cup L_{k-1} \cup L_{k+2} \cup L_{k+4} \cdots \cup L_{2k+1},$$

which is an independent set of $C(C_{2k+1})$ (by Propositions 7.1 and 7.2). We have that

$$\begin{aligned} |L| &= 2 + 4 + \cdots + (k-1) + (k+2) + (k+4) + \cdots + (2k+1) \\ &= k^2 + \frac{3}{2}k + \frac{1}{2} \\ &= k^2 + k + \left\lfloor \frac{k+1}{2} \right\rfloor, \end{aligned}$$

which shows that $\alpha(C(C_{2k+1})) \geq k^2 + k + \lfloor (k+1)/2 \rfloor$.

Now we prove that $\alpha(C(C_{2k+1})) \leq k^2 + k + \left\lfloor \frac{k+1}{2} \right\rfloor$.

Let $n = 2k+1$. Let $A = L_1 \cup L_2 \cup \cdots \cup L_k$ and $B = L_{k+1} \cup L_{k+2} \cup \cdots \cup L_n$. Let I be an independent set in $C(C_{2k+1})$. By Proposition 7.4 we can assume that $\{1, n\} \notin I$. Let $I_1 = I \cap A$ and $I_2 = I \cap B$.

We will construct an independent set I' of $C(C_{2k+1})$ such that $|I'| \geq |I|$ and $I' \subseteq L$. First we list the steps and then we say how to do each step.

Step 1. Construct an independent set I_2' from I_2 such that $|I_2'| \geq |I_2|$ and

$$I_2' \subseteq L_{k+2} \cup L_{k+4} \cup \cdots \cup L_{2k+1}$$

Step 2. Construct a set I'' from $I_1 \cup I_2'$ by replacing every vertex $\{i, n\} \in I \cap (L_1 \cup L_3 \cup \cdots \cup L_k)$ (if any) with $\{1, i\}$.

Step 3. Construct I' from I'' by replacing every vertex in $I''\cap L_3\cup L_5\cup\cdots\cup L_k$ with a vertex in $L_2\cup L_4\cup\cdots\cup L_{k-1}$ in such a way that $I'\subseteq L$.

The set I'_2 in Step 1 is obtained in the following way:

Let $W=\{k+1,k+3,\dots,2k\}$. Let i_1 be the greatest integer in W such that $L_{i_1}\cap I_2\neq\varnothing$. That is, $L_w\cap I_2=\varnothing$, for every $w\in W$ with $w>i_1$. To construct J_1 from I_2 by replacing every vertex in $L_{i_1}\cap I_2$ with its right neighbor in L_{i_1+1}. Now, Let i_2 be the greatest integer in $W-\{i_1\}$ such that $L_{i_2}\cap J_1\neq\varnothing$. To construct J_2 from J_1 by replacing every vertex in $L_{i_2}\cap J_1$ with its right neighbor in L_{i_2+1}. We will continue this procedure until we find a set J_r such that $W-\{i_1,\dots,i_r\}=\varnothing$ or $L_w\cap J_r=\varnothing$, for any $w\in W-\{i_1,\dots,i_r\}$. That is, we will finish until all the vertices in $I_2\cap(L_{k+1}\cup L_{k+3}\cup\cdots\cup L_{2k})$ have been replaced with vertices in $L_{k+2}\cup L_{k+4}\cup\cdots\cup L_{2k+1}$. We obtain the desired independent set by making $I'_2=J_{i_r}$.

Let $X=L_1\cup L_3\cup\cdots\cup L_k$. Notice that Step 2 can be done because if $\{i,n\}\in I_1\cap X$, then $\{1,i\}\notin I'_2$. We have that $\{1,i\}\in L_{k+2}\cup L_{k+4}\cup\cdots\cup L_{2k+1}$ because $\{i,n\}\in X$ and hence $I''\cap B\subseteq L_{k+2}\cup L_{k+4}\cup\cdots\cup L_{2k+1}$.

Finally we show how to realize Step 3. Let $I'_1=I''\cap X$. First, we construct M_1 from I'' by replacing all the vertices in I'_1 as follows: select the smallest integer i in $\{3,5,\dots,k\}$ such that $I'_1\cap L_i\neq\varnothing$ and $I'_1\cap L_j=\varnothing$, for every $j\in\{3,5,\dots,k\}$ with $j<i$. To obtain M_1 from I'' by replacing every vertex in $I'_1\cap L_i$ with its respective right neighbor in L_{i-1}. To repeat this process to obtain M_2 from M_1 but now with the smallest integer in $\{3,5,\dots,k\}-\{i_1\}$ such that $I'_1\cap L_i\neq\varnothing$ and $I'_1\cap L_j=\varnothing$, for every $j\in\{3,5,\dots,k\}-\{i_1\}$ with $j<i$. To continue in this way until a set M_r is obtained in which all the vertices in $I'_1\cap X$ has been replaced by vertices in $L_2\cup L_4\cup\cdots\cup L_{k-1}$. Now to make $I'=M_s$. Notice that $|I'|\geq|I|$ and $I'\subseteq L$ which implies that $|I|\leq|L|$ as desired.

Case k even.

First we show that $\alpha(C(C_{2k+1}))\geq k^2+k+\lfloor(k+1)/2\rfloor$. Let

$$L=L_2\cup L_4\cup\cdots\cup L_k\cup L_{k+3}\cup L_{k+5}\cup\cdots\cup L_{2k+1},$$

which is an independent set of $C(C_{2k+1})$. We have that

$$\begin{aligned}|L|&=2+4+\cdots+k+(k+3)+(k+5)+\cdots+(2k+1)\\&=\frac{1}{4}k(k+2)+\frac{1}{4}(4k+3k^2)\\&=k^2+\frac{3}{2}k=k^2+k+\left\lfloor\frac{k+1}{2}\right\rfloor.\end{aligned}$$

Now we prove that $\alpha(C(C_{2k+1}))\leq k^2+k+\lfloor(k+1)/2\rfloor$.

Let I be any independent set of $C(C_{2k+1})$. By Proposition 7.4 we can assume that $\{1, n\} \notin I$. We will obtain an independent set I' such that $I' \subseteq L$ and $|I| \leq |I'|$. Let $I_1 = I \cap (L_2 \cup L_3 \cup \cdots \cup L_{k+1})$ and $I_2 = I \cap (L_{k+2} \cup L_{k+3} \cup \cdots \cup L_{2k+1})$.

Now we obtain I' with the following steps.

Step 1. Obtain I'_2 from I_2 by interchanging every vertex in $I_2 \cap L_i$ with its right neighbor in L_{i+1}, for every $i \in \{k+2, k+4, \ldots, 2k\}$, in the same way Step 1 for the case k odd.

Step 2. Obtain I'' from $I_1 \cup I'_2$ by interchanging all the vertices of the form $\{a, n\} \in I \cap L_i$ with $i \in \{3, 5, \ldots, k-1\}$ (if any) with $\{1, a\}$.

Step 3. Obtain I''' from I'' by interchange every vertex in $I'' \cap L_i$, for $i \in \{3, 5, \ldots, k-1\}$ with its right neighbor in L_{i-1} in the same way that in Step 3 of the case k odd.

Step 4. Obtain I' from I''' by interchange the vertices in $I \cap L_{k+1}$ as follows: as L_{k+1} is linked with L_{k+1} by the edge $[\{1, k+1\}, \{k+1, n\}]$, then only one of this vertices could be in I. If $\{\{1, k+1\}, \{k+1, n\}\} \cap I = \varnothing$, then move all the vertices in $I''' \cap L_{k+1}$ to its right neighbors in L_k. If $\{1, k+1\} \in I$, then move all the vertices in $I''' \cap L_{k+1}$ to its right neighbors in L_k, and if $\{k+1, n\} \in I$, then move all the vertices in $I'' \cap L_{k+1}$ to its left neighbors in L_k.

☑

8. Proof of Theorem 1.6

In this proof, the vertices of C_m in $W_{m,1}$ are $1, \ldots, m$ and the vertex in K_1 is $m+1$. We use U_m to denote the subgraph $C(C_m)$ of $C(W_{m,1})$. The sets R_i are defined as in the proof of Theorem 1.4. Let

$$B = \{\{i, m+1\} : 1 \leq i \leq m+1\}.$$

Notice that the subgraph of $C(W_{m,1})$ induced by B is isomorphic to $W_{m,1}$ that has $\alpha(W_{m,1}) = \lfloor m/2 \rfloor$.

Proposition 8.1. *Let $m \geq 2$ be an integer. If S is a subset of $\{1, \ldots, m\}$, then $\alpha(U_m - \cup_{i \in S} R_i) \leq \lfloor m^2/4 \rfloor$.*

Proof. We know that $U_m - x \simeq C(P_{m-1}) \simeq P_m^{(2)}$, for any $x \in S$. Therefore

$$\alpha(U_m - x) = \alpha(P_m^{(2)}) = \lfloor m^2/4 \rfloor$$

and the result follows because $U_m - \cup_{i \in S} R_i$ is an induced subgraph of $U_{m+1} - R_x$, for any $x \in S$. ☑

Theorem 8.2. 1.6 *Let $m \geq 3$. Then*

$$\alpha(C(W_{m,1})) = \alpha(C(C_m)) + 1.$$

Proof. Let I be any maximal independent set of U_m. As $I \cup \{m+1, m+1\}$ is an independent set in $C(W_{m,1})$ then $\alpha(C(W_{m,1})) \geq \alpha(C(C_m)) + 1$. Now we prove that $\alpha(C(W_{m,1})) \leq \alpha(C(C_m)) + 1$. Let I be an independent set in $C(W_{m,1})$. We have several cases: if $I \subset U_m$ then $|I| \leq \alpha(C(C_m))$. If $I \subset B$ then $|I| \leq \lfloor m/2 \rfloor \leq \alpha(C(C_m))$. The last case is when $I = I' \cup I''$, where $I' \subset U_m$, $I'' \subset B$, and with I', I'' both non-empty sets. As the subgraph of $C(W_{m,1})$ induced by B is isomorphic to $W_{m,1}$ then $|I''| \leq \lfloor m/2 \rfloor$. Let

$$S = \{i \in \{1, \ldots, m\} \colon \{i, m+1\} \in I''\}.$$

First consider the case when $|S| = 1$. By Proposition 8.1 we have that

$$|I| \leq \lfloor m^2/4 \rfloor + 1.$$

If $m = 2k$, then

$$|I| \leq \lfloor m^2/4 \rfloor + 1 \leq \lfloor k^2 \rfloor + 1 \leq k(k+1) + 1 = \alpha(C(C_{2k}) + 1.$$

If $m = 2k+1$, then

$$\lfloor m^2/4 \rfloor + 1 \leq \lfloor k^2 + k + 1/4 \rfloor + 1 \leq k(k+1) + \lfloor (k+1)/2 \rfloor + 1 = \alpha(C(C_{2k+1}) + 1.$$

When $|S| \geq 2$ it can be shown, in a similar way that for the case of the double vertex graph of the wheel graph, that

$$\alpha(U_m - \cup_{i \in S} R_i) \leq \lfloor (m-1)^2/4 \rfloor.$$

Therefore

$$|I| \leq \lfloor (m-1)^2/4 \rfloor + \lfloor m/2 \rfloor$$

If $m = 2k$, then

$$|I| \leq \lfloor (2k-1)^2/4 \rfloor + \lfloor 2k/2 \rfloor \leq k(k+1) + 1.$$

If $m = 2k+1$, then

$$|I| \leq \lfloor (2k)^2/4 \rfloor + \lfloor (2k+1)/2 \rfloor \leq k^2 + k + \left\lfloor \frac{k+1}{2} \right\rfloor + 1$$

and the proof is completed. ☑

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