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On k-Pell numbers close to power of 2

Números de k-Pell cercanos a potencias de 2

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Abstract. For $k \geq 2$, let $\left(P_n^{(k)}\right)_{n \geq 2-k}$ be the k-generalized Pell sequence which starts with $0, \cdots, 0, 1$ (k terms) and each term afterwards is given by the linear recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{for all } n \geq 2.$$

An integer n is said to be close to a positive integer m if n satisfies $|n - m| < \sqrt{m}$. In this paper, we solve the Diophantine inequality

$$\left| P_n^{(k)} - 2^m \right| < 2^{m/2},$$

in positive unknowns k, n, and m.

Key words and phrases. Diophantine equations, k-Pell numbers, linear forms in logarithms, reduction method.

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Resumen. Para $k \geq 2$ sea $\left(P_n^{(k)}\right)_{n \geq 2-k}$ la k-sucesión generalizada de Pell que comienza en los valores $0, \cdots, 0, 1$ (k términos en total) y que satisface la relación de recurrencia

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{para todo } n \geq 2.$$

Un entero n se denomina cercano a otro entero m si n satisface $|n - m| < \sqrt{m}$. En este artículo se resuleve la desigualdad Diofantina

$$\left| P_n^{(k)} - 2^m \right| < 2^{m/2},$$

para las indeterminadas enteras k, n, y m.

Palabras y frases clave. Ecuaciones Diofantina, números k-Pell, formas lineales en logaritmos, metodo de reducción.

1. Introduction

The Fibonacci sequence $(F_m)_{m\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$ and

$$F_{m+2} = F_{m+1} + F_m, \quad \text{for all } m \geq 0.$$

Its first few terms are given by

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, \ldots$$

In the same way, the sequence of Pell numbers $(P_n)_{n\geq 0}$ is one of these sequences. It verifies the linear recurrence

$$P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 2,$$

with $P_0 = 0$ and $P_1 = 1$ as initial conditions.

Let $k \geq 2$ be an integer. We consider a generalization of the Pell sequence $\left(P_n^{(k)}\right)_{n\geq 2-k}$ defined as

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)}, \quad \text{for all } n \geq 2,$$

with the initial conditions

$$P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0 \ \text{ and } \ P_1^{(k)} = 1.$$

This sequence is called the k-generalized Pell sequence or the k-Pell sequence. We note that $P_n^{(k)}$ is the n^{th} k-Pell number. This sequence generalizes the usual Pell sequence which corresponds to $k = 2$. Below we present the values of these numbers for the first few values of k and $n \geq 1$.

k	Name	First non-zero terms
2	Pell	1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, ...
3	3-Pell	1, 2, 5, 13, 33, 84, 214, 545, 1388, 3535, 9003, 22929, ...
4	4-Pell	1, 2, 5, 13, 34, 88, 228, 591, 1532, 3971, 10293, 26680, ...
5	5-Pell	1, 2, 5, 13, 34, 89, 232, 605, 1578, 4116, 10736, 28003, ...
6	6-Pell	1, 2, 5, 13, 34, 89, 233, 609, 1592, 4162, 10881, 28447, ...
7	7-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1596, 4176, 10927, 28592, ...
8	8-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4180, 10941, 28638, ...
9	9-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10945, 28652, ...
10	10-Pell	1, 2, 5, 13, 34, 89, 233, 610, 1597, 4181, 10946, 28656, ...

Let $k \geq 2$ be an integer. The k-generalized Fibonacci sequence or for simplicity, the k-Fibonacci sequence is a sequence given by the recurrence relation

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{for all } n \geq 2,$$

with the initial conditions

$$F^{(k)}_{-(k-2)} = F^{(k)}_{-(k-3)} = \cdots = F^{(k)}_0 = 0 \text{ and } F^{(k)}_1 = 1.$$

This sequence generalizes the usual Fibonacci sequence. We obtain the Fibonacci sequence for $k = 2$.

We need the following definition of closeness.

Definición 1.1. An integer n is said to be close to a positive integer m if n satisfies

$$|n - m| < \sqrt{m}.$$

Several papers study the occurrence of specific subsets of $\mathbb{Z}$ in well-known linear recurrence sequences. Bravo and Luca [7] showed that $F^{(2)}_6 = 8$ provides the only non-trivial solution to the Diophantine equation $F^{(k)}_n = 2^m$. In 2014, Chern and Cui [10] introduced the previous definition, and found all the Fibonacci numbers close to a power of 2. Bravo, Gomez and Herrera [3] extended the main result in [10] by determining all $F^{(k)}_n$, which are close to a power of 2. Recently, Açikel, Irmak and Szalay [1] determined the k-generalized Lucas numbers which are close to a power of 2. This leads to ask the following question:

What are the k-Pell numbers which are close to a power of 2?

To answer this question, we will prove our principal result that is formulated as follows.

Theorem 1.2. *All the solutions $(P^{(k)}_n, k, n, m)$ of the inequality*

$$\left| P^{(k)}_n - 2^m \right| < 2^{m/2}, \tag{1}$$

in positive integers k, n, m with $k \geq 2$, are given by

$$(1, k, 1, 1),\ k \geq 2, \quad (2, k, 2, 1),\ k \geq 2, \quad (5, k, 3, 2),\ k \geq 2,$$
$$(13, k, 4, 4),\ k \geq 3, \quad (34, k, 5, 5),\ k \geq 4,$$
$$(29, 2, 5, 5), \quad (70, 2, 6, 6), \quad (33, 3, 5, 5), \quad \text{and} \quad (4116, 5, 10, 12).$$

Note that this theorem gives the solutions to the Diophantine equation

$$P^{(k)}_n = 2^m + t \quad \text{with} \quad |t| < 2^{m/2}. \tag{2}$$

The proof of Theorem 1.2 is mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport [2]. Here, we use a modified version of the result due to Dujella-Pethő [11]. There is a slight difference between our method and that of [1] because we don't need the LLL algorithm but instead we use Baker and Davenport's method. The properties in Subsection 2.3 help us to handle well the linear forms in logarithms.

2. Preliminary Results

This section is devoted to collect a few definitions, notations, proprieties, and results, which will be used in the remaining of this work.

2.1. Linear form in logarithms

We will use Baker's theory of linear forms in logarithms of algebraic numbers for the proof of our result. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\alpha = \alpha^{(1)}, \ldots, \alpha^{(d)}$ denote its conjugates. We denote by

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\left(\max\{|\alpha^{(i)}|, 1\}\right)\right)$$

the logarithmic height of α. In particular, if $\eta = p/q$ is a rational number with $\gcd(p, q) = 1$ and $q > 0$, then $h(\eta) = \log \max\{|p|, q\}$. The following properties of the logarithmic height function $h(\cdot)$, which will be used in the next section without special reference, are known. For any algebraic numbers α and β, we have the following properties [[15], Property 3.3]

$$h(\alpha\beta) \leq h(\alpha) + h(\beta),$$
$$h(\alpha \pm \beta) \leq \log 2 + h(\alpha) + h(\beta).$$

Moreover, for any integer n,

$$h(\alpha^n) \leq |n| h(\alpha).$$

Now, let $\mathbb{K}$ be an algebraic number field of degree $d_{\mathbb{K}}$. Let $\eta_1, \ldots, \eta_l \in \mathbb{K}$ and $d_1, \ldots, d_l$ be nonzero integers. Let $D \geq \max\{|d_1|, \ldots, |d_l|\}$, and

$$\Gamma = \prod_{i=1}^{l} \eta_i^{d_i} - 1.$$

Let $A_1, \ldots, A_l$ be real numbers such that

$$A_j \geq \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for } j = 1, \ldots, l.$$

The first tool we need is the following result due to Matveev [[12], Corollary 2.3]; also see Bugeaud, Mignotte and Siksek [[8], Theorem 9.4].

Theorem 2.1. *If $\Gamma \neq 0$, then*

$$\log |\Gamma| \geq -1.4 \times 30^{l+3} \times l^{4.5} \times d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log D) A_1 \ldots A_l.$$

2.2. The reduction method

Using Theorem 2.1, we get an upper bound on the variable n which is too large, thus we need to reduce that bound. To do this, we need to recall a variant of the reduction method of Baker and Davenport [[2], Lemma] due to Dujella and Pethő [[11], Lemma 5]. We use the one given by Bravo, Gómez, and Luca [[4], Lemma 1].

Lemma 2.2. *Let M be a positive integer, let p/q be a convergent of the continued fraction of an irrational number τ such that $q > 6M$, and let $A,\ B,\ \mu$ be some eal numbers with $A > 0$ and $B > 1$. Let further $\varepsilon = ||\mu q|| - M \cdot ||\tau q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < |m\tau - n + \mu| < AB^{-k},$$

in positive integers $m,\ n$ and k with

$$m \leq M \text{ and } k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The following result of Guzmán and Luca [[14], Lemma 7] will also be very useful.

Lemma 2.3. *If $\ell \geq 1,\ T > (4\ell^2)^\ell$ and $T > x/(\log x)^\ell$, then*

$$x < 2^\ell T(\log T)^\ell.$$

2.3. Properties of k-generalized Pell sequence

In this subsection, we recall some facts and properties of k-Pell sequence. The characteristic polynomial of this sequence is

$$\varphi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1.$$

In [[6], Section 2], it is proved that $\varphi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle. This root is real and positive and satisfies $\alpha(k) > 1$. The other roots are strictly inside the unit circle. Furthermore, the authors proved [[6], Lemma 3.2] that

$$\phi^2(1 - \phi^{-k}) < \alpha(k) < \phi^2, \quad \text{for } k \geq 2, \tag{3}$$

where $\phi = \frac{1+\sqrt{5}}{2}$. To simplify the notation, in general, we omit the dependence of $\alpha(k)$ on k and use α. For $s \geq 2$, let

$$f_s(x) := \frac{x-1}{(s+1)x^2 - 3sx + s - 1} = \frac{x-1}{s(x^2 - 3x + 1) + x^2 - 1}. \tag{4}$$

In [[5], Lemma 1], it is also proved that the inequalities

$$0.276 < f_k(\alpha) < 0.5 \quad \text{and} \quad \left|f_k(\alpha^{(i)})\right| < 1, \quad \text{with } 2 \leq i \leq k \tag{5}$$

hold, where $\alpha := \alpha^{(1)}, \ldots, \alpha^{(k)}$ (the conjugates of α) are all the zeros of $\varphi_k(x)$. It was inferred in [[13], Section 2.4] that $f_k(\alpha)$ is not an algebraic integer. In addition, the authors from [[5], Lemma 3] proved that the logarithmic height of $f_k(\alpha)$ satisfies

$$h(f_k(\alpha)) < 4k \log \phi + k \log(k+1), \quad \text{for} \quad k \geq 2. \tag{6}$$

With the above notations, the authors of [[6], Theorem 3.1] showed that

$$P_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha^{(i)}) {\alpha^{(i)}}^n \quad \text{and} \quad \left|P_n^{(k)} - f_k(\alpha)\alpha^n\right| < \frac{1}{2}, \tag{7}$$

which is valid for $n \geq 1$ and $k \geq 2$. So, for $n \geq 1$ and $k \geq 2$, we have

$$P_n^{(k)} = f_k(\alpha)\alpha^n + e_k(n), \quad \text{where} \quad |e_k(n)| \leq \frac{1}{2}. \tag{8}$$

Furthermore, it was shown that

$$\alpha^{n-2} \leq P_n^{(k)} \leq \alpha^{n-1}, \quad \text{for } n \geq 1 \text{ and } k \geq 2. \tag{9}$$

Finally, we conclude this subsection by giving the following estimate [[5], Lemma 2]. If $k \geq 30$ and $n > 1$ are integers satisfying $n < \phi^{k/2}$, then

$$f_k(\alpha)\alpha^n = \frac{\phi^{2n}}{\phi+2}(1+\zeta), \quad \text{where} \quad |\zeta| < \frac{4}{\phi^{k/2}}. \tag{10}$$

3. Proof of Theorem 1.2

In this section, we give all details about the proof of Theorem 1.2. For this, two cases will be considered according to the values of n.

3.1. The case $1 \leq n \leq k+1$

It is known in [[9], Section 2] that for $1 \leq n \leq k+1$, we have

$$P_n^{(k)} = F_{2n-1}.$$

Hence, inequality (1) becomes

$$|F_{2n-1} - 2^m| < 2^{m/2}. \tag{11}$$

We recall the following theorem [[10], Theorem 1.1].

Theorem 3.1. *There are only* 8 *Fibonacci numbers which are close to a power of* 2*. Namely, the solutions* $(F_n, 2^m)$ *of the inequality*

$$|F_n - 2^m| < 2^{m/2}$$

are $(1,2),(2,2),(3,2),(3,4),(5,4),(8,8),(13,16),$ *and* $(34,32)$.

By this theorem, we deduce that the solutions $(F_{2n-1}, 2^m)$ of the inequality (11) are $(1,2),(2,2),(5,4),(13,16),$ and $(34,32)$. Namely, the solutions $(P_n^{(k)}, k, n, m)$ of the inequality (1) are

$$(1,k,1,1),\ k\geq 2, \quad (2,k,2,1),\ k\geq 2, \quad (5,k,3,2),\ k\geq 2,$$

$$(13,k,4,4),\ k\geq 3, \quad (34,k,5,5),\ k\geq 4.$$

3.2. The case $n \geq k+2$

We start this subsection by assuming that $n \geq k+2$. We have the following result which gives us the bounds of m in terms of n.

Lemma 3.2. *If* (m,n,k) *is a solution of the Diophantine inequality* (1) *with* $n \geq 1$*,* $m \geq 2$*, and* $n \geq k+2$*, then we have the following inequalities*

$$0.69n - 2.38 < m < 1.39n - 0.39 < 1.5n. \tag{12}$$

Proof. Combining inequalities (9) with equation (1), we have

$$2^{m-1} \leq 2^m - 2^{m/2} < P_n^{(k)} \leq \alpha^{n-1}$$

and

$$\alpha^{n-2} \leq P_n^{(k)} < 2^m + 2^{m/2} < 2^{m+1}.$$

Taking the logarithm of both sides of the two above inequalities, we get

$$(n-2)\frac{\log\alpha}{\log 2} - 1 < m \leq (n-1)\frac{\log\alpha}{\log 2} + 1.$$

Because $\phi^2(1-\phi^{-2}) < \alpha < \phi^2$, for $k \geq 2$, we deduce that

$$0.69n - 2.38 < m < 1.39n - 0.39 < 1.5n.$$

This finishes the proof. ☑

Furthermore, with Theorem 3.1 and checking for the small values for n, we may assume that $n \geq 9$. Next, we get the following result which gives an upper bound of m and n in terms of k.

Lemma 3.3. *If the integers n, k, and m satisfy Diophantine equation (2) with $n \geq k+2$, then we have the following estimates*

$$n < 7.71 \times 10^{14} \cdot k^5 \cdot \log^3 k \quad \text{and} \quad m < 1.16 \times 10^{15} \cdot k^5 \cdot \log^3 k.$$

Proof. Substituting the formula (8) in (2), we have

$$2^m - f_k(\alpha)\alpha^n = -t + e_k(n).$$

Taking the absolute value of both sides, we obtain

$$|2^m - f_k(\alpha)\alpha^n| = |-t + e_k(n)| < |t| + |e_k(n)| < 2^{m/2} + \frac{1}{2}.$$

Dividing through by $f_k(\alpha)\alpha^n$, we get

$$\left|\frac{1}{f_k(\alpha)}\alpha^{-n}2^m - 1\right| < \frac{1}{2f_k(\alpha)\alpha^n} + \frac{2^{m/2}}{f_k(\alpha)\alpha^n}.$$

Since the inequalities $0.276 < f_k(\alpha) < 0.5$ hold, for all $k \geq 2$ and $2^{m-1} < \alpha^{n-1}$, then we deduce that

$$\left|\frac{1}{f_k(\alpha)}\alpha^{-n}2^m - 1\right| < \frac{1.82}{\alpha^n} + 3.63 \times \frac{2^{1/2}\left(\alpha^{n-1}\right)^{1/2}}{\alpha^n} < \frac{6}{\alpha^{n/2}}. \tag{13}$$

Let Γ_1 be the expression between the absolute value on the left-hand side of (13). Observe that $\Gamma_1 \neq 0$. If $\Gamma_1 = 0$, then $f_k(\alpha) = 2^m\alpha^{-n}$ and so $f_k(\alpha)$ is an algebraic integer, which is a contradiction. Hence $\Gamma_1 \neq 0$ and we can apply Theorem 2.1 to it. Let us consider

$$\eta_1 = \frac{1}{f_k(\alpha)}, \ \eta_2 = \alpha, \ \eta_3 = 2, \quad d_1 = 1, \ d_2 = -n, \ d_3 = m.$$

Since η_1, η_2, η_3 are elements of the number field $\mathbb{K} := \mathbb{Q}(\alpha)$ and $d_{\mathbb{K}} = k$. We have

$$h(\eta_2) = \frac{\log\alpha}{k} < \frac{2\log\phi}{k} \quad \text{and} \quad h(\eta_3) = \log 2.$$

Moreover, we get

$$\max\{kh(\eta_2), |\log\eta_2|, 0.16\} < 0.97 = A_2,$$

and

$$\max\{kh(\eta_3), |\log\eta_3|, 0.16\} < k\log 2 = A_3.$$

Using the properties of the logarithmic height and (6), we obtain

$$h(\eta_1) = h(f_k(\alpha)) < 4k\log\phi + k\log(k+1) < 4.5k\log k,$$

for $k \geq 2$. So, we can take

$$\max\{kh(\eta_1), |\log \eta_1|, 0.16\} < 4.5k^2 \log k = A_1.$$

Finally, from Lemma 3.2, we can choose $D = 2n \geq \max\{1, m, n\}$. Thus, Theorem 2.1 tells us that

$$\log |\Gamma_1| \geq -4.34 \times 10^{11} \cdot k^5 \cdot \log k \cdot (1 + \log k)(1 + \log(2n)).$$

By the facts $1 + \log(2n) < 1.8 \log n$ and $1 + \log k < 2.5 \log k$, which hold, for $n \geq 9$ and $k \geq 2$, we obtain

$$\log |\Gamma_1| > -1.96 \times 10^{12} \cdot k^5 \cdot \log^2 k \cdot \log n. \tag{14}$$

Combining this inequality with (13), we get

$$n < 8.16 \times 10^{12} \cdot k^5 \cdot \log^2 k \cdot \log n. \tag{15}$$

Thus, we get

$$\frac{n}{\log n} < 8.16 \times 10^{12} \cdot k^5 \cdot \log^2 k.$$

Applying Lemma 2.3 with $T = 8.16 \times 10^{12} \cdot k^5 \cdot \log^2 k$, $x = n$, and $\ell = 1$, we have

$$\begin{aligned} n &< 2 \cdot (8.16 \times 10^{12} \cdot k^5 \cdot \log^2 k) \cdot \log(8.16 \times 10^{12} \cdot k^5 \cdot \log^2 k) \\ &< (1.64 \times 10^{13} \cdot k^5 \cdot \log^2 k) \cdot (29.8 + 5 \log k + 2 \log \log k) \\ &< 7.71 \times 10^{14} \cdot k^5 \cdot \log^3 k. \end{aligned} \tag{16}$$

In the above inequalities, we have used the fact that $29.8+5\log k+2\log(\log k) < 47 \log k$ holds, for $k \geq 2$. Finally, using inequality (16) and Lemma 3.2, we obtain

$$m < 1.16 \times 10^{15} \cdot k^5 \cdot \log^3 k.$$

This completes the proof of Lemma 3.3. ☑

Subsequently, we will discuss the cases the value of k is small or large.

3.2.1. The case $2 \leq k \leq 350$

To reduce the above bound on n, we put

$$\Lambda_1 := n \log \alpha - m \log 2 + \log \left(f_k(\alpha)\right).$$

Note that $e^{-\Lambda_1} - 1 = \Gamma_1 \neq 0$ and then $\Lambda_1 \neq 0$. If $\Lambda_1 < 0$, then

$$0 < |\Lambda_1| < e^{|\Lambda_1|} - 1 = |\Gamma_1| < \frac{6}{\alpha^{n/2}}$$

according to inequality (13). If $\Lambda_1 > 0$, then we have $1 - e^{-\Lambda_1} = \left|e^{-\Lambda_1} - 1\right| < 1/2$. Thus, we get

$$0 < \Lambda_1 < e^{\Lambda_1} - 1 = e^{\Lambda_1}|\Gamma_1| < \frac{12}{\alpha^{n/2}}.$$

So, in both cases we have

$$0 < |\Lambda_1| < \frac{12}{\alpha^{n/2}}.$$

Dividing through by $\log 2$ we get

$$|n\tau - m + \mu| < \frac{18}{\alpha^{n/2}}, \tag{17}$$

where

$$\tau = \frac{\log \alpha}{\log 2} \text{ and } \mu = \frac{\log\left(f_k(\alpha)\right)}{\log 2}.$$

Now, we apply Lemma 2.2 to (17) for $2 \leq k \leq 350$ by putting

$$M = M_k := \lfloor 7.71 \times 10^{14} \cdot k^5 \cdot \log^3 k \rfloor, \ A = 18 \ \text{ and } \ B = \alpha.$$

A quick computation with Mathematica reveals that, for all $k \in [2, 350]$,

$$\frac{n}{2} \leq 79. \tag{18}$$

Finally, we write a program with Maple to determine all the solutions of equation (2) for $2 \leq k \leq 350$, $0 \leq n \leq 158$ and $0 \leq m \leq 219$ (because $m < 1.39n - 0.39$) with $n \geq k + 2$ and we find the remaining solutions cited in our main theorem.

3.2.2. The case $k > 350$

In this case, we prove the following lemma, which will complete the proof of Theorem 1.2.

Lemma 3.4. *There is no solution for inequality* (1) *with* $k > 350$ *and* $n \geq k+2$.

Proof. Referring to Lemma 3.3, we have

$$n < 7.71 \times 10^{14} \cdot k^5 \cdot \log^3 k < \phi^{k/2}, \ \text{ for } k > 350.$$

Thus, from (8), (10), and (2), we have

$$\left|2^m - \frac{\phi^{2n}}{\phi + 2}\right| = \left|\frac{\phi^{2n}}{\phi + 2}\zeta + e_k(n) - t\right| < \frac{\phi^{2n}}{\phi + 2} \times \frac{4}{\phi^{k/2}} + \frac{1}{2} + 2^{m/2}.$$

Multiplying through by $(\phi+2)/\phi^{2n}$ and using the facts that $2<\phi^2$, and $m<1.5n$ according to Lemma 3.2, we obtain

$$\begin{aligned}|\Gamma_2| &< \frac{4}{\phi^{k/2}}+\frac{1}{2}\times\frac{\phi+2}{\phi^{2n}}+2^{m/2}\times\frac{\phi+2}{\phi^{2n}}\\ &< \frac{4}{\phi^{k/2}}+\frac{1.81}{\phi^{2n}}+\frac{3.62\phi^{m}}{\phi^{2n}}\\ &< \frac{5.81}{\phi^{k/2}}+\frac{3.62}{\phi^{n/2}}\\ &< \frac{10}{\phi^{k/2}},\end{aligned}\tag{19}$$

where

$$\Gamma_2 := (\phi+2)\phi^{-2n}2^m-1.$$

Observe that $\Gamma_2\neq 0$. Indeed, we have $2^m=\frac{\phi^{2n}}{\phi+2}$, which is impossible. So, we can apply Theorem 2.1 to Γ_2 with

$$\eta_1=\phi+2,\quad \eta_2=\phi,\quad \eta_3=2,\quad d_1=1,\quad d_2=-2n,\ \text{and}\ d_3=m.$$

Since $\mathbb{K}:=\mathbb{Q}(\eta_1,\eta_2,\eta_3)=\mathbb{Q}(\phi)$, then $d_{\mathbb{K}}=2$. Also, we have

$$h(\eta_2)=\frac{\log\phi}{2}\quad\text{and}\quad h(\eta_3)=\log 2.$$

Moreover, one has

$$h(\eta_1)\leq h(\phi)+h(2)+\log 2\leq\frac{\log\phi}{2}+2\log 2<1.63.$$

Thus, we can take

$$A_1:=3.26,\quad A_2:=\log\phi\quad\text{and}\quad A_3:=2\log 2.$$

Here again, we can take $D=2n$. Using the fact that $1+\log(2n)<1.8\log n$ holds for $n\geq 9$ and from Theorem 2.1, we get

$$\log|\Gamma_2|>-3.8\cdot 10^{12}\cdot\log n.\tag{20}$$

Next, we put (19) and (20) together to obtain

$$k<1.58\cdot 10^{13}\cdot\log n.$$

By Lemma 3.3, we have

$$\log n<\log(7.71\cdot 10^{14}\cdot k^5\cdot\log^3 k)<11.8\log k,\quad \text{for } k\geq 350.$$

Using the above inequality, we obtain

$$\frac{k}{\log k} < 1.87 \cdot 10^{14}.$$

We apply again Lemma 2.3 and we obtain

$$k < 1.23 \cdot 10^{16}.$$

It turns out that

$$n < 1.11 \cdot 10^{100}.$$

In order to reduce the above bounds of n, we put

$$\Lambda_2 = \log\left(\frac{1}{\phi + 2}\right) + 2n \log \phi - m \log 2.$$

We have $e^{-\Lambda_2} - 1 = \Gamma_2 \neq 0$. Hence, $\Lambda_2 \neq 0$. So, we get

$$0 < |\Lambda_2| < \frac{20}{\phi^{k/2}}.$$

Dividing through by $\log 2$ we get

$$|2n\tau - m + \mu| < \frac{29}{\phi^{k/2}}, \tag{21}$$

with

$$\tau = \frac{\log \phi}{\log 2} \text{ and } \mu = \frac{\log\left(\frac{1}{\phi+2}\right)}{\log 2}.$$

Now we apply Lemma 2.2 with $A = 29$ and $B = \phi$ and $M = 2.22 \cdot 10^{100}$. Using Maple, we find that q_{197} satisfying the hypotheses of Lemma 2.2, and we get

$$\frac{k}{2} \leq 493. \tag{22}$$

Thus, for $k \leq 986$, using Lemma 3.3, we obtain

$$n < 2.36 \cdot 10^{32}.$$

We apply again Lemma 2.2 to inequality (21) with $A = 29$, $B = \phi$, $M = 4.72 \cdot 10^{32}$, and we obtain

$$\frac{k}{2} \leq 173. \tag{23}$$

We obtain a contradiction to the fact that $k > 350$. Therefore, we deduce that inequality (1) doesn't admit any solution for $k > 350$. This completes the proof of Lemma 3.4 and that of our main result. ☑

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