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New results regarding the existence, uniqueness and convergence of the solution for nonlinear fractional Volterra integro-differential equations via Caputo-Fabrizio operator

Nuevos resultados sobre la existencia, unicidad y convergencia de la solución para ecuaciones integro-diferenciales fraccionarias no lineales de Volterra mediante el operador Caputo-Fabrizio

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Abstract. In this paper, we study new results regarding the existence, uniqueness and convergence of the solution of nonlinear fractional Volterra integro-differential equations via Caputo-Fabrizio operator. The main results of this paper are based on the Banach contraction principle. Furthermore, we investigate the approximate analytical solutions of the proposed problem using a new combination method called Khalouta decomposition method. Some illustrated examples of our results are provided with some numerical simulations of the solutions.

Key words and phrases. Fractional Volterra integro-differential equations, Caputo-Fabrizio fractional operator, Banach contraction principle, Khalouta transform method, Adomian decomposition method.

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Resumen. En este artículo estudiamos nuevos resultados sobre la existencia, unicidad y convergencia de la solución de ecuaciones integro-diferenciales fraccionarias no lineales de Volterra mediante el operador Caputo-Fabrizio. Los principales resultados de este artículo se basan en el principio de contracción de Banach. Además, investigamos las soluciones analíticas aproximadas del problema propuesto utilizando un nuevo método de combinación llamado método de descomposición de Khalouta. Se proporcionan algunos ejemplos ilustrados de nuestros resultados con algunas simulaciones numéricas de las soluciones.

Palabras y frases clave. Ecuaciones diferenciales integro-fraccionales de Volterra, operador fraccionario de Caputo-Fabrizio, principio de contracción de Banach, método de la transformada de Khalouta, método de descomposición de Adomian.

1. Introduction

In recent years, the theory of nonlinear fractional integro-differential equations has witnessed great advancement, as these equations have appeared in various fields of physical sciences, fluid mechanics, engineering, electromagnetic, optimal control theory, biology, economics, and applied mathematics, in addition to other fields of science [1, 2, 4, 13, 15].

In general, nonlinear fractional integro-differential equations do not have an exact analytical solution, so many researchers resort to using approximation and numerical techniques.

In the literature, there are several techniques that have been employed to solve fractional intego-differential equations. For example, in [7] the modified variational iteration method was utilized to find the approximate solution of Caputo fractional Volterra-Fredholm integro-differential equations. In [3] the homotopy perturbation method was presented for solving fourth-order fractional Volterra integral-differential equations involving the Caputo fractional derivative. Also, [9] the modified Adomian decomposition method was applied to solve fractional integro-differential equations and systems of fractional integro-differential equations. In [8] the authors use the homotopy analysis method for higher-order fractional Volterra-Fredholm integro-differential equations. In [10] the hybrid functions and the collocation method were applied to obtain the numerical solution of the nonlinear Fredholm integral-differential equations.

The main objective of this paper is to prove the existence, uniqueness and convergence results of the solution of nonlinear fractional Volterra integro-differential equations involving the Caputo-Fabrizio fractional derivative. Furthermore, we study the solution behaviour of our problem which can be formally obtained by a new hybrid method called Khalouta decomposition method (KHDM).

The nonlinear fractional Volterra integro-differential equation of is given by

$$ {}^{CF}\mathfrak{D}^{\alpha}\mathfrak{X}(\varpi) = f(\varpi) + \int_0^{\varpi} K(\varpi,\varsigma)\mathcal{F}(\mathfrak{X}(\varsigma))d\varsigma, \tag{1} $$

under the condition

$$ \mathfrak{X}(0) = \mathfrak{X}_0, \tag{2} $$

where ${}^{CF}\mathfrak{D}^{\alpha}$ is the Caputo-Fabrizio fractional operator of order $0 < \alpha \leq 1$ with $\mathfrak{X} : I = [0,1] \to \mathbb{R}$ is the continuous function which has to be determined,

$f : I \to \mathbb{R}$ and $K : I \times I \to \mathbb{R}$ are continuous functions and $\mathcal{F} : \mathbb{R} \to \mathbb{R}$ is nonlinear function of $\mathfrak{X}(\varpi)$.

The KHDM, which is based on the coupling of the Adomian decomposition method with the Khalouta transform method, has advantages over other existing methods, which can be summarized in the following points:

- The ability to transform the proposed problem into Khalouta space and create solutions in the form of algebraic equations that can be easily solved.

- The method provides analytical and approximate solutions in the form of rapidly convergent series with minimal computational effort.

- The KHDM takes less time and provides higher accuracy with minor computational requirements.

- The proposed method is free from any restrictive assumption, perturbations, discretization or linearization.

- The method is straightforward, accurate, and suitable to investigate the solutions of the nonlinear physical and engineering problems.

The framework of this paper is as follows: In Section 2, we recall some essential definitions and properties related to fractional calculus and Khalouta transform. In Section 3, we talk about our main findings regarding the existence, uniqueness and convergence analysis. In addition, we present the Khalouta decomposition method and apply this method to nonlinear fractional Volterra integro-differential equations. Some numerical examples are presented in Section 4. Conclusions are presented in Section 5.

2. Definitions and preliminary results

In this section, we present the basic definitions and several properties of the theory of fractional calculus which has been recently developed by [5, 12].

Definición 2.1. The Caputo-Fabrizio fractional integral of order $0 < \alpha \leq 1$ for a function $\mathfrak{X} \in H^1(0,1)$ is defined by

$$^{CF}\mathfrak{J}^{\alpha}\mathfrak{X}(\varpi) = \frac{2(1-\alpha)}{(2-\alpha)\mathfrak{M}(\alpha)}\mathfrak{X}(\varpi) + \frac{2\alpha}{(2-\alpha)\mathfrak{M}(\alpha)}\int_0^{\varpi}\mathfrak{X}(\varsigma)d\varsigma,$$

where $\mathfrak{M}(\alpha)$ is a normalization function that satisfies $\mathfrak{M}(0) = \mathfrak{M}(1) = 1$.

Definición 2.2. The Caputo-Fabrizio fractional derivative of order $0 < \alpha \leq 1$ for a function $\mathfrak{X} \in H^1(0,1)$ is defined by

$$^{CF}\mathfrak{D}^{\alpha}\mathfrak{X}(\varpi) = \frac{(2-\alpha)\mathfrak{M}(\alpha)}{2(1-\alpha)}\int_0^{\varpi}\mathfrak{X}'(\varsigma)\exp\left(-\frac{\alpha(\varpi-\varsigma)}{1-\alpha}\right)d\varsigma. \qquad (3)$$

For $\mathfrak{M}(\alpha) = \dfrac{2}{2-\alpha}$ in equation (3), we have

$$^{CF}\mathfrak{D}^{\alpha}\mathfrak{X}(\varpi) = \frac{1}{1-\alpha}\int_0^{\varpi}\mathfrak{X}'(\varsigma)\exp\left(-\frac{\alpha(\varpi-\varsigma)}{1-\alpha}\right)d\varsigma. \qquad (4)$$

Definición 2.3. The Caputo-Fabrizio fractional derivative of order $\alpha+n$ when $0<\alpha\leq 1$ and $n\geq 1$ is defined by

$$^{CF}\mathfrak{D}^{\alpha+n}\mathfrak{X}(\varpi) = ^{CF}\mathfrak{D}^{\alpha}(\mathfrak{D}^{n}\mathfrak{X}(\varpi)). \tag{5}$$

Theorem 2.4. [18] (Banach contraction principle). *Let (E,d) be a complete metric space, then each contraction mapping $\mathfrak{T}: E\rightarrow E$ has a unique fixed point ω of $\mathfrak{T}$ in E i.e.*

$$\mathfrak{T}\omega = \omega.$$

Now, we present a new result related to the Khalouta transform of the Caputo-Fabrizio fractional derivative. The Khalouta transform is a new integral transform that is applied to solve ordinary and partial differential equations, defined and developed by [11].

Definición 2.5. [11] Let $\mathfrak{X}(\varpi)$ be a integrable function defined for $\varpi\geq 0$. We define the Khalouta transform $\mathcal{K}$ of $\mathfrak{X}(\varpi)$ by the formula

$$\mathbb{KH}\left[\mathfrak{X}(\varpi)\right] = \mathcal{K}(s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{s\varsigma}{\gamma\eta}\right)\mathfrak{X}(\varsigma)d\varsigma,$$

provided the integral exists.

Some basic properties of the Khalouta transform are given as follows [11].

Property 1: The Khalouta transform is a linear operator

$$\mathbb{KH}\left[a\mathfrak{X}(\varpi)+b\mathfrak{Y}(\varpi)\right] = a\mathbb{KH}\left[\mathfrak{X}(\varpi)\right]+b\mathbb{KH}\left[\mathfrak{Y}(\varpi)\right], a,b\in\mathbb{R}.$$

Property 2: If $\mathfrak{X}(\varpi)$ is n^{th} differentiable, then

$$\mathbb{KH}\left[\mathfrak{X}(\varpi)\right] = \left(\frac{s}{\gamma\eta}\right)^n\mathcal{K}(s,\gamma,\eta) - \sum_{k=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{n-k}\mathfrak{X}^{(k)}(0).$$

Property 3: (Convolution) Let $\mathfrak{X}(\varpi)$ and $\mathfrak{Y}(\varpi)$ have Khalouta transforms $\mathcal{K}(s,\gamma,\eta)$ and $\mathcal{H}(s,\gamma,\eta)$ respectively. Then the Khalouta transform of the convolution of $\mathfrak{X}$ and $\mathfrak{Y}$ is

$$\mathbb{KH}\left[\left(\mathfrak{X}*\mathfrak{Y}\right)(\varpi)\right] = \int_0^{+\infty}\mathfrak{X}(\varsigma)\mathfrak{Y}(\varpi-\varsigma)d\varsigma = \frac{\gamma\eta}{s}\mathcal{K}(s,\gamma,\eta)\mathcal{H}(s,\gamma,\eta).$$

Property 4: Some special Khalouta transforms :

$$\begin{aligned}
\mathbb{KH}\left[1\right] &= 1,\\
\mathbb{KH}\left[\varpi\right] &= \frac{\gamma\eta}{s},\\
\mathbb{KH}\left[\frac{\varpi^n}{n!}\right] &= \left(\frac{\gamma\eta}{s}\right)^n, n=0,1,2,...\\
\mathbb{KH}\left[\exp(a\varpi)\right] &= \frac{s}{s-a\gamma\eta}.
\end{aligned}$$

Theorem 2.6. *The Khalouta transform of the Caputo-Fabrizio fractional derivative operator of the function $\mathfrak{X}(\varpi)$ of order $\alpha+n$, where $0<\alpha\leq 1$ and $n\in\mathbb{N}\cup\{0\}$, is given by*

$$\mathbb{KH}\left[{}^{CF}\mathfrak{D}^{\alpha+n}\mathfrak{X}(\varpi)\right]=\frac{s}{(1-\alpha)\,s+\alpha\gamma\eta}\left(\left(\frac{s}{\gamma\eta}\right)^{n}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-k}\mathfrak{X}^{(k)}(0)\right).$$

Proof. Using equations (4) and (5), we have

$$\begin{aligned}\mathbb{KH}\left[{}^{CF}\mathfrak{D}^{\alpha+n}\mathfrak{X}(\varpi)\right]&=\mathbb{KH}\left[{}^{CF}\mathfrak{D}^{\alpha}(\mathfrak{D}^{n}\mathfrak{X}(\varpi))\right]\\&=\frac{1}{1-\alpha}\frac{s}{\gamma\eta}\int_0^{\infty}\exp\left(-\frac{st}{\gamma\eta}\right)\left(\int_0^{\varpi}\mathfrak{X}^{(n+1)}(\varsigma)\exp\left(-\frac{\alpha\,(\varpi-\varsigma)}{1-\alpha}\right)d\varsigma\right)d\varpi\\&=\frac{1}{1-\alpha}\frac{s}{\gamma\eta}\int_0^{\infty}\exp\left(-\frac{st}{\gamma\eta}\right)\left(\mathfrak{X}^{(n+1)}(\varpi)*\exp\left(-\frac{\alpha\varpi}{1-\alpha}\right)\right)d\varpi\\&=\frac{1}{1-\alpha}\mathbb{KH}\left[\mathfrak{X}^{(n+1)}(\varpi)*\exp\left(-\frac{\alpha\varpi}{1-\alpha}\right)\right].\end{aligned}$$

Using Properties (2), (3) and (4), we have

$$\begin{aligned}\mathbb{KH}\left[{}^{CF}\mathfrak{D}^{\alpha+n}\mathfrak{X}(\varpi)\right]&=\frac{1}{1-\alpha}\frac{\gamma\eta}{s}\mathbb{KH}\left[\mathfrak{X}^{(n+1)}(\varpi)\right]\mathbb{KH}\left[\exp\left(-\frac{\alpha\varpi}{1-\alpha}\right)\right]\\&=\frac{\gamma\eta}{s\,(1-\alpha)+\alpha\gamma\eta}\left(\left(\frac{s}{\gamma\eta}\right)^{n+1}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-k+1}u^{(k)}(0)\right)\\&=\frac{s}{s-\alpha\,(s-\gamma\eta)}\left(\left(\frac{s}{\gamma\eta}\right)^{n}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-k}\mathfrak{X}^{(k)}(0)\right)\\&=\frac{s}{(1-\alpha)\,s+\alpha\gamma\eta}\left(\left(\frac{s}{\gamma\eta}\right)^{n}\mathcal{K}(s,\gamma,\eta)-\sum_{k=0}^{n}\left(\frac{s}{\gamma\eta}\right)^{n-k}\mathfrak{X}^{(k)}(0)\right).\end{aligned}$$

The proof is complete. ☑

3. Main results

3.1. Existence and uniqueness results

In this subsection, we shall prove an existence and uniqueness results for equation (1) under the condition (2).

Let $\mathfrak{B}=\{\mathfrak{X}(\varpi)/\mathfrak{X}(\varpi)\in C(I)\}$ be the Banach space equipped with the norm $\|\mathfrak{X}(\varpi)\|_{\mathfrak{B}}=\max\limits_{\varpi\in I}|\mathfrak{X}(\varpi)|$. To prove the main results, we need the following hypotheses

(**H1**) There exists a constant $L_{\mathcal{F}}>0$ such that, for any $\mathfrak{X},\mathfrak{Y}\in C(I,\mathbb{R})$

$$|\mathcal{F}(\mathfrak{X}(\varpi))-\mathcal{F}(\mathfrak{Y}(\varpi))|\leq L_{\mathcal{F}}\,|\mathfrak{X}(\varpi)-\mathfrak{Y}(\varpi)|.$$

(**H2**) There exists a function $K^* \in C(D, \mathbb{R}^+)$, the set of all positive function continuous on $D = \{(\varpi, \varsigma) \in \mathbb{R} \times \mathbb{R} : 0 \le \varsigma \le \varpi \le 1\}$ such that

$$K^* = \sup_{\varpi \in [0,1]} \int_0^{\varpi} |K(\varpi, \varsigma)| \, d\varsigma < \infty.$$

(**H3**) The function $f : I \to \mathbb{R}$ is continuous.

Lemma 3.1. *If a function $\mathfrak{X}(\varpi) \in C(I, \mathbb{R})$ satisfies (1)-(2), then the problems (1)-(2) are equivalent to the problem of finding a continuous solution of the integral equation*

$$\begin{aligned} \mathfrak{X}(\varpi) &= \mathfrak{X}(0) + \frac{2(1-\alpha)}{(2-\alpha)\mathfrak{M}(\alpha)} \left(f(\varpi) + \int_0^{\varpi} K(\varpi, \varsigma) \mathcal{F}(\mathfrak{X}(\varsigma)) d\varsigma \right) \\ &\quad + \frac{2\alpha}{(2-\alpha)\mathfrak{M}(\alpha)} \int_0^{\varpi} \left(f(\varsigma) + \int_0^{\varsigma} K(\varsigma, \tau) \mathcal{F}(\mathfrak{X}(\tau)) d\tau \right) d\varsigma. \end{aligned}$$

Theorem 3.2. *Suppose that the hypotheses (**H1**)-(**H3**) are satisfied and*

$$(A(\alpha) + B(\alpha))\, K^* L_{\mathcal{F}} < 1. \tag{6}$$

Then there is a unique solution $\mathfrak{X}(\varpi) \in C(I, \mathbb{R})$ to the equations (1)-(2).

Proof. We define an operator $\mathfrak{T} : C(I, \mathbb{R}) \to C(I, \mathbb{R})$ by

$$\begin{aligned} \mathfrak{T}\mathfrak{X}(\varpi) &= \mathfrak{X}(0) + A(\alpha) \left(f(\varpi) + \int_0^{\varpi} K(\varpi, \varsigma) \mathcal{F}(\mathfrak{X}(\varsigma)) d\varsigma \right) \\ &\quad + B(\alpha) \int_0^{\varpi} \left(f(\varsigma) + \int_0^{\varsigma} K(\varsigma, \tau) \mathcal{F}(\mathfrak{X}(\tau)) d\tau \right) d\varsigma, \end{aligned}$$

where

$$A(\alpha) = \frac{2(1-\alpha)}{(2-\alpha)\mathfrak{M}(\alpha)} \text{ and } B(\alpha) = \frac{2\alpha}{(2-\alpha)\mathfrak{M}(\alpha)}.$$

Let $\mathfrak{X}(\varpi), \mathfrak{Y}(\varpi) \in C(I, \mathbb{R})$, then we have

$$\begin{aligned} |\mathfrak{T}\mathfrak{X}(\varpi) - \mathfrak{T}\mathfrak{Y}(\varpi)| &\le A(\alpha) \left(\int_0^{\varpi} |K(\varpi, \varsigma)| \, |\mathcal{F}(\mathfrak{X}(\varsigma)) - \mathcal{F}(\mathfrak{Y}(\varsigma))| \, d\varsigma \right) \\ &\quad + B(\alpha) \int_0^{\varpi} \left(\int_0^{\varsigma} |K(\varsigma, \tau)| \, |\mathcal{F}(\mathfrak{X}(\tau)) - \mathcal{F}(\mathfrak{Y}(\tau))| \, d\tau \right) d\varsigma \\ &\le A(\alpha) K^* L_{\mathcal{F}} |\mathfrak{X}(\varsigma) - \mathfrak{Y}(\varsigma)| + B(\alpha) K^* L_{\mathcal{F}} |\mathfrak{X}(\varsigma) - \mathfrak{Y}(\varsigma)| \int_0^{\varpi} 1 d\varsigma \\ &\le (A(\alpha) + B(\alpha))\, K^* L_{\mathcal{F}} |\mathfrak{X}(\varsigma) - \mathfrak{Y}(\varsigma)| \, . \end{aligned}$$

Hence, we have

$$\|\mathfrak{T}\mathfrak{X}(\varpi) - \mathfrak{T}\mathfrak{Y}(\varpi)\|_{\mathfrak{B}} \le (A(\alpha) + B(\alpha))\, K^* L_{\mathcal{F}} \|\mathfrak{X}(\varsigma) - \mathfrak{Y}(\varsigma)\|_{\mathfrak{B}} \, .$$

Given (6), $\mathfrak{T}$ is a contraction. By the Banach contraction principle (See. Theorem 2.4), $\mathfrak{T}$ has only one fixed point $\mathfrak{X}$ in $C(I,\mathbb{R})$, then $\mathfrak{X}$ is a solution of the equations (1)-(2). ☑

3.2. Analysis of the Khalouta decomposition method (KHDM)

In this subsection, we explore the essential facts of the KHDM. Let us consider the nonlinear fractional Volterra integro-differential equation (1) under the condition (2). We consider the kernel $K(\varpi,\varsigma)$ of equation (1) as a difference kernel that depends on the difference $\varpi-\varsigma$. Then the nonlinear fractional Volterra integro-differential equation (1) can be expressed as

$$^{CF}\mathfrak{D}^{\alpha}\mathfrak{X}(\varpi)=f(\varpi)+\int_0^{\varpi}K(\varpi-\varsigma)\mathcal{F}(\mathfrak{X}(\varsigma))d\varsigma, \tag{7}$$

Operating the Khalouta transform to (7) and using Theorem 2.6, we get

$$\begin{aligned}\mathbb{KH}\left[\mathfrak{X}(\varpi)\right]&=\mathfrak{X}(0)+\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(\varpi)+\int_0^{\varpi}K(\varpi-\varsigma)\mathcal{F}(\mathfrak{X}(\varsigma))d\varsigma\right]\\&=\mathfrak{X}(0)+\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(\varpi)\right]\\&\quad+\left(\frac{(1-\alpha)\,s\gamma\eta+\alpha\left(\gamma\eta\right)^2}{s^2}\right)\mathbb{KH}\left[K(\varpi)\right]\mathbb{KH}\left[\mathcal{F}(\mathfrak{X}(\varsigma))\right].\end{aligned} \tag{8}$$

Taking the inverse Khalouta transform of (8), we get

$$\begin{aligned}\mathfrak{X}(\varpi)&=\mathfrak{X}(0)+\mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(\varpi)\right]\right]\\&\quad+\mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s\gamma\eta+\alpha\left(\gamma\eta\right)^2}{s^2}\right)\mathbb{KH}\left[K(\varpi)\right]\mathbb{KH}\left[\mathcal{F}(\mathfrak{X}(\varsigma))\right]\right].\end{aligned} \tag{9}$$

Now, we represent the solution in an infinite series form

$$\mathfrak{X}(\varpi)=\sum_{n=0}^{\infty}\mathfrak{X}_n(\varpi), \tag{10}$$

and the nonlinear terms can be decomposed as

$$\mathcal{F}(\mathfrak{X}(\varsigma))=\sum_{n=0}^{\infty}\mathcal{A}_n(\varsigma), \tag{11}$$

where $\mathcal{A}_n$ are the Adomian polynomials [19] of $\mathfrak{X}_0,\mathfrak{X}_1,\mathfrak{X}_2,..,\mathfrak{X}_n$, and it can be calculated by formula given by

$$\mathcal{A}_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[\mathcal{F}\left(\sum_{i=0}^{\infty}\lambda^i\mathfrak{X}_i\right)\right]_{\lambda=0},n=0,1,2,...$$

Substituting (10) and (11) into (9) leads to

$$\sum_{n=0}^{\infty}\mathfrak{X}_n(\varpi) = \mathfrak{X}(0) + \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(\varpi)\right]\right] + \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s\gamma\eta+\alpha\,(\gamma\eta)^2}{s^2}\right)\mathbb{KH}\left[K(\varpi)\right]\mathbb{KH}\left[\sum_{n=0}^{\infty}\mathcal{A}_n(\varpi)\right]\right].$$

So that, the recursive relation is given by

$$\mathfrak{X}_0(\varpi) = \mathfrak{X}(0) + \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[f(\varpi)\right]\right],$$

$$\mathfrak{X}_{n+1}(\varpi) = \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)\,s\gamma\eta+\alpha\,(\gamma\eta)^2}{s^2}\right)\mathbb{KH}\left[K(\varpi)\right]\mathbb{KH}\left[\mathcal{A}_n(\varpi)\right]\right], n \geq 0.$$

Using the obtained recurrence relation, we get the components of $\mathfrak{X}_n(\varpi)$).

We define the $m-$terms approximte of the solution $\mathfrak{X}(\varpi)$ by

$$\phi_m\left[\mathfrak{X}(\varpi)\right] = \sum_{n=0}^{m-1}\mathfrak{X}_n(\varpi),$$

with

$$\lim_{m\to\infty}\phi_m\left[\mathfrak{X}(\varpi)\right] = \mathfrak{X}(\varpi).$$

3.3. Convergence analysis of the KHDM

In this subsection, we give a sufficient condition for the convergence analysis of our method.

Theorem 3.3. *Let $\mathfrak{X}_n(\varpi)$ and $\mathfrak{X}(\varpi)$ be in the Banach space $\mathfrak{B}$. Then the KHDM series solutions $\mathfrak{X}_n(\varpi)$ defined by equation (10) converges to the solution of equations (1) and (2) provided that $0 < \varrho < 1$ and $\mathfrak{X}_0(\varpi) \in \mathfrak{B}$ is bounded.*

Proof. Considering the sequence of partial sums $\{\mathcal{P}_n(\varpi)\}_{n=0}^{\infty}$ of the form

$$\begin{aligned}
\mathcal{P}_0(\varpi) &= \mathfrak{X}_0(\varpi),\\
\mathcal{P}_1(\varpi) &= \mathfrak{X}_0(\varpi)+\mathfrak{X}_1(\varpi),\\
\mathcal{P}_2(\varpi) &= \mathfrak{X}_0(\varpi)+\mathfrak{X}_1(\varpi)+\mathfrak{X}_2(\varpi),\\
&\vdots\\
\mathcal{P}_n(\varpi) &= \mathfrak{X}_0(\varpi)+\mathfrak{X}_1(\varpi)+\mathfrak{X}_2(\varpi)+...+\mathfrak{X}_n(\varpi).
\end{aligned}$$

To achieve the desired result, we will prove that $\{\mathcal{P}_n(\varpi)\}_{n=0}^{\infty}$ is a Cauchy sequence in $\mathfrak{B}$. From the last hypothesis of the theorem, we have $\varrho \in (0,1)$ then

$$\begin{aligned} \|\mathcal{P}_{n+1}(\varpi) - \mathcal{P}_n(\varpi)\| &\leq \|\mathfrak{X}_{n+1}(\varpi)\| \leq \varrho \|\mathfrak{X}_n(\varpi)\| \\ &\leq \varrho^2 \|\mathfrak{X}_{n-1}(\varpi)\| \leq ... \leq \varrho^{n+1} \|\mathfrak{X}_0(\varpi)\| . \end{aligned}$$

For any $n, m \in \mathbb{N}$ with $n \geq m$, we have

$$\begin{aligned} \|\mathcal{P}_n(\varpi) - \mathcal{P}_m(\varpi)\| &= \|\mathcal{P}_n(\varpi) - \mathcal{P}_{n-1}(\varpi) + \mathcal{P}_{n-1}(\varpi) - \mathcal{P}_{n-2}(\varpi) \\ &\quad +... + \mathcal{P}_{m+1}(\varpi) - \mathcal{P}_m(\varpi)\| \\ &\leq \|\mathcal{P}_n(\varpi) - \mathcal{P}_{n-1}(\varpi)\| + \|\mathcal{P}_{n-1}(\varpi) - \mathcal{P}_{n-2}(\varpi)\| \\ &\quad +... + \|\mathcal{P}_{m+1}(\varpi) - \mathcal{P}_m(\varpi)\| \\ &\leq \varrho^n \|\mathfrak{X}_0(\varpi)\| + \varrho^{n-1} \|\mathfrak{X}_0(\varpi)\| + ... + \varrho^{m+1} \|\mathfrak{X}_0(\varpi)\| \\ &= \varrho^{m+1} \left(1 + \varrho + ... + \varrho^{n-m-1}\right) \|\mathfrak{X}_0(\varpi)\| \\ &\leq \varrho^{m+1} \left(\frac{1 - \varrho^{n-m}}{1 - \varrho}\right) \|\mathfrak{X}_0(\varpi)\| . \end{aligned}$$

Since $0 < \varrho < 1$, we have $1 - \varrho^{n-m} < 1$, then

$$\|\mathcal{P}_n(\varpi) - \mathcal{P}_m(\varpi)\| \leq \frac{\varrho^{m+1}}{1 - \varrho} \|\mathfrak{X}_0(\varpi)\| .$$

Since $\mathfrak{X}_0(\varpi)$ is bounded, then $\|\mathfrak{X}_0(\varpi)\| < \infty$. So

$$\lim_{n,m \to \infty} \|\mathcal{P}_n(\varpi) - \mathcal{P}_m(\varpi)\| = 0$$

Hence, $\{\mathcal{P}_n(\varpi)\}_{n=0}^{\infty}$ is a Cauchy sequence in Banach space $\mathfrak{B}$. Consequently, the series $\sum\limits_{n=0}^{\infty} \mathfrak{X}_n(\varpi)$ is convergent. The proof is complete. ☑

Theorem 3.4. *The maximum absolute truncation error of the series solution (10) for equations (1) and (2) is estimated to be*

$$\left\| \mathfrak{X}(\varpi) - \sum_{n=0}^{r} \mathfrak{X}_n(\varpi) \right\| \leq \frac{\varrho^{r+1}}{1 - \varrho} \|\mathfrak{X}_0(\varpi)\| .$$

Proof. Consider the partial sum $\sum_{n=0}^{r}\mathfrak{X}_n(\varpi)$. Then

$$\begin{aligned}\left\|u(x,t)-\sum_{n=0}^{r}\mathfrak{X}_n(\varpi)\right\| &\leq \left\|\sum_{n=r+1}^{\infty}\mathfrak{X}_n(\varpi)\right\| \\ &\leq \sum_{n=r+1}^{\infty}\|\mathfrak{X}_n(\varpi)\| \\ &\leq \sum_{n=r+1}^{\infty}\varrho^r\|\mathfrak{X}_0(\varpi)\| \\ &\leq \varrho^{r+1}\left(1+\varrho+\varrho^2+\varrho^3+...\right)\|\mathfrak{X}_0(\varpi)\| \\ &\leq \frac{\varrho^{r+1}}{1-\varrho}\|\mathfrak{X}_0(\varpi)\|.\end{aligned}$$

The proof is complete. ☑

4. Illustrative Examples

In this section, we present the analytical technique based on KHDM for solving nonlinear fractional Volterra integro-differential equations with non-singular kernel derivative in the Caputo-Fabrizio sense. It should be noted that the $m-$term approximate solution using KHDM is given by

$$\mathfrak{X}(\varpi)=\sum_{n=0}^{m-1}\mathfrak{X}_n(\varpi)=\mathfrak{X}_0(\varpi)+\mathfrak{X}_1(\varpi)+\mathfrak{X}_2(\varpi)+...+\mathfrak{X}_{m-1}(\varpi).$$

Example 4.1. Let us consider the nonlinear fractional Volterra integro-differential

$$^{CF}\mathfrak{D}^{\alpha}\mathfrak{X}(\varpi)=-1+\int_0^{\varpi}\mathfrak{X}^2(\varsigma)d\varsigma, \tag{12}$$

under the condition

$$\mathfrak{X}(0)=1, \tag{13}$$

To solve this problem by the proposed method, we apply the steps involved in KHDM as presented in subsection 3.2 to equations (12)-(13), we obtain the following recursive relation

$$\begin{aligned}\mathfrak{X}_0(\varpi) &= 1+\mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)s+\alpha\gamma\eta}{s}\right)\mathbb{KH}[1]\right], \\ \mathfrak{X}_{n+1}(\varpi) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[\int_0^{\varpi}\mathcal{A}_n(\varpi)\right]\right],\end{aligned} \tag{14}$$

where $\mathcal{A}_n, n\geq 0$ are the Adomian polynomials of the nonlinear term $\mathfrak{X}^2$.

The first few nonlinear terms are given by

$$\begin{aligned} A_0 &= \mathfrak{X}_0^2, \\ A_1 &= 2\mathfrak{X}_0\mathfrak{X}_1, \\ A_2 &= 2\mathfrak{X}_0\mathfrak{X}_2 + \mathfrak{X}_1^2. \end{aligned}$$

From (14), we have

$$\begin{aligned} \mathfrak{X}_0(\varpi) &= -\left((1-\alpha)+\alpha\varpi\right), \\ \mathfrak{X}_1(\varpi) &= (1-\alpha)^3\varpi + 3\alpha(1-\alpha)^2\frac{\varpi^2}{2!} + 4\alpha^2(1-\alpha)\frac{\varpi^3}{3!} + 2\alpha^3\frac{\varpi^4}{4!}, \\ &\vdots \end{aligned}$$

and so on.

Therefore we obtain the approximate solution as

$$\begin{aligned} \mathfrak{X}(\varpi) &= -(1-\alpha) + \left((1-\alpha)^3 - \alpha\right)\varpi + 3\alpha(1-\alpha)^2\frac{\varpi^2}{2!} \\ &\quad + 4\alpha^2(1-\alpha)\frac{\varpi^3}{3!} + 2\alpha^3\frac{\varpi^4}{4!} + \ldots \end{aligned} \tag{15}$$

Taking $\alpha = 1$ in equation (15) we get

$$\mathfrak{X}(\varpi) = -\varpi + \frac{\varpi^4}{12} - \frac{\varpi^7}{252} + \ldots$$

which is the same result as in [16].

The 2D plots of the approximate solutions using KHDM for equations (12)-(13) for different values of α are given in Figure 1. Additionally, the numerical values of the approximate solutions using KHDM for different values of α are provided in Table 1. The comparison of KHDM with LGLQM [6], OPM [14], and IRKA [17] at $\alpha = 1$ is shown in Table 2.

ϖ	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$
0.1	-0.36625	-0.27863	-0.18970	-0.099992
0.2	-0.42999	-0.35573	-0.27873	-0.199870
0.3	-0.49052	-0.43063	-0.36653	-0.29933
0.4	-0.54708	-0.50257	-0.45243	-0.39787
0.5	-0.59884	-0.57067	-0.53558	-0.49482

Table 1. Numerical values of the KHDM-approximate solutions

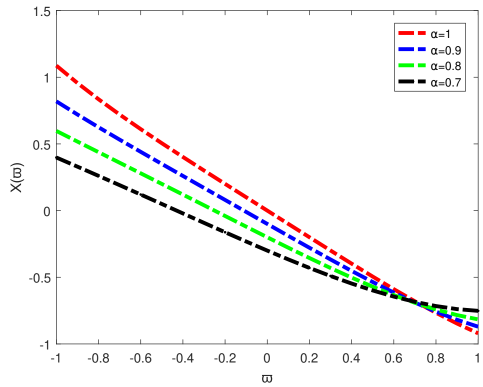

FIGURE 1. The 2D plots of the KHDM-approximate solutions

ϖ	$LGLQM$	OPM	$IRKA$	$KHDM$
0.1	−0.099992	−0.099992	−0.099992	−0.099992
0.2	−0.199870	−0.199870	−0.199870	−0.199870
0.3	−0.29933	−0.29933	−0.29933	−0.29933
0.4	−0.39787	−0.39787	−0.39787	−0.39787
0.5	−0.49482	−0.49482	−0.49482	−0.49482

TABLE 2. Comparison of the proposed method with LGLQM, OPM, and IRKA at $\alpha = 1$

Example 4.2. Let us consider the nonlinear fractional Volterra integro-differential

$$^{CF}\mathfrak{D}^{\alpha}\mathfrak{X}(\varpi) = 1 + \int_0^{\varpi} \mathfrak{X}'(\varsigma)\mathfrak{X}(\varsigma)d\varsigma, \tag{16}$$

under the condition

$$\mathfrak{X}(0) = 0, \tag{17}$$

The exact solution of equations (12)-(13) for $\alpha = 1$ is [16]

$$\mathfrak{X}(\varpi) = \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}\varpi\right).$$

To solve this problem by the proposed method, we apply the steps involved in KHDM as presented in subsection 3.2 to equations (16)-(17), we obtain the following recursive relation

$$\begin{aligned} \mathfrak{X}_0(\varpi) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)s+\alpha\gamma\eta}{s}\right)\mathbb{KH}[1]\right], \\ \mathfrak{X}_{n+1}(\varpi) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[\int_0^{\varpi}\mathcal{A}_n(\varpi)\right]\right], \end{aligned} \tag{18}$$

where $\mathcal{A}_n, n \geq 0$ are the Adomian polynomials of the nonlinear term $\mathfrak{X}'(\varsigma)\mathfrak{X}(\varsigma)$.

The first few nonlinear terms are given by

$$\begin{aligned} A_0 &= \mathfrak{X}_0'\mathfrak{X}_0, \\ A_1 &= \mathfrak{X}_0'\mathfrak{X}_1 + \mathfrak{X}_0\mathfrak{X}_1', \\ A_2 &= \mathfrak{X}_0'\mathfrak{X}_2 + \mathfrak{X}_1\mathfrak{X}_1' + \mathfrak{X}_0\mathfrak{X}_2', \end{aligned}$$

From (18), we have

$$\begin{aligned} \mathfrak{X}_0(\varpi) &= (1-\alpha)+\alpha\varpi \\ \mathfrak{X}_1(\varpi) &= \alpha(1-\alpha)^2\varpi + 2\alpha^2(1-\alpha)\frac{\varpi^2}{2} + \alpha^3\frac{\varpi^3}{3!} \\ &\vdots \end{aligned}$$

and so on.

Therefore we obtain the approximate solution as

$$\mathfrak{X}(\varpi) = (1-\alpha) + \left(\alpha+\alpha(1-\alpha)^2\right)\varpi + 2\alpha^2(1-\alpha)\frac{\varpi^2}{2} + \alpha^3\frac{\varpi^3}{3!} + ... \tag{19}$$

Taking $\alpha = 1$ in equation (19) we get

$$\begin{aligned} \mathfrak{X}(\varpi) &= \varpi + \frac{\varpi^3}{6} + \frac{\varpi^5}{30} + ... \\ &= \sqrt{2}\tan\left(\frac{\sqrt{2}}{2}\varpi\right). \end{aligned}$$

which is the same result as in [16].

The 2D plots of the approximate solutions using KHDM and the exact solution for equations (16)-(17) for different values of α are given in Figure 2. Additionally, the numerical values of the approximate solutions using KHDM and the exact solution for different values of α are provided in Table 3. The absolute error comparison of the KHDM with LGLQM [6], OPM [14], and IRKA [17] at $\alpha = 1$ is shown in Table 4.

Example 4.3. Let us consider the nonlinear fractional Volterra integro-differential

$$^{CF}\mathfrak{D}^{\alpha}\mathfrak{X}(\varpi) = -\frac{1}{2} + \int_0^{\varpi}\left(\mathfrak{X}'(\varsigma)\right)^2 d\varsigma, \tag{20}$$

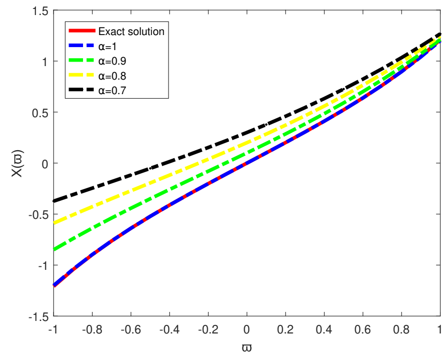

FIGURE 2. The 2D plots of KHDM-approximate solutions and exact solution

ϖ	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	*Exact solution*
0.1	0.37783	0.28457	0.19183	0.10017	0.10017
0.2	0.45894	0.37220	0.28601	0.20134	0.20134
0.3	0.54367	0.46342	0.38327	0.30458	0.30458
0.4	0.63238	0.55874	0.48434	0.41101	0.41102
0.5	0.72540	0.65867	0.58994	0.52188	0.52193

TABLE 3. Numerical values of the KHDM-approximate solutions and exact solution

ϖ	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{LGLQM}\rvert$	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{OPM}\rvert$	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{IRKA}\rvert$	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{KHDM}\rvert$
0.1	6.7597×10^{-10}	6.7597×10^{-10}	6.7597×10^{-10}	6.7597×10^{-10}
0.2	8.7055×10^{-8}	8.7055×10^{-8}	8.7055×10^{-8}	8.7055×10^{-8}
0.3	1.5028×10^{-6}	1.5028×10^{-6}	1.5028×10^{-6}	1.5028×10^{-6}
0.4	1.1423×10^{-5}	1.1423×10^{-5}	1.1423×10^{-5}	1.1423×10^{-5}
0.5	5.5515×10^{-5}	5.5515×10^{-5}	5.5515×10^{-5}	5.5515×10^{-5}

TABLE 4. Comparison of the proposed method with LGLQM, OPM, and IRKA in terms of absolute error at $\alpha = 1$

under the condition

$$\mathfrak{X}(0) = 0, \tag{21}$$

The exact solution of equations (20)-(21) for $\alpha = 1$ is [16]

$$\mathfrak{X}(\varpi) = -\ln\left(\frac{1}{2}\varpi + 1\right). \tag{22}$$

To solve this problem by the proposed method, we apply the steps involved in KHDM as presented in subsection 3.2 to equations (20)-(21), we obtain the following recursive relation

$$\begin{aligned} \mathfrak{X}_0(\varpi) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[-\frac{1}{2}\right]\right], \\ \mathfrak{X}_{n+1}(\varpi) &= \mathbb{KH}^{-1}\left[\left(\frac{(1-\alpha)s+\alpha\gamma\eta}{s}\right)\mathbb{KH}\left[\int_0^{\varpi}\mathcal{A}_n(\varsigma)d\varsigma\right]\right], \end{aligned} \tag{23}$$

where $\mathcal{A}_n, n \geq 0$ are the Adomian polynomials of the nonlinear term $(\mathfrak{X}'(\varsigma))^2$.

The first few nonlinear terms are given by

$$\begin{aligned} A_0 &= (\mathfrak{X}_0')^2, \\ A_1 &= 2\mathfrak{X}_0'\mathfrak{X}_1'+, \\ A_2 &= 2\mathfrak{X}_0'\mathfrak{X}_2' + \mathfrak{X}_1'\mathfrak{X}_1'. \end{aligned}$$

From (23), we have

$$\begin{aligned} \mathfrak{X}_0(\varpi) &= -\frac{1}{2}((1-\alpha)+\alpha\varpi) \\ \mathfrak{X}_1(\varpi) &= \frac{1}{4}\alpha^2(1-\alpha)\varpi + \frac{1}{8}\alpha^3\varpi^2, \\ \mathfrak{X}_2(\varpi) &= -\left(\frac{1}{4}\alpha^3(1-\alpha)^2\varpi + \frac{1}{4}\alpha^4(1-\alpha)\varpi^2 + \frac{1}{24}\alpha^5\varpi^3\right), \\ &\vdots \end{aligned}$$

and so on.

Therefore we obtain the approximate solution as

$$\begin{aligned} \mathfrak{X}(\varpi) &= -\frac{1}{2}(1-\alpha) - \frac{1}{4}\left(\alpha^3(1-\alpha)^2 - \alpha^2(1-\alpha) + 2\alpha\right)\varpi \\ &\quad + \frac{1}{8}\left(\alpha^3 - 2\upsilon^4(1-\alpha)^2\right)\varpi^2 - \frac{1}{24}\alpha^5\varpi^3 + ... \end{aligned} \tag{24}$$

Taking $\alpha = 1$ in equation (24) we get

$$\begin{aligned} \mathfrak{X}(\varpi) &= -\frac{1}{2}\varpi + \frac{1}{8}\varpi^2 - \frac{1}{24}\varpi^3 + ... \\ &= -\ln\left(\frac{1}{2}\varpi + 1\right), \end{aligned}$$

which is the same result as in [16].

The numerical values of the approximate solutions using KHDM and the exact solution for equations (20)-(21) for different values of α are provided in Table 5. The absolute error comparison of the KHDM with LGLQM [6], OPM [14], and IRKA [17] at $\alpha = 1$ is shown in Table 6.

ϖ	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$	*Exact solution*
0.1	-0.13673	-0.092287	-0.0049875	-0.0049875
0.2	-0.17234	-0.13293	-0.0099503	-0.0099503
0.3	-0.20691	-0.17208	-0.014889	-0.014889
0.4	-0.24054	-0.20989	-0.019803	-0.019803
0.5	-0.27329	-0.24649	-0.024693	-0.024693

TABLE 5. Numerical values of the KHDM-approximate solutions and exact solution

ϖ	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{LGLQM}\rvert$	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{OPM}\rvert$	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{IRKA}\rvert$	$\lvert\mathfrak{X}_{exact} - \mathfrak{X}_{KHDM}\rvert$
0.1	1.5563×10^{-10}	1.5563×10^{-10}	1.5563×10^{-10}	1.5563×10^{-10}
0.2	2.4802×10^{-9}	2.4802×10^{-9}	2.4802×10^{-9}	2.4802×10^{-9}
0.3	1.2506×10^{-8}	1.2506×10^{-8}	1.2506×10^{-8}	1.2506×10^{-8}
0.4	3.9370×10^{-8}	3.9370×10^{-8}	3.9370×10^{-8}	3.9370×10^{-8}
0.5	9.5743×10^{-8}	9.5743×10^{-8}	9.5743×10^{-8}	9.5743×10^{-8}

TABLE 6. Comparison of the proposed method with LGLQM, OPM, and IRKA in terms of absolute error at $\alpha = 1$

5. Conclusion

In this paper, we proposed a new technique known as Khalouta decomposition method (KHDM) for solving nonlinear fractional Volterra integro-differential equations. Fractional derivatives are described in the Caputo-Fabrizio sense recently introduced by Michele Caputo and Mauro Fabrizio. The existence, uniqueness and convergence results are performed based on the Banach contraction principle. Finally, the results obtained are verified through three numerical examples. The proposed method has shown to be a powerful and effective technique for obtaining an approximate analytical solution for nonlinear fractional Volterra integro-differential equations. The results obtained will be treated as benchmarks for our future studies on solving nonlinear fractional Volterra-Fredholm integro-differential equations.

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