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Entropy solutions for variable exponents nonlinear anisotropic elliptic equations with natural growth terms

Soluciones de entropía para ecuaciones elípticas anisotrópicas no lineales de exponente variable con términos de crecimiento natural

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Abstract. In this paper, we prove existence results for entropy solutions of a nonlinear boundary value problems represented by a class of nonlinear elliptic anisotropic equations with variable exponents and natural growth terms. The functional setting involves variable exponents anisotropic Sobolev spaces.

Key words and phrases. Nonlinear, elliptic equation, natural growth term, Anisotropic Sobolev spaces, Variable exponents, Entropy solution.

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Resumen. En este artículo, probamos la existencia de soluciones de entropía para problemas de frontera no lineales correspondientes a una clase de ecuaciones anisotrópicas elípticas no lineales con exponentes variables y términos de crecimiento natural.

Palabras y frases clave. ecuaciones no lineales, ecuación elíptica, término de crecimiento natural, espacios de Sobolev anisotrópicos, exponentes variables, soluciones de entropía.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open set with Lipschitz boundary $\partial\Omega$, and $f \in L^1(\Omega)$. In this paper we will prove that there is at least one entropy solution for the following $\overrightarrow{p}(x)-$ nonlinear elliptic anisotropic equations :

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i \left(| \partial_i u |^{p_i(x)-2} \partial_i u \right) + \sum_{i=1}^{N} | \partial_i u |^{p_i(x)-1} &= f, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where, $\partial_i u = \frac{\partial u}{\partial x_i}$, $i = 1, \dots, N$.

Our L^1-problem 1 has a main part characterized by a lower order term with a natural growth on the gradient $\sum_{i=1}^{N} \mid \partial_i u \mid^{p_i(x)-1}$, and it also includes the $\overrightarrow{p}(x)-$anisotropic Laplacian operator which represents a generalization of $p(x)-$Laplacian differential operator. The $\overrightarrow{p}(x)-$anisotropic Laplacian is defined between the space $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$(which will be further discussed in section 2) and its dual space, by the formula:

$$u \mapsto -\sum_{i=1}^{N} \partial_i \big(\mid \partial_i u \mid^{p_i(x)-2} \partial_i u \big).$$

There are several differential equations that include this type of operators that appear in natural applications, among them modeling of image processing and electro-rheological fluids. The digital image is a model of signal diffusion, smoothed by this operator by formulating it in a non-linear diffusion equation based on a criterion that determines the degree of smoothing by controlling the distribution of pixel values across the image and adjusting its degree of homogeneity, we refer to [12, 7] for more details. As for the importance of electro-rheological fluids, it lies in studying their motion, which is given by a nonlinear differential equation that includes the $p(x)-$Laplace operator in the coercive case (see [13, 19]).

The main aim of this paper is to prove the existence results of entropy solutions in the variable exponents anisotropic Sobolev space with zero boundary $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$. Many results of existence for these kinds of problems appeared in several works, for example see [4, 2, 1, 15].

The proof in this paper is based on a sequence of suitable approximate solutions (u_n), the existence of which was proven using the main Theorem on pseudo-monotone operators (Theorem 27.A in [20]). Then we presented a priori estimates for them and its truncated sequence $(T_t(u_n))(T_t,\, t > 0$ the truncation function will be defined later at the end of section 2) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ by proving its convergence in measure and the a.e. convergence of $T_t(u_n)$ in Ω. We then obtain strong L^1-convergence, and we pass to the limit in the strong L^1 sense for $\mid \partial_i u_n \mid^{p_i(x)-2} \partial_i u_n$, and for $|\partial_i u_n|^{p_i(x)-1}$, and finally we conclude the convergence of u_n to the solution of 1.

2. Mathematical preliminaries

In this section we provide some basics definitions and properties about isotropic and anisotropic variable exponent Lebesgue-Sobolev spaces (see [10, 11, 8, 9]).

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded open subset , we denote

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R}, : \quad p^- > 1\},$$

where

$$p^- = \min_{x\in\overline{\Omega}} p(x) \qquad \text{and} \qquad p^+ = \max_{x\in\overline{\Omega}} p(x).$$

Let $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. Then the following version of Young's inequality holds for all $a, b \in \mathbb{R}$ and all $\varepsilon > 0$,

$$|ab| \leq \varepsilon |a|^{p(x)} + c(\varepsilon)|b|^{p'(x)}, \tag{2}$$

where, $p'(\cdot)$ denotes the Sobolev conjugate of $p(\cdot)$ (i.e. $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$ in $\overline{\Omega}$).

In addition, for any two real a, b ($(a, b) \neq (0, 0)$) we also have:

$$(|a|^{p(x)-2}a - |b|^{p(x)-2}b)(a-b) \geq \begin{cases} 2^{2-p^+}|a-b|^{p(x)}, & \text{if } p(x) \geq 2, \\ (p^- - 1)\frac{|a-b|^2}{(|a|+|b|)^{2-p(x)}}, & \text{if } 1 < p(x) < 2. \end{cases} \tag{3}$$

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ defined by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R}; \rho_{p(\cdot)}(u) < \infty\}$$

where the function

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)} dx \quad \text{is called the convex modular.}$$

It is a Banach space, and reflexive if $p^- > 1$, and it is endowed with the Luxemburg norm given by:

$$\|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 \mid \rho_{p(\cdot)}(u/\lambda) \leq 1 \right\}.$$

A Hölder type inequality:

$$\left| \int_\Omega uv\, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)} \|v\|_{p'(\cdot)},$$

in this setting holds.

We define also the Banach space

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\},$$

which is equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

We write $W_0^{1,p(\cdot)}(\Omega)$ for the Banach space defined by

$$W_0^{1,p(\cdot)}(\Omega) := \left\{ f \in L^{p(\cdot)}(\Omega) : |\nabla f| \in L^{p(\cdot)}(\Omega) \text{ and } f = 0 \text{ on } \partial\Omega \right\}$$

under the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)}$. Moreover, is reflexive and separable if $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$.

If (u_n), $u \in L^{p(\cdot)}(\Omega)$, then by Theorems 1.2 and 1.3 in [11] we get:

$$\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \le \|u\|_{p(\cdot)} \le \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right), \quad (4)$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \le \rho_{p(\cdot)}(u) \le \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \quad (5)$$

Now we will introduce the variable exponents anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

Let $p_i(\cdot) \in C\big(\overline{\Omega}, [1,+\infty)\big),\ i = 1,\dots,N$, and we set for every x in $\overline{\Omega}$

$$\overrightarrow{p}(x) = (p_1(x),\dots,p_N(x)), \quad p_+(x) = \max_{1\le i\le N} p_i(x),\ p_-(x) = \min_{1\le i\le N} p_i(x),$$

$$p_-^- = \min_{x\in\overline{\Omega}} p_-(x), \quad p_+^+ = \max_{x\in\overline{\Omega}} p_+(x),$$

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^{N}\frac{1}{p_i(x)}}, \quad \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N,\\ +\infty, & \text{for } \overline{p}(x) \ge N.\end{cases}$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega), \partial_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1,\dots,N\right\},$$

under the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}.$$

The spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ are defined as follow

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)}, \quad \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

The folowing embedding results where proved as Theorem 2.5 in [10].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$.

$(*)$ If $r \in \mathcal{C}_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $r(x) < \max(p_+(x),\ \overline{p}^\star(x))$. Then the embedding

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact.} \quad (6)$$

$(*)$ If we have

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^\star(x). \quad (7)$$

Then the following inequality (see Theorem 2.6 in [10]) holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \le C\sum_{i=1}^{N}\|\partial_i u\|_{L^{p_i(\cdot)}(\Omega)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \quad (8)$$

where $C>0$ independent of u.Thus,

$$u \mapsto \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i(\cdot)}(\Omega)} \text{ is an equivalent norm to } \|\cdot\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} \text{ on } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{9}$$

The scalar truncation function $T_t : \mathbb{R} \longrightarrow \mathbb{R}$, $t>0$) is defined as

$$T_t(r) := \begin{cases} r, & \text{if } |r| \le t, \\ \frac{r}{|r|}t, & \text{if } |r| > t. \end{cases} \tag{10}$$

This function will be used to derive a priori estimates for our approximate solutions. We also need its derivative (see [17, 16, 14])

$$(DT_t)(r) = \begin{cases} 1, & |r| < t, \\ 0, & |r| > t. \end{cases} \tag{11}$$

Wewill also need the following function defined for $s \in \mathbb{R}$ by

$$G_t(s) = \begin{cases} 0, & if\ |s| \le t, \\ s-t, & if\ s > t, \quad t > 0 \\ s+t, & if\ s < -t, \end{cases}$$

as a test function in the approximate weak formulation.

3. Statement of Results

Definition 3.1. The function u is a entropy solution of 1 if and only if :

$\sum_{i=1}^{N} \mid \partial_i u \mid^{p_i(x)-1} \in L^1(\Omega)$, and for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and all $t>0$,

$$\int_\Omega \sum_{i=1}^{N} \mid \partial_i u \mid^{p_i(x)-2} \partial_i u \partial_i \big(T_t(u-\varphi)\big)\, dx + \int_\Omega \sum_{i=1}^{N} \mid \partial_i u \mid^{p_i(x)-1} T_t(u-\varphi)\, dx = \int_\Omega f(x) T_t(u-\varphi)\, dx.$$

The next Theorem is our main result.

Theorem 3.2. *Let $\overrightarrow{p}(\cdot) \in (\mathcal{C}_+(\overline{\Omega}))^N$ be such that $\overline{p} < N$ and 7 holds, and assume that f is in $L^1(\Omega)$. Then the problem 1 has at least one entropy solution.*

3.1. Approximate solutions

We define, $f_n = T_n(f)$, $n \in \mathbb{N}^*$.

Lemma 3.3. *Let $f(\cdot)$, and $p_i(\cdot) > 1$, $i = 1, \dots, N$, be as in Theorem 3.2. Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problem*

$$
\begin{aligned}
-\sum_{i=1}^{N} \partial_i \left(| \partial_i u_n |^{p_i(x)-2} \partial_i u_n \right) + \sum_{i=1}^{N} | \partial_i u_n |^{p_i(x)-1} = f_n, \quad \text{in } \Omega, \\
u_n = 0, \quad \text{on } \partial\Omega.
\end{aligned} \tag{12}
$$

in the following sense : $\forall \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$,

$$
\begin{aligned}
\sum_{i=1}^{N} \int_\Omega | \partial_i u_n |^{p_i(x)-2} \partial_i u_n \partial_i \varphi \, dx + \sum_{i=1}^{N} \int_\Omega | \partial_i u_n |^{p_i(x)-1} \varphi \, dx \\
= \int_\Omega f_n(x) \varphi \, dx.
\end{aligned} \tag{13}
$$

Proof. We consider the operator $\mathbf{K_n}$ defined between $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and its dual space by :

$\mathbf{K_n} : u \mapsto \mathbf{K_n}(u)$, where

$$
\begin{aligned}
\langle \mathbf{K_n}(u), v \rangle = & \int_\Omega \sum_{i=1}^{N} | \partial_i u |^{p_i(x)-2} \partial_i u \partial_i v \, dx + \int_\Omega \sum_{i=1}^{N} | \partial_i u |^{p_i(x)-1} v \, dx \\
= & \mathbf{C_n} + \mathbf{D_n},
\end{aligned}
$$

where

$$
\begin{aligned}
\langle \mathbf{C_n}(u), v \rangle &= \int_\Omega \sum_{i=1}^{N} | \partial_i u |^{p_i(x)-2} \partial_i u \partial_i v, \\
\langle \mathbf{D_n}(u), v \rangle &= \int_\Omega \sum_{i=1}^{N} | \partial_i u |^{p_i(x)-1} v \, dx.
\end{aligned}
$$

From Hölder inequality, 4, and 5, we obtain, for all $u, v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}
|\langle \mathbf{C_n}(u), v\rangle| &\leq n \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)-1} |\partial_i v| \, dx \\
&\leq 2n \sum_{i=1}^{N} \left\| |\partial_i u|^{p_i(x)-1} \right\|_{p_i'(\cdot)} \|\partial_i v\|_{p_i(\cdot)} \\
&\leq 2n \sum_{i=1}^{N} \left(1 + \int_\Omega |\partial_i u|^{p_i(x)} \, dx\right) \sum_{i=1}^{N} \|\partial_i v\|_{p_i(\cdot)} \\
&\leq 2n \sum_{i=1}^{N} \left(2 + \|\partial_i u\|_{p_i(\cdot)}^{p_+^+}\right) \|v\|_{\overrightarrow{p}(\cdot)} \\
&\leq 2n \left(2N + \sum_{i=1}^{N} \|\partial_i u\|_{p_i(\cdot)}^{p_+^+}\right) \|v\|_{\overrightarrow{p}(\cdot)} \\
&\leq 2n \left(2N + \|u\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right) \|v\|_{\overrightarrow{p}(\cdot)}.
\end{aligned}$$

Which implies the boundedness of $\mathbf{C_n}$. Also, by Hölder inequality, 4, and 5, we can get for all $u, v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\begin{aligned}
|\langle \mathbf{D_n}(u), v\rangle| \leq & n \int_\Omega \left(\sum_{i=1}^{N} |\partial_i u|^{p_i(x)-1}\right) |v| \, dx \\
\leq & 2n \sum_{i=1}^{N} \||\partial_i u|^{p_i(x)-1}\|_{p_i'(\cdot)} \|v\|_{p_i(\cdot)} \\
& \leq 2n \sum_{i=1}^{N} \left(1 + \int_\Omega |\partial_i u|^{p_i(x)} \, dx\right) \|v\|_{p_+(x)} \\
& \leq 2n \sum_{i=1}^{N} \left(2 + \|\partial_i u\|_{p_i(\cdot)}^{p_+^+}\right) \|v\|_{\overrightarrow{p}(\cdot)} \\
& \leq 2n \left(2N + \|u\|_{\overrightarrow{p}(\cdot)}^{p_+^+}\right) \|v\|_{\overrightarrow{p}(\cdot)}.
\end{aligned}$$

Which implies the boundedness of $\mathbf{D_n}$.

Using equation 5, and after dropping the nonnegative term, we get

$$\frac{\langle \mathbf{K_n}(u), u\rangle}{\|u\|_{\overrightarrow{p}(\cdot)}} \geq \frac{\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)} \, dx}{\|u\|_{\overrightarrow{p}(\cdot)}}. \tag{14}$$

On the other hand, we have $\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i(x)} \, dx \geq \sum_{i=1}^{N} \min\{\left\|\partial_i u\right\|_{p_i(x)}^{p_i^-}, \left\|\partial_i u\right\|_{p_i(x)}^{p_i^+}\}$.

If we define for all $i = 1, \dots, N$; $\quad \xi_i = \begin{cases} p_+^+, & \text{si } \|\partial_i u\|_{p_i(\cdot)} < 1 \\ p_-^-, & \text{si } \|\partial_i u\|_{p_i(\cdot)} \geq 1 \end{cases}$, we obtain

$$\begin{aligned}\sum_{i=1}^{N} \min\{\|\partial_i u\|_{p_i(.)}^{p_i^-}, \|\partial_i u\|_{p_i(.)}^{p_i^+}\} &\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(.)}^{\xi_i} \\ &\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(.)}^{p_-^-} - \sum_{\{i, \xi_i = p_+^+\}} (\|\partial_i u\|_{p_i(.)}^{p_-^-} - \|\partial_i u\|_{p_i(.)}^{p_+^+}) \\ &\geq \sum_{i=1}^{N} \|\partial_i u\|_{p_i(.)}^{p_-^-} - \sum_{\{i, \xi_i = p_+^+\}} \|\partial_i u\|_{p_i(.)}^{p_-^-} \geq (\frac{1}{N} \sum_{i=1}^{N} \|\partial_i u\|_{p_i(.)})^{p_-^-} - N.\end{aligned}$$

Then, we get

$$\sum_{i=1}^{N} \int_{\Omega} |\partial_i u|^{p_i(x)} \, dx \geq \left(\frac{1}{N} \|u\|_{\vec{p}(\cdot)} \right)^{p_-^-} - N. \tag{15}$$

By combining 15 and 14, we get

$$\frac{\langle \mathbf{K_n}(u), u \rangle}{\|u\|_{\vec{p}(\cdot)}} \geq \frac{\left(\frac{1}{N} \|u\|_{\vec{p}(\cdot)} \right)^{p_-^-} - N}{\|u\|_{\vec{p}(\cdot)}}.$$

This proves the coerciveness of $\mathbf{K_n}$.

Now we deal with the pseudo-monotonicity of $\mathbf{K_n}$:

Let $(u_k)_k$ be a sequence in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$u_k \rightharpoonup u \text{ in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega), \tag{16}$$

and

$$\limsup_{k \to \infty} \langle \mathbf{K_n}(u_k), u_k - u \rangle \leq 0. \tag{17}$$

We have to prove that,

$$\liminf_{k \to \infty} \langle \mathbf{K_n}(u_k), u_k - v \rangle \geq \langle \mathbf{K_n}(u), u - v \rangle \text{ for every } v \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega). \tag{18}$$

Note that

$$\begin{aligned}\langle \mathbf{K_n}(u_k), u_k - v \rangle &= \sum_{i=1}^{N} \int_{\Omega} | \partial_i u_k |^{p_i(x)-2} \partial_i u_k (\partial_i u_k - \partial_i v) \, dx \\ &+ \int_{\Omega} \left(\sum_{i=1}^{N} | \partial_i u_k |^{p_i(x)-1} \right) (u_k - v) \, dx.\end{aligned}$$

First, prove that

$$\lim_{k\to+\infty}\sum_{i=1}^{N}\int_\Omega \big(\mid \partial_i u_k \mid^{p_i(x)-2} \partial_i u_k - \mid \partial_i u \mid^{p_i(x)-2} Du\big)(\partial_i u_k - \partial_i u)\, dx = 0. \tag{19}$$

Note that,

$$\langle \mathbf{K_n}(u_k), u_k - u\rangle = \sum_{i=1}^{N}\int_\Omega \mid \partial_i u_k \mid^{p_i(x)-2} \partial_i u_k(\partial_i u_k - \partial_i u)\, dx + \int_\Omega \left(\sum_{i=1}^{N} \mid \partial_i u_k \mid^{p_i(x)-1}\right)(u_k - u)\, dx. \tag{20}$$

From 16, 7, and 6 using that $p_i(\cdot) \le p_+(\cdot) \le \overline{p}^*(\cdot)$ in $\overline{\Omega}$, $i = 1, \dots, N$, we get

$$u_k \longrightarrow u \text{ Strongly in } L^{p_i(\cdot)}(\Omega). \tag{21}$$

Also we have, for all $i = 1, \dots, N$

$$\int_\Omega \left|\left(\sum_{i=1}^{N} \mid \partial_i u_k \mid^{p_i(x)-1}\right)\right|^{p_i'(x)} dx \le cn^{p'^{+}_{+}} \sum_{i=1}^{N}\int_\Omega \mid \partial_i u_k \mid^{p_i(x)} dx \le C. \tag{22}$$

Then, 21 and 22 imply that

$$\lim_{k\longrightarrow+\infty}\int_\Omega \left(\sum_{i=1}^{N} \mid \partial_i u_k \mid^{p_i(x)-1}\right)(u_k - u) = 0. \tag{23}$$

Using equations 17, 23, and $\partial_i u_k \rightharpoonup \partial_i u$ up to a further subsequence, we find that equation 20 will give us

$$\limsup_{k\to+\infty}\sum_{i=1}^{N}\int_\Omega \Big(\mid \partial_i u_k \mid^{p_i(x)-2} \partial_i u_k - \mid \partial_i u \mid^{p_i(x)-2} Du\Big)(\partial_i u_k - \partial_i u)\, dx \le 0. \tag{24}$$

Using 3 we can get, for all $i = 1, \dots, N$

$$\Big(\mid \partial_i u_k \mid^{p_i(x)-2} \partial_i u_k - \mid \partial_i u \mid^{p_i(x)-2} \partial_i u\Big)(\partial_i u_k - \partial_i u) \ge 0. \tag{25}$$

Equation 24 together with 25, give the desired result 19.

Then, we can obtain, for all $i = 1, \dots, N$

$$\lim_{k\to+\infty}\Big(\mid \partial_i u_k \mid^{p_i(x)-2} \partial_i u_k - \mid \partial_i u \mid^{p_i(x)-2} \partial_i u\Big)(\partial_i u_k - \partial_i u) = 0. \tag{26}$$

From 26 and in a similar way as was proved Lemma 3.4. in [3], we can get, for all $i = 1, \dots, N$,

$$\partial_i u_k \longrightarrow \partial_i u \quad \text{a.e. in}\, \Omega. \tag{27}$$

By 27 we have

$$| \partial_i u_k |^{p_i(x)-2} Du_k \rightharpoonup | \partial_i u |^{p_i(x)-2} \partial_i u \text{ Weakly in } L^{p_i'(\cdot)}(\Omega). \tag{28}$$

From 16 and Lebesgue's dominated convergence theorem, we obtain for all $v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and all $i = 1, \ldots, N$

$$\partial_i v \longrightarrow \partial_i v \text{ Strongly in } L^{p_i(\cdot)}(\Omega). \tag{29}$$

Then we have, for every $v \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$

$$\sum_{i=1}^{N} \int_{\Omega} | \partial_i u_k |^{p_i(x)-2} \partial_i u_k \partial_i v \longrightarrow \sum_{i=1}^{N} \int_{\Omega} | \partial_i u |^{p_i(x)-2} \partial_i u \partial_i v. \tag{30}$$

On the other hand,from 28, 27, and 21, Fatou's Lemma implies that

$$\liminf_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega} | \partial_i u_k |^{p_i(x)-2} \partial_i u_k \partial_i u_k \geq \sum_{i=1}^{N} \int_{\Omega} | \partial_i u |^{p_i(x)-2} \partial_i u \partial_i u. \tag{31}$$

From 30 and 31, we deduce that

$$\begin{aligned} \liminf_{k\to+\infty} \sum_{i=1}^{N} \int_{\Omega} & | \partial_i u_k |^{p_i(x)-2} \partial_i u_k (\partial_i u_k - \partial_i v) \\ & \geq \sum_{i=1}^{N} \int_{\Omega} | \partial_i u |^{p_i(x)-2} \partial_i u (\partial_i u - \partial_i v). \end{aligned} \tag{32}$$

Using 21 and 27,we obtain

$$\left(\sum_{i=1}^{N} | \partial_i u_k |^{p_i(x)-1} \right) \rightharpoonup \left(\sum_{i=1}^{N} | \partial_i u |^{p_i(x)-1} \right) \text{ weakly in } L^{p_i'(\cdot)}(\Omega). \tag{33}$$

Then, 33 and 21 implies that

$$\liminf_{k\to+\infty} \int_{\Omega} \left(\sum_{i=1}^{N} | \partial_i u_k |^{p_i(x)-1} \right) (u_k - v) \geq \int_{\Omega} \left(\sum_{i=1}^{N} | \partial_i u |^{p_i(x)-1} \right) (u - v). \tag{34}$$

Combining 32 and 34, we obtain 18. So, $\mathbf{K_n}$ is pseudo-monotone.

• From the results proven above, we find that $\mathbf{K_n}$ fulfills the conditions of the main Theorem on pseudo-monotone operators (Theorem 27.A in [20]). Thus, the proof of Lemma 3.3 is completed ☑

3.1.1. A priori estimates

Lemma 3.4. *Let $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be a weak solution to problem 12. Then, there exists a constant C, such that, for all $i = 1, \dots, N$ and all $t > 0$*

$$\|T_t(u_n)\|_{\overrightarrow{p}(\cdot)} \leq C\left(1 + t^{\frac{1}{p_-^-}}\right). \tag{35}$$

Proof. Taking $\varphi = T_t(u_n)$ as test function in 13, we obtain

$$\int_\Omega \sum_{i=1}^N |\ \partial_i u_n\ |^{p_i(x)-2}\ \partial_i u_n \partial_i T_t(u_n)\, dx + \int_\Omega \sum_{i=1}^N |\ \partial_i u_n\ |^{p_i(x)-1}\ T_t(u_n)\, dx = \int_\Omega f_n(x) T_t(u_n)\, dx. \tag{36}$$

After dropping the non negative term in 36, we get

$$\sum_{i=1}^N \int_\Omega |\ \partial_i T_t(u_n)\ |^{p_i(x)}\ dx \leq t\|f_n\|_{L^1(\Omega)}.$$

Using 15, we obtain

$$\left(\frac{1}{N}\left\|T_t(u_n)\right\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-} - N \leq t\|f_n\|_{L^1(\Omega)}.$$

Then, we have

$$\left(\left\|T_t(u_n)\right\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-} \leq ct + c',$$

and this implies 35. ☑

Lemma 3.5. *Let $u_n \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be a weak solution to problem 12. Then,*

$$u_n \text{ converges in measure to some function } u. \tag{37}$$

Moreover, for every $t > 0$

$$T_t(u_n) \rightharpoonup T_t(u) \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{38}$$

$$T_t(u_n) \longrightarrow T_t(u) \text{strongly in } L^{p_i(\cdot)}(\Omega) \text{ and a.e. in } \Omega. \tag{39}$$

Proof. Using 35, we can obtain, for every $t > 0$

$$\begin{aligned}\left|\{|u_n| > t\}\right| &= t^{-1}\int_{\{|u_n|>t\}} |T_t(u_n)|\, dx \\ &\leq t^{-1}\|T_t(u_n)\|_{L^1(\Omega)} \\ &\leq ct^{-1}\left\|T_t(u_n)\right\|_{\overrightarrow{p}(\cdot)} \\ &\leq C'\left(t^{-1} + t^{\frac{1}{p_-^-}-1}\right).\end{aligned} \tag{40}$$

Since $\frac{1}{p_-^-}-1<0$, then we have

$$\lim_{t\longrightarrow+\infty}\left|\{|u_n|>t\}\right|=0.$$

Now, using the fact that, for every $\delta>0$, $t>0$

$$\{|u_n-u_m|>\delta\}\subset\{|u_n|>t\}\cup\{|u_m|>t\}\cup\{|T_t(u_n)-T_t(u_m)|>\delta\},$$

we get

$$\left|\{|u_n-u_m|>\delta\}\right|\leq\left|\{|u_n|>t\}\right|+\left|\{|u_m|>t\}\right|+\left|\{|T_t(u_n)-T_t(u_m)|>\delta\}\right|. \tag{41}$$

From 35 it follows that $T_t(u_n)$ is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for all $t>0$.

Then, up to a subsequence there exists some ξ_t in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ for all $t>0$ such that

$$T_t(u_n)\rightharpoonup\xi_t \text{ Weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega).$$

By 7 and 6 (since $p_i(\cdot)\leq p_+(\cdot)\leq\overline{p}^*(\cdot)$ in $\overline{\Omega}$), we obtain

$$T_t(u_n)\longrightarrow\xi_t \text{ Strongly in } L^{p_i(\cdot)}(\Omega).$$

Consequently, we conclude that

$$T_t(u_n) \text{ is a Cauchy sequence in measure in } \Omega.$$

Now, let $\varepsilon>0$, then through 40 and 41, there exists $t(\varepsilon)$ such that

$$\forall n,m\geq n_0(t(\varepsilon),\delta):\left|\{|u_n-u_m|>\delta\}\right|<\varepsilon,$$

and this proves 37.

Moreover, 37 implies that, the sequence (u_n) converges almost everywhere to some measurable function u. Then we can prove 38 and 39. ☑

Lemma 3.6. *Let $u_n\in\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be a weak solution to problem 12. Then, for all $t>0$*

$$T_t(u_n)\longrightarrow T_t(u), \text{ a.e. in } \Omega. \tag{42}$$

We apply to our problem the approach of [6] in part 3.2.

Proof. Let us define for all $i=1,\dots,N$ and a fixed $t>0$

$$I_{i,n}^t(x)=\big(|\partial_i(T_t(u_n))|^{p_i(x)-2}\partial_i(T_t(u_n))-\partial_i(T_t(u))|^{p_i(x)-2} \partial_i(T_t(u))\big)\big(\partial_i(T_t(u_n))-\partial_i(T_t(u))\big)$$

For $0<\theta<1$, $0<h<t$, and all $i=1,\dots,N$, we get

$$\begin{aligned}\int_\Omega \big(I^t_{i,n}(x)\big)^\theta dx &= \int_{\{|T_t(u_n)-T_t(u)|>h\}} \big(I^t_{i,n}(x)\big)^\theta dx + \int_{\{|T_t(u_n)-T_t(u)|\le h\}} \big(I^t_{i,n}(x)\big)^\theta dx \\ &\le \left(\int_\Omega I^t_{i,n}(x)\,dx\right)^\theta \big|\{|T_t(u_n)-T_t(u)|>h\}\big|^{1-\theta} \\ &+ \left(\int_{\{|T_t(u_n)-T_t(u)|\le h\}} I^t_{i,n}(x)\,dx\right)^\theta |\Omega|^{1-\theta}. \end{aligned} \tag{43}$$

Then, we can write

$$\int_\Omega \big(I^t_{i,n}(x)\big)^\theta\, dx \le J_1 + J_2. \tag{44}$$

Where,

$$\begin{aligned} J_1 &= \left(\int_\Omega I^t_{i,n}(x)\,dx\right)^\theta \big|\{|T_t(u_n)-T_t(u)|>h\}\big|^{1-\theta} \\ J_2 &= \left(\int_{\{|T_t(u_n)-T_t(u)|\le h\}} I^t_{i,n}(x)\,dx\right)^\theta |\Omega|^{1-\theta}. \end{aligned} \tag{45}$$

For every fixed h, thanks to 38, and the convergence in measure of $T_t(u_n)$, we can get

$$\lim_{n\longrightarrow+\infty} J_1 = 0 \tag{46}$$

Now, choosing $T_h(u_n - T_t(u))$ (with $0<h<t$) as test function in 13, we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega |\partial_i(T_t(u_n))|^{p_i(x)-2}\partial_i(T_t(u_n))\partial_i T_h\big(T_t(u_n)-T_t(u)\big)\,dx \\ &\quad - \sum_{i=1}^N \int_{\{|u_n-T_t(u)|<h\}} |\partial_i(G_t(u_n))|^{p_i(x)-2}\partial_i(G_t(u_n))\partial_i\big(T_t(u)\big)\,dx \\ &\quad + \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i(x)-1} T_h(u_n-T_t(u))\,dx \le ch. \end{aligned} \tag{47}$$

Then, we get

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega |\partial_i(T_t(u_n))|^{p_i(x)-2}\partial_i(T_t(u_n))\partial_i T_h\big(T_t(u_n)-T_t(u)\big)\,dx \\ &\quad \le c_1 h + c_2 \sum_{i=1}^N \int_{\{|u_n-T_t(u)|<h\}} |\partial_i(G_t(u_n))|^{p_i(x)-2}\partial_i(G_t(u_n))\partial_i\big(T_t(u)\big)\,dx \\ &\quad - c_3 \sum_{i=1}^N \int_\Omega |\partial_i u_n|^{p_i(x)-1} T_h(u_n-T_t(u))\,dx. \end{aligned} \tag{48}$$

Then, by using 48 and the fact that $I_{i,n}^t(x) \geq 0$ (which follows from 3), we deduce

$$\begin{aligned}
0 \leq & \sum_{i=1}^{N} \int_{\{|T_t(u_n)-T_t(u)|\leq h\}} I_{i,n}^t(x)\,dx \\
= & \sum_{i=1}^{N} \int_{\Omega} \big(|\partial_i(T_t(u_n))|^{p_i(x)-2}\partial_i(T_t(u_n)) - |\partial_i(T_t(u))|^{p_i(x)-2}\partial_i(T_t(u))\big)\partial_i T_h\big(T_t(u_n)-T_t(u)\big)dx \\
\leq & c_1 h - c_3 \sum_{i=1}^{N} \int_{\Omega} |\partial_i u_n|^{p_i(x)-1} T_h(u_n - T_t(u))\,dx \\
& + c_2 \sum_{i=1}^{N} \int_{\{|u_n|>t\}\cap\{|u_n-T_t(u)|<h\}} |\partial_i(u_n)|^{p_i(x)-2}\partial_i(u_n)\partial_i\big(T_t(u)\,dx \\
& - \sum_{i=1}^{N} \int_{\Omega} |\partial_i(T_t(u))|^{p_i(x)-2}\partial_i(T_t(u))\partial_i T_h\big(T_t(u_n)-T_t(u)\big)\,dx.
\end{aligned} \tag{49}$$

By noticing that $\{|u_n - T_t(u)| < h\} \subset \{|u_n| \leq h + t\} \subset \{|u_n| \leq 2t\}$, and that $\big\{|\partial_i(T_{2t}(u_n))|^{p_i(x)-2}\partial_i(T_{2t}(u_n))\big\}$, and $\big\{|\partial_i u_n|^{p_i(x)-1}\big\}$ are bounded in $L^{p_i'(x)}(\Omega)$, and using 38, we can pass to the limit with respect to n in 49 when $n \longrightarrow +\infty$, we get

$$\begin{aligned}
\limsup_{n\longrightarrow+\infty} \sum_{i=1}^{N} \int_{\{|T_t(u_n)-T_t(u)|\leq h\}} I_{i,n}^t(x)\,dx \leq c_1 h - c_3 \sum_{i=1}^{N} \int_{\Omega} \nu T_h(G_t(u))\,dx \\
+ c_2 \sum_{i=1}^{N} \int_{\{t<|u|<t+h\}} \tau_t \partial_i\big(T_t(u)\big),
\end{aligned} \tag{50}$$

where τ_t, $\nu \in L^{p_i'(x)}(\Omega)$ are the weak limits of $\big\{|\partial_i(T_{2t}(u_n))|^{p_i(x)-2}\partial_i(T_{2t}(u_n))\big\}_n$ and $\big\{|\partial_i u_n|^{p_i(x)-1}\big\}_n$ respectively.

After letting $h \longrightarrow 0$ in 50, we obtain

$$\lim_{n\longrightarrow+\infty} J_2 = 0, \tag{51}$$

where J_2 was defined in 45.

We combine 44, 46, 51, and recalling 3, we get

$$\lim_{n\longrightarrow+\infty} \int_{\Omega} \left(I_{i,n}^t(x)\right)^{\theta}\,dx = 0. \tag{52}$$

From 52 we deduce, like in [5], that: for every $t > 0$ and every $i = 1, \ldots, N$

$$\partial_i(T_t(u_n)) \longrightarrow \partial_i(T_t(u)),\ a.e. \text{ in } \overline{\Omega}. \tag{53}$$

And through the results obtained in [18] we can get 42. ☑

3.2. Proof of the Theorem 3.2 :

Choosing $\varphi = T_t(u_n - \phi)$ as test function in 13 where $\phi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, we get for all $n > \gamma_t = t + \|\phi\|_{L^\infty(\Omega)}$

$$\sum_{i=1}^{N} \int_\Omega |\ \partial_i u_n\ |^{p_i(x)-2}\ \partial_i u_n \partial_i T_t(u_n - \phi)\, dx + \sum_{i=1}^{N} \int_\Omega |\ \partial_i u_n\ |^{p_i(x)-1}\ T_t(u_n - \phi)\, dx = \int_\Omega f_n(x) T_t(u_n - \phi)\, dx.$$

Since the inequality $|\ u_n\ | > \gamma_t$ means that $|\ u_n\ | - \|\phi\|_{L^\infty(\Omega)} > t$, then by the fact that $|\ u_n - \phi\ | \geq |\ u_n\ | - \|\phi\|_{L^\infty(\Omega)}$, we obtain

$$\{|\ u_n - \phi\ | \leq t\} \subset \{|\ u_n\ | \leq \gamma_t\}.$$

From this we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega &|\ \partial_i T_{\gamma_t}(u_n)\ |^{p_i(x)-2}\ \partial_i T_{\gamma_t}(u_n) \partial_i T_t(u_n - \phi)\, dx \\ &+ \sum_{i=1}^{N} \int_\Omega |\ \partial_i T_{\gamma_t}(u_n)\ |^{p_i(x)-1}\ T_t(u_n - \phi)\, dx \\ &= \int_\Omega f_n(x) T_t(u_n - \phi)\, dx.\end{aligned}$$

Using 42, and the boundedness of both sequences :

$\{|\ \partial_i T_{\gamma_t}(u_n)\ |^{p_i(x)-2}\ \partial_i T_{\gamma_t}(u_n)\}$, $\{|\ \partial_i T_{\gamma_t}(u_n)\ |^{p_i(x)-1}\}$ in $L^{p_i'(\cdot)}(\Omega)$, we can get

$$\begin{aligned}&|\ \partial_i T_{\gamma_t}(u_n)\ |^{p_i(x)-2}\ \partial_i T_{\gamma_t}(u_n) \rightharpoonup |\ \partial_i T_{\gamma_t}(u)\ |^{p_i(x)-2}\ \partial_i T_{\gamma_t}(u),\ \text{weakly in } L^{p_i'(\cdot)}(\Omega),\\ &|\ \partial_i T_{\gamma_t}(u_n)\ |^{p_i(x)-1} \rightharpoonup |\ \partial_i T_{\gamma_t}(u)\ |^{p_i(x)-1}, \text{weakly in } L^{p_i'(\cdot)}(\Omega),\\ &T_t(u_n - \phi) \rightharpoonup T_t(u - \phi),\ \text{strongly in } L^{p_i(\cdot)}(\Omega).\end{aligned}$$

So, we can easily pass to the limit in 13 for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Now, since the sequence $\{|\ \partial_i T_{\gamma_t}(u_n)\ |^{p_i(x)-1}\}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, we can conclude its bondedness in $L^1(\Omega)$, then by Fatou's Lemma, we deduce that, $\sum_{i=1}^{N} |\ \partial_i u\ |^{p_i(x)-1} \in L^1(\Omega)$. Thus, the Theorem 3.2 was proven.

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