

Minimality of the \mathcal{D} -groupoid of symmetries of a projective structure

Minimalidad del \mathcal{D} -grupoide de simetrías de una estructura proyectiva

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ABSTRACT. In this article we study Kummer's \mathcal{D} -groupoid, which is the groupoid of symmetries of a meromorphic projective structure. We give necessary and sufficient conditions for its minimality, in the sense of not having infinite sub- \mathcal{D} -groupoids. The condition that we find turns out to be equivalent to the strong minimality of the non-linear Schwarzian equation and the non-integrability by means of Liouvillian functions of the linear Schwarzian equation.

Key words and phrases. Schwarzian equation, Schwarzian derivative, Strong minimality, Symmetric Power, Lie groupoid, \mathcal{D} -groupoid.

2020 Mathematics Subject Classification. 53C21, 53C42.

RESUMEN. En este artículo estudiamos el \mathcal{D} -grupoide de Kummer, el cual es el grupoide de simetrías de una estructura proyectiva meromorfa. Damos condiciones necesarias y suficientes para su simplicidad, en el sentido de no tener sub- \mathcal{D} -grupos no finitos. La condición que encontramos resulta ser equivalente a la fuerte minimalidad de la ecuación schwarziana no lineal y la no integrabilidad mediante funciones liouvillianas de la ecuación schwarziana lineal.

Palabras y frases clave. Ecuación Schwarziana, derivada Schwarziana, fuerte minimalidad, potencia simétrica, grupoide de Lie, \mathcal{D} -grupoide.

1. Introduction

The *Schwarzian derivative* of a meromorphic function $f \in \mathcal{K}$ (the field of meromorphic functions over an open subset of a Riemann surface) with respect to a coordinate z is defined by the expression:

$$S_z(f) = \left(\frac{f''}{f'} \right)' - \frac{1}{2} \left(\frac{f''}{f'} \right)^2 \quad \text{where } ' = \frac{d}{dz}$$

Particularly, the Schwarzian derivative classifies the functions related by the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\overline{\mathbb{C}}$:

$$S_z(f) = S_z(g) \iff \exists \sigma \in \mathrm{PSL}_2(\mathbb{C}) \text{ s.t. } f = \sigma \circ g$$

The Schwarzian derivative is an object of great interest as it appears in the differential equations satisfied by modular and uniformization functions (see, for instance [7, Section IV]). In particular, modular functions satisfy a differential equation of the form:

$$S_\tau(\lambda(\tau)) + \lambda'(\tau)^2 R(\lambda(\tau)) = 0, \quad (1)$$

where $R(\lambda) \in \mathbb{C}(\lambda)^{\mathrm{alg}}$ is an algebraic function [21]. This implies that R can be seen as a rational function on an algebraic curve which is seen as a finite ramified covering of the Riemann sphere through λ . The particular cases,

$$R(\lambda) = \frac{\lambda^2 - \lambda + 1}{2\lambda^2(1 - \lambda)^2}, \quad R(\lambda) = \frac{36\lambda^2 - 41\lambda + 32}{72\lambda^2(\lambda - 1)^2},$$

correspond to Schwarz's modular function and Klein's j -invariant respectively [9, Equations 1.11, 1.13].

K. Mahler proved that these functions do not satisfy lower order equations [13]. The chain rule for the Schwarzian derivative allows us to invert the above equation obtaining a linear-Schwarzian equation that is geometrically equivalent:

$$S_\lambda(\tau(\lambda)) = R(\lambda). \quad (2)$$

Even if this equation is not linear, it is classical fact explained in section 1.1 that this equation is linearizable. Differential algebraic properties of modular functions are relevant and have been recently studied in connection with model theory and transcendence theory [1, 10, 16, 17]. In particular, it was shown in [3, 5] that, for any particular algebraic function $R(\lambda) \in \mathbb{C}(\lambda)^{\mathrm{alg}}$, the strong minimality of the equation (1), is equivalent to the non-integrability of equation (2) by means of Liouvillian functions.

In this article, we study the symmetries of differential equations (1) and (2) given by change of variable $(\lambda, \tau) \mapsto (\varphi(\lambda), \sigma(\tau))$ with $\sigma \in \mathrm{PSL}_2(\mathbb{C})$ and φ a local biholomorphism. The object that describes these symmetries is a

\mathcal{D} -groupoid defined by a third order equation on $\varphi(\lambda)$ that was first studied by Kummer in relation with hypergeometric functions [12]. It is given by the differential equation:

$$S_\lambda(\varphi(\lambda)) + \varphi'(\lambda)^2 R(\varphi(\lambda)) = R(\lambda) \quad (3)$$

whose solutions are stable by compositions. We refer to this object as the Kummer groupoid.

Our main result (Theorem 3.3) states that, if equation (2) has no Liouvillian solutions then the Kummer groupoid (3) is minimal in the sense of not having non-trivial sub- \mathcal{D} -groupoids of order greater than 0. We note here that the minimality of this \mathcal{D} -groupoid is, in principle, a weaker assumption than of the strong minimality of the definable set defined by the same equation (3), as not all definable subsets need to be sub- \mathcal{D} -groupoids. The criterion for the minimality of this \mathcal{D} -groupoid coincides with a recently discovered criterion for the strong minimality of (1). As a consequence, we obtain (see Proposition 3.6 numerals 2 and 4):

Theorem 1.1. *The Schwarz equation (1) is strongly minimal if and only if the Kummer groupoid (3) is minimal.*

Whether there exists a direct connection between the strong minimality of differential equations and the minimality of their symmetry \mathcal{D} -groupoids in a broader context remains unclear and poses an intriguing question.

In subsection 1.1 we explain our take on Schwarzian equations using the jet spaces and invariant connections. The interested reader may find additional information in [5, 3]. Subsection 1.2 contains some well known tools of Picard-Vessiot theory that we will use in the proof of our result, main sources are [11, 6, 19]. Section 2 is devoted to the definition of \mathcal{D} -groupoid and its linearization. Here we give a synthetic exposition with the purpose of just giving the necessary tools to the reader. The main reference is [15] and there is also a general presentation of the subject in [2]. Finally in section 3 we go into the exploration of Kummer's groupoid. The computation of the equation for Kummer's groupoid and its linearization is classical. We then proceed to prove some preliminary results, followed by the presentation of our main result.

1.1. The Schwarzian equation as a PSL_2 -connection

We consider an algebraic curve X as a suitable ramified covering of $\bar{\mathbb{C}}$ so that the algebraic function $R(\lambda) \in \mathbb{C}(X)$ is seen as a rational function on X . We also remove from X all the zeroes and poles of $d\lambda$ and poles of $R(\lambda)$ so that the vector field $\frac{d}{d\lambda}$ has neither zeroes nor poles on X and $R(\lambda)$ is a regular function on X .

We see equations (1) and (2) geometrically as foliations in the jet space. Thus, we consider $J = J_*^2(\bar{\mathbb{C}}, X)$ the variety of 2-jets of local biholomorphisms

from $\overline{\mathbb{C}}$ to X . An affine open subset of this jet space is $\mathbb{C} \times X \times \mathbb{C}^* \times \mathbb{C}$ where the 4-tuple $(\tau, \lambda, \lambda_\tau, \lambda_{\tau\tau})$ represents the order 2 development of a biholomorphism sending τ to λ with derivatives λ_τ , and $\lambda_{\tau\tau}$. Differential equation (1) is seen geometrically as the equations of the integral curves of the foliation $\mathcal{F} = \langle D_\tau \rangle$ generated by the vector field D_τ in the jet space J :

$$D_\tau = \frac{\partial}{\partial \tau} + \lambda_\tau \frac{\partial}{\partial \lambda} + \lambda_{\tau\tau} \frac{\partial}{\partial \lambda_\tau} + \left(\frac{3}{2} \frac{\lambda_{\tau\tau}^2}{\lambda_\tau} + \lambda_\tau^3 R(\lambda) \right) \frac{\partial}{\partial \lambda_{\tau\tau}}.$$

The integral curves of \mathcal{F} are the graphs of the 2-jet prolongations

$$j^2 \hat{\lambda}: \overline{\mathbb{C}} \dashrightarrow J, \quad \tau \mapsto (\tau, \hat{\lambda}(\tau), \hat{\lambda}_\tau(\tau), \hat{\lambda}_{\tau\tau}(\tau))$$

of solutions $\hat{\lambda}(\tau)$ of (1).

On the other hand, note that the inversion of biholomorphisms gives an isomorphism $J \simeq J_*^2(X, \overline{\mathbb{C}})$ that corresponds just to the interchange between the dependent and independent variable λ and τ . We consider this isomorphism just as change of coordinates in J giving rise to new coordinates $\lambda, \tau, \tau_\lambda, \tau_{\lambda\lambda}$ for which the same foliation \mathcal{F} represents equation (2).

A projective structure on X is a maximal atlas $\{(\mathcal{U}_i, \tau_i)\}_{i \in I}$ of coordinates $\tau_i: X \rightarrow \overline{\mathbb{C}}$ with the property that transition functions in $\overline{\mathbb{C}}$ are elements of $\mathrm{PSL}_2(\mathbb{C})$. An important feature about equation (2) is that it defines a projective structure on X . Note that, if $\tau(\lambda)$ is a solution of (2) then any other solution with the same domain of definition is of the form $g(\tau(\lambda))$ for some $g \in \mathrm{PSL}_2(\mathbb{C})$.

This structure of the solution space of (2) is reflected in the jet space. The action of PSL_2 on $\overline{\mathbb{C}}$ lifts to an action on $J \simeq J_*^2(X, \overline{\mathbb{C}})$. The natural projection $\pi: J \rightarrow X$ is a principal bundle with structure group PSL_2 . The foliation $\langle D_\tau \rangle$ turns out to be a PSL_2 -invariant connection. Therefore, the theory of strongly normal extensions can be applied to the equation (2); its Galois group will be an algebraic subgroup of PSL_2 . Indeed, there is a well known explicit relation between the linear-Schwarzian equation (2) and the second order linear differential equation,

$$\frac{d^2 \psi}{d\lambda^2} = -\frac{1}{2} R(\lambda) \psi. \quad (4)$$

Namely, the quotient $\tau = \psi_1/\psi_2$ between any two linearly independent solutions ψ_1, ψ_2 of (4) is a solution of (2). This relation can be seen geometrically as an equivariant 2-cover of principal bundles,

$$\mathrm{SL}_2(\mathbb{C}) \times X \rightarrow J$$

$$\left(\begin{bmatrix} a & b \\ c & e \end{bmatrix}, \lambda \right) \mapsto j_\lambda^2 \left(\frac{a\tau + b}{c\tau + e} \right)$$

that maps the companion system

$$\frac{d}{d\lambda} \begin{bmatrix} \psi_1 & \psi_2 \\ \psi'_1 & \psi'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} R(\lambda) & 0 \end{bmatrix} \begin{bmatrix} \psi_1 & \psi_2 \\ \psi'_1 & \psi'_2 \end{bmatrix} \quad (5)$$

of equation (4) to the linear-Schwarzian equation (2). Note that this map is well defined except on the singularities of $d\lambda$ that we already removed from our algebraic curve X . In [3] and [5] it is shown that the equation (1) is strongly minimal if and only if the Galois group of the equation (4) is exactly SL_2 . This last condition is equivalent to the non-integrability of either (2) or (4) by means of Liouvillian functions.

1.2. Some relevant facts of Picard-Vessiot theory

Let us discuss here some facts around the integrability of equations (2) and (4) in the context of Picard-Vessiot theory, which is the part of differential Galois theory that deals with linear differential equations. The interested reader may consult section 7.3 of [6] or [11] for the original source. There is also a slightly more detailed exposition of these facts in [3]. In our case the field of coefficients is $\mathcal{K} = \mathbb{C}(X)$ endowed with the derivation $\frac{d}{d\lambda}$. All the facts explained in this section hold for any differential field of characteristic zero with algebraically closed field of constants.

Let us consider the differential field extension $\mathcal{K} \subseteq \mathcal{K}\langle\psi_1, \psi_2\rangle$ spanned by two linearly independent solutions of (4). The differential Galois group of equation (4) is the group of differential field automorphisms $G = \mathrm{Aut}(\mathcal{K}\langle\psi_1, \psi_2\rangle/\mathcal{K})$. This group is naturally represented as an algebraic subgroup of $\mathrm{SL}_2(\mathbb{C})$. For each $\sigma \in G$ we have,

$$\begin{bmatrix} \sigma(\psi_1) & \sigma(\psi_2) \end{bmatrix} = \begin{bmatrix} \psi_1 & \psi_2 \end{bmatrix} \cdot \begin{bmatrix} a_\sigma & c_\sigma \\ b_\sigma & e_\sigma \end{bmatrix} \quad (6)$$

The fundamental theory of integrability by Liouvillian functions in differential Galois theory states that a linear differential equation such as (2) is integrable by means of Liouvillian functions if and only if the Lie algebra of G is solvable. On the other hand, all proper subgroups of $\mathrm{SL}_2(\mathbb{C})$ have solvable Lie algebra. Eigenvectors of the action of Lie algebra of G on the vector space of solutions are related with algebraic solutions of the auxiliary Riccati equation,

$$u' + u^2 + \frac{1}{2}R(\lambda) = 0 \quad (7)$$

satisfied by $u = \frac{d \log \psi}{d\lambda}$, the logarithmic derivative of a solution of (4). The following proposition accounts for the preliminary considerations before Kovacic's algorithm in [11] and is a well-known fact in the context of Picard-Vessiot theory.

Proposition 1.2. *Let us consider the differential equation (4). There are the following four mutually exclusive possibilities for the solutions:*

(Case 1) *The Riccati equation (7) has at least a solution $u \in \mathcal{K}$. $\psi = e^{\int u}$ is a Liouvillian solution of (4) and the Galois group G is conjugated to a group of triangular matrices.*

(Case 2) *The Ricatti equation (7) has a pair of conjugated solutions u_{\pm} that are algebraic of degree 2 over \mathcal{K} . $\psi_{\pm} = e^{\int u_{\pm}}$, are algebraically independent Liouvillian solutions of (4) and the Galois group G is conjugated to a subgroup of the infinite dihedral group.*

(Case 3) *All solutions of (4) are algebraic over \mathcal{K} and the Galois group G is conjugated to a finite crystallographic group.*

(Case 4) *Equation (4) has no Liouvillian solution and the Galois group is $G = \mathrm{SL}_2(\mathbb{C})$.*

Let us consider now $\tau = \psi_1/\psi_2$, which is a solution of (2), and the tower of differential fields,

$$\mathcal{K} \subseteq \mathcal{K}\langle\tau\rangle \subseteq K\langle\psi_1, \psi_2\rangle.$$

From equation (6) we have that,

$$\sigma(\tau) = \frac{a_{\sigma}\tau + b_{\sigma}}{c_{\sigma}\tau + d_{\sigma}} \in \mathbb{C}\langle\tau\rangle$$

It follows that $\mathcal{K} \subset \mathcal{K}\langle\tau\rangle$ is a Picard-Vessiot extension whose differential Galois group is the quotient of G by the stabilizer of τ ,

$$\mathrm{Aut}(\mathcal{K}\langle\tau\rangle/\mathcal{K}) = \overline{G} = G/(G \cap \{I, -I\}).$$

Note that cases 1, 2, 3 of Proposition 1.2 will lead us to a Liouvillian extension $\mathcal{K} \subseteq \mathcal{K}\langle\tau\rangle$ and \overline{G} a proper subgroup of $\mathrm{PSL}_2(\mathbb{C})$ and case 4 implies the non-existence of Liouvillian solutions for (2) and $\overline{G} = \mathrm{PSL}_2(\mathbb{C})$.

2. \mathcal{D} -Groupoids

The notion of \mathcal{D} -groupoid was introduced by B. Malgrange in [14] in the context of non-linear differential Galois theory. We may also call then algebraic Lie pseudogroups, as they are systems of algebraic differential equations whose solutions form a Lie pseudogroup. We refer the reader to [14, 15] and references therein for the notion of pseudogroup and Lie pseudogroup. The original proposal allowed differential equations that were analytic in the base and algebraic on the derivatives, but later formulations restricted the definition to algebraic differential equations. Here, we will give a definition that is equivalent to definition 5.2 in [15] (Definition 2.2); the equivalence between these definitions can be found in appendix A of [2].

From this part on, in order to make the notation simple the derivative symbol f' will be used with the meaning of the derivative with respect to λ .

2.1. Jets of biholomorphisms

The jet space $J^k(X, X)$ is defined as the set of equivalence classes of contact of order $\geq k$ at points of X of local biholomorphisms from open subsets of X to open subsets of X .¹ This space $J^k(X, X)$ has a natural structure of algebraic variety (see, for instance [2]), and it contains the open subset of jets of biholomorphism:

$$\text{Aut}_k(X) := J_*^k(X, X)$$

Jets of biholomorphisms may be composed and inverted taking into account their sources and targets, so that $\text{Aut}_k(X)$ is an algebraic groupoid over X ; the groupoid of k -jets of invertible local biholomorphisms on X .

In order to clarify the algebraic structure of $\text{Aut}_k(X)$ let us examine the case in which $X \subseteq \mathbb{C}$ is an affine subset of the complex numbers, with coordinate λ . The general case is recovered by gluing coverings of that case. A $(k+2)$ -tuple $(\lambda_0, \varphi_0, \varphi'_0, \dots, \varphi_0^{(k)})$ corresponds to the k -jet in λ_0 of any biholomorphism of the form:

$$\varphi : \lambda \mapsto \varphi_0 + \varphi'_0(\lambda - \lambda_0) + \frac{\varphi''_0}{2}(\lambda - \lambda_0)^2 + \dots + \frac{\varphi_0^{(k)}}{k!}(\lambda - \lambda_0)^k + O(\lambda - \lambda_0)^{k+1}.$$

From now on we do not mention the subindex zero, so $\lambda, \varphi, \varphi', \dots, \varphi^{(k)}$ is a system of coordinates in $\text{Aut}_k(X) \simeq X \times X \times \mathbb{C}^* \times \mathbb{C}^{k-1}$. The same direct product decomposition is possible for an affine algebraic curve X , providing that the vector field $\frac{d}{d\lambda}$ has neither zeros nor poles in X . The composition law in $\text{Aut}_k(X)$ is given by Faà di Bruno formula², and thus, it is polynomial in the coordinates of $\text{Aut}_k(X)$ which turns out to be an algebraic groupoid over X . Truncations $\text{Aut}_k(X) \rightarrow \text{Aut}_{k-1}(X)$ are compatible with composition and inversion. Hence, the projective limit $\text{Aut}(X) := \varprojlim \text{Aut}_k(X)$ inherits the groupoid structure. Elements of $\text{Aut}(X)$ are formal local biholomorphisms, not necessarily convergent.

Definition 2.1. The following types of algebraic subvarieties of $\text{Aut}_k(X)$ will be considered.

- (1) A strict algebraic subgroupoid of $\text{Aut}_k(X)$ is a Zariski closed subset $Z \subset \text{Aut}_k(X)$ which is also a smooth subgroupoid.
- (2) A rational subgroupoid $\mathcal{G}_k \subset \text{Aut}_k(X)$ is a Zariski closed subset such that on an open $U \subset X$, $\mathcal{G}_k|_U$ is dense in \mathcal{G}_k and is a strict algebraic subgroupoid of $\text{Aut}_k(U)$.

¹To be more precise, in general, taking a $\pi : E \rightarrow M$ a submersion between varieties; φ_1, φ_2 local sections defined around $p \in M$ have contact of order $\geq k$ if $d\varphi_1$, and $d\varphi_2$ have contact of order $\geq k-1$, that is, $\varphi_1(p) = \varphi_2(p), \dots, d_p(d^{k-1}\varphi_1) = d_p(d^{k-1}\varphi_2)$. A local holomorphism is seen as a local section of the trivial bundle $\pi_1 : X \times X \rightarrow X$.

²Faà di Bruno formula gives explicit expressions for the higher order derivatives of the composition of functions. See, for instance [20].

2.2. Zariski Topology of $\text{Aut}(X)$

Let us consider the case in which X is an affine curve and the vector field $\frac{d}{d\lambda}$ has neither zeroes nor poles, so that $\text{Aut}_k(X) \simeq X \times X \times \mathbb{C}^* \times \mathbb{C}^{k-1}$ is an affine algebraic manifold with ring of regular functions $\mathcal{O}_{\text{Aut}_k(X)}$. Taking limit $k \rightarrow \infty$ we obtain the proalgebraic variety $\text{Aut}(X)$ with ring of regular functions,

$$\mathcal{O}_{\text{Aut}(X)} = \bigcup_k \mathcal{O}_{\text{Aut}_k(X)}.$$

The general case is similar but instead of ideals of the ring of regular functions $\mathcal{O}_{\text{Aut}_k(X)}$ we would be forced to use coherent sheaves of ideals of its structural sheaf.

A *Zariski closed* subset Z of $\text{Aut}(X)$ is defined by a radical ideal $\mathcal{E} \subset \mathcal{O}_{\text{Aut}(X)}$. The k -order truncation of Z is the Zariski closed Z_k of $\text{Aut}_k(X)$, defined by the ideal $\mathcal{E}_k = \mathcal{E} \cap \mathcal{O}_{\text{Aut}_k(X)}$, $Z_k := V(\mathcal{E}_k)$. In this way, Z can be viewed as a sequence of closed Zariski sets $\{Z_k\}_{k \in \mathbb{N}}$ such that each projection $Z_k \rightarrow Z_{k-1}$ is dominant, that means

$$\cdots \longrightarrow Z_k \longrightarrow Z_{k-1} \longrightarrow \cdots \longrightarrow Z_0 \subset X \times X$$

is a dominant sequence.

2.3. Kolchin Topology and definition of \mathcal{D} -groupoid

The total derivative is a canonical way of extending derivations in \mathcal{O}_X to derivations of $\mathcal{O}_{\text{Aut}(X)}$. Given a vector field \vec{w} in X , we define its total derivative,

$$\vec{w}^{tot}: \mathcal{O}_{\text{Aut}_k(X)} \rightarrow \mathcal{O}_{\text{Aut}_{k+1}(X)}, \quad (\vec{w}^{tot} f)(j_x^{k+1} \varphi) = \vec{w}_x(f \circ j^k \varphi).$$

Let us recall that $\frac{d}{d\lambda}$ is a vector field without zeros nor poles in X . Through the total derivation mechanism it extends to a derivation of $\mathcal{O}_{\text{Aut}(X)}$ that we denote by the same symbol (without the *tot* superscript), so that $\mathcal{O}_{\text{Aut}(X)}$, endowed with

$$\frac{d}{d\lambda} = \frac{\partial}{\partial \lambda} + \varphi' \frac{\partial}{\partial \varphi} + \varphi'' \frac{\partial}{\partial \varphi'} + \cdots$$

is a differential ring.

An ideal \mathcal{I} of $\mathcal{O}_{\text{Aut}(X)}$ is an \mathcal{D} -ideal if for every differential function f in \mathcal{I} the total derivatives of f are also in \mathcal{I} . A subset $Y \subset \text{Aut}(X)$ is *Kolchin closed* if it is a Zariski closed, whose ideal is a \mathcal{D} -ideal. That is, a closed set in Kolchin topology is given by the zeros of a radical \mathcal{D} -ideal.

Definition 2.2. A \mathcal{D} -groupoid $\mathcal{G} = \{\mathcal{G}_k\}_{k \in \mathbb{N}} \subset \text{Aut}(X)$ is a Kolchin closed set such that for all k , \mathcal{G}_k is a rational subgroupoid. The smaller k such that $\mathcal{G}_k \subsetneq \text{Aut}_k(X)$ is called the order of \mathcal{G} .

- a) Given a \mathcal{D} -groupoid \mathcal{G} then there is an open set $\mathcal{U} \subset X$ such that $\mathcal{G}|_{\mathcal{U}}$ is a groupoid over \mathcal{U} , see [2] appendix A.
- b) Solutions of \mathcal{G} , meaning, local biholomorphisms $f: X \dashrightarrow X^3$ such that for all x in its domain $j_x f \in \mathcal{G}$ form a pseudogroup of transformations of X .
- c) In the general setting \mathcal{G} is defined by a system of algebraic PDE with as many independent variables and unknowns as the dimension of the base variety. As X is one-dimensional then \mathcal{G} is defined by ODE. Let us consider \mathcal{G} of order k . As \mathcal{G}_k dominates $\text{Aut}_{k-1}(X)$ it must be an hypersurface of $\text{Aut}_k(X)$. As $\text{Aut}_k(X)$ is affine then the ideal of \mathcal{G}_k is spanned by a single element $F(\lambda, \varphi, \dots, \varphi^{(k)}) \in \mathcal{O}_{\text{Aut}_k(X)}$. From the differential equation $F = 0$ we deduce,

$$\frac{d}{d\lambda} F = \varphi^{(k+1)} \frac{\partial F}{\partial \varphi^{(k)}} + Q = 0; \quad \varphi^{(k+1)} = -\frac{Q}{F_{\varphi^{(k)}}}$$

a $(k+1)$ -order differential equation where the $(k+1)$ -th derivative is expressed as a rational function of the lower degree derivatives, and same for higher order derivatives. Summarizing, a \mathcal{D} -groupoid of order k on an affine algebraic curve X with a non vanishing vector field is always determined by a single k -th order differential equation.

2.4. \mathcal{D} -algebra of a \mathcal{D} -groupoid

Consider the tangent bundle $TX \rightarrow X$, and define $\text{aut}_k(X) = J^k(TX/X)$ as the bundle of k -jets of sections of the tangent bundle. The vector bundle $\text{aut}_k(X) \rightarrow X$ with its anchor $\text{aut}_k(X) \rightarrow \text{aut}_0(X) = TX$ is the Lie algebroid of $\text{Aut}_k(X)$. Let us explore how elements of $\text{aut}_k(X)$ are identified with vectors tangent to the identity in $\text{Aut}_k(X)$. For this, consider $\vec{w} = f(\lambda) \frac{\partial}{\partial \lambda}$, $j_x^k \vec{w}^{(k)} \in \text{aut}_k(X)$ so that:

$$f(\lambda) = f(x) + f'(x)(\lambda - \lambda(x)) + f''(x) \frac{(\lambda - \lambda(x))^2}{2} + \dots$$

For ε varying in \mathbb{C} , $j_x^k(\varepsilon \vec{w})$ is a curve on $\text{aut}_k(X)$. If ε is sufficiently small, The existence theorem for ordinary differential equations implies that the exponential

$$\exp(\varepsilon \vec{w}): (X, x) \rightarrow (X, y(\varepsilon))$$

is an analytic map, where $y(\varepsilon) = \exp(\varepsilon \vec{w})x$. Thus, by taking the k -jet of exponential $j_x^k \exp(\varepsilon \vec{w})$ as a curve in $\text{Aut}_k(X)$,

$$\Phi_k: \text{aut}_k(X) \hookrightarrow T(\text{Aut}_k(X)), \quad \vec{w} \mapsto \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} j_x^k(\exp(\varepsilon \vec{w})),$$

³The dash arrow is used to denote maps defined on a open subset of X .

so that each element of $\text{aut}_k(X)$ is seen as a vector tangent to the identity in $\text{Aut}_k(X)$ preserving the source map. Taking the projective limit $k \rightarrow \infty$ we obtain,

$$\Phi: \text{aut}(X) \hookrightarrow T(\text{Aut}(X))$$

whose expression in coordinates is

$$j_x \left(f(\lambda) \frac{\partial}{\partial \lambda} \right) \mapsto f(x) \frac{\partial}{\partial \varphi} + f'(x) \frac{\partial}{\partial \varphi_\lambda} + f''(x) \frac{\partial}{\partial \varphi_{\lambda\lambda}} + \dots$$

Now, we are studying certain \mathcal{D} -groupoid $\mathcal{G} \subset \text{Aut}(X)$. We may ask for which germs vector fields $j_x \vec{w} \in \text{aut}(X)$ have exponential $j_x(\varepsilon \vec{w})$ in \mathcal{G} . This happens if $\Phi(j_x \vec{w})$ is a vector tangent to \mathcal{G} . This is the problem of linearizing the groupoid \mathcal{G} .

Consider some differential equation

$$F(\lambda, \varphi, \varphi', \varphi'', \dots, \varphi^{(k)}) = 0$$

which vanish on the \mathcal{G} , and here $F \in \mathcal{O}_{\text{Aut}(X)}$. We linearize F by taking,

$$\ell F \left(j_x^k f(\lambda) \frac{\partial}{\partial \lambda} \right) = dF \left(\Phi_k \left(j_x^k f(\lambda) \frac{\partial}{\partial \lambda} \right) \right)$$

so that $\ell F \in \mathcal{O}_{\text{aut}(X)}$ is in fact a linear form in the coefficients of the power series development of $f(\lambda)$ and thus a linear differential equation for the unknown f .

We can now define $\text{Lie}(\mathcal{G}_k) \subset \text{aut}_k(X)$ as the rational linear bundle defined by all the linearizations ℓF of functions vanishing on \mathcal{G}_k , and the \mathcal{D} -Lie algebra of \mathcal{G} as the projective limit of this system $\text{Lie}(\mathcal{G}) \subset \text{aut}(X)$. It is a system of linear differential equations whose solutions are vector fields in X . Without discussing the structure of these objects, let us take note of the two following relevant facts:

- a) The Lie bracket of two solutions of $\text{Lie}(\mathcal{G})$, when defined, is also a solution of $\text{Lie}(\mathcal{G})$.
- b) If \mathcal{G} is determined by single differential equation of order k then $\text{Lie}(\mathcal{G})$ is determined by a single linear differential equation of order k .

More theoretical results on \mathcal{D} -Lie algebras in relation with \mathcal{D} -Lie groupoid can be found in [15].

3. Kummer's groupoid

Let us consider $\varphi: X \dashrightarrow X$ a local biholomorphism. We want to check if σ is compatible with the projective structure induced by equation (2) in the sense that composition with σ sends local projective charts on local projective charts

of the same structure. In other words, for each solution τ of (2) the composition $\tau \circ \sigma$ is also a solution. From the chain rule for the Schwarzian derivative we obtain:

$$S_\lambda(\tau \circ \varphi) = (S_\lambda(\tau) \circ \varphi)\varphi_\lambda^2 + S_\lambda(\varphi).$$

Now, from equation (2) after composition with φ we obtain $S_\lambda(\tau) \circ \varphi = R(\varphi)$. Finally we obtain a differential equation for φ as function of λ :

$$S_\lambda(\varphi) = R(\lambda) - R(\varphi)\varphi_\lambda^2. \quad (8)$$

The above differential equation characterizes the symmetries of (1) and then it defines a \mathcal{D} -groupoid of X which is the Kolchin closed subset $\mathcal{G} \subset \text{Aut}(X)$ determined by the radical \mathcal{D} -ideal generated by $S_\lambda(\varphi) - R(\lambda) + R(\varphi)\varphi_\lambda^2$. As this equation was first presented by Kummer in [12], we refer to \mathcal{G} as Kummer's groupoid. This equation also appear in various different works during the last century: in Ritt's work on hypertranscendancy of Koenings's linearisations [18], in Ecalle synthesis of binary parabolic diffeomorphisms [8], or in the classification of rational transformations of \mathbb{CP}_1 preserving an rational geometric structure [4].

In $\text{Aut}_3(X)$, the rational subgroupoid \mathcal{G}_3 consisting of 3-jets of solutions of the Kummer equation is:

$$\mathcal{G}_3 := \{j_\lambda^3 \varphi | \varphi: \lambda \mapsto \varphi(\lambda), R(\varphi)\varphi_\lambda^2 + S_\lambda(\varphi) = R(\lambda)\}.$$

It corresponds to the 3-jets of local biholomorphisms of X that are compatible with the projective structure. Note that \mathcal{G} is defined by a single differential equation of order 3, which is seen as a function on $\text{Aut}_3(X)$. By applying total derivative with respect to λ we obtain that all derivatives of order higher than two can be written as a function of $\lambda, \varphi, \varphi_\lambda, \varphi_{\lambda\lambda}$ and therefore:

$$\mathcal{G} \simeq \mathcal{G}_k \simeq \dots \mathcal{G}_3 \simeq \mathcal{G}_2 = \text{Aut}_2(X).$$

as algebraic varieties. Hence \mathcal{G} is an algebraic Lie groupoid of complex dimension 4 and it is isomorphic, as Lie groupoid, to $\text{Aut}_2(X)$.

3.1. Linearization of the Kummer's equation

Let us write the Kummer equation (8)

$$R(\varphi)\varphi'^2 + S - R(\lambda) = 0$$

where the Schwarzian derivative is now seen as a function of the coordinates in $\text{Aut}_3(X)$

$$S(\lambda, \varphi, \varphi', \varphi'', \varphi''') = \frac{\varphi'''}{\varphi'} - \frac{3}{2} \left(\frac{\varphi''}{\varphi'} \right)^2.$$

Let us consider $\vec{w} = f(\lambda) \frac{\partial}{\partial \lambda}$. We obtain by direct computation:

$$\begin{aligned} \frac{\partial S}{\partial \varphi} &= 0; \\ \frac{\partial S}{\partial \varphi'} &= -\frac{\varphi'''}{\varphi'^2} + 2\frac{\varphi''}{\varphi'} \frac{\varphi''}{\varphi'^2} \equiv 0 \quad (\text{along } \mathcal{G}); \\ \frac{\partial S}{\partial \varphi''} &= -3\frac{\varphi''}{\varphi'^2} \equiv 0 \quad (\text{along } \mathcal{G}); \\ \frac{\partial S}{\partial \varphi'''} &= \frac{1}{\varphi'} \equiv 1 \quad (\text{along } \mathcal{G}); \\ \Phi(j_x \vec{w})S &= f'''(x); \\ \Phi(j_x \vec{w})R(\lambda) &= 0; \\ \Phi(j_x \vec{w})R(\varphi)\varphi'^2 &= f(x)R'(\lambda(x))\lambda'^2 + 2R(\lambda(x))f'(x)\lambda' \\ &\equiv f(x)R'(\lambda) + 2R(\lambda(x))f'(x) \quad (\text{along } \mathcal{G}). \end{aligned}$$

And thus, the linearized differential equation is

$$f''' + 2R(\lambda)f' + R'(\lambda)f = 0. \quad (9)$$

It provides the necessary and sufficient conditions for the vector field $f(\lambda) \frac{\partial}{\partial \lambda}$ to be tangent to \mathcal{G} . This linear differential equation, defines a linear and closed Kolchin sub-bundle $\text{Lie}(\mathcal{G})$ of $\text{aut}(X)$, known as *\mathcal{D} -Lie algebra of the Kummer's groupoid \mathcal{G}* .

3.2. Symmetric Power

In this section we show the relation between the equation (9) of $\text{Lie}(\mathcal{G})$ and the Riccati equation (7). For this purpose, we need to introduce the symmetric power of the second order linear differential equation (4). As before, let \mathcal{K} the differential field consisting of $\mathbb{C}(X)$ endowed with $\frac{d}{d\lambda}$, and let ψ_1, ψ_2 be a pair of linearly independent solutions, so that we have the following extensions of differential fields.

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\quad\quad\quad} & \mathcal{K}\langle\psi_1, \psi_2\rangle \\ & \searrow \quad \quad \swarrow & \\ & \mathcal{K}\langle\psi_1\psi_2, \psi_1^2, \psi_2^2\rangle & \end{array}$$

The three functions $\psi_i\psi_j$ for $i, j = 1, 2$ generate the solution space of a third order equation, which is known as the *second symmetric power*. To derive it,

consider $f = \psi^2$, where $\psi = a\psi_1 + b\psi_2$ is any solution of (4), we have:

$$\begin{aligned}
 f' &= 2\psi\psi' \\
 f'' &= 2\psi'^2 + 2\psi\psi'' = 2\psi'^2 - R(\lambda)a \\
 f''' &= 4\psi'\psi'' - R'(\lambda)a - R(\lambda)a' \\
 &= 4\psi' \left(-\frac{1}{2}R(\lambda)\psi \right) - R'(\lambda)a - R(\lambda)a' \\
 &= -2\psi\psi'R(\lambda) - R'(\lambda)a - R(\lambda)a' \\
 &= -2R(\lambda)a' - R'(\lambda)a
 \end{aligned}$$

So, rewriting

$$f''' + 2R(\lambda)f' + R'(\lambda)f = 0. \quad (10)$$

Which is the same third order differential equation that (9). In this way, we say that the third order linear differential equation (9) is the second symmetric power of the second order linear equation (4). From the above diagram, we have that $\mathcal{K} \subset \mathcal{K}\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle$ is the Picard-Vessiot extension of the third order equation (10).

Proposition 3.1. *The Galois groups $\text{Aut}(\mathcal{K}\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle/\mathcal{K})$ and $\text{Aut}(\mathcal{K}\langle\psi_1, \psi_2\rangle/\mathcal{K})$ have isomorphic Lie algebras.*

Proof. Let us examine the diagram of extensions of differential fields, with $G = \text{Aut}(\mathcal{K}\langle\psi_1, \psi_2\rangle/\mathcal{K})$, $H = \text{Aut}(\mathcal{K}\langle\psi_1, \psi_2\rangle/\mathcal{K}\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle)$ so that, by Galois correspondence we have that $\text{Aut}(\mathcal{K}\langle\psi_1, \psi_2\rangle/\mathcal{K}) \simeq G/H$.

$$\begin{array}{ccc}
 K & \xrightarrow{G} & K\langle\psi_1, \psi_2\rangle \\
 & \searrow^{G/H} & \nearrow H \\
 & K\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle &
 \end{array}$$

It suffices to observe that the second inclusion is either the identity or an algebraic extension of finite degree two. By the Galois correspondence, the Galois group of the small extension is a quotient of that of the large extension by a finite normal subgroup and therefore has the same Lie algebra. \square

Proposition 3.2. *The following statements are equivalent:*

- (a) $\text{Aut}(\mathcal{K}\langle\psi_1, \psi_2\rangle/\mathcal{K}) = \text{SL}_2$.
- (b) $\text{Aut}(\mathcal{K}\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle/\mathcal{K}) = \text{PSL}_2$.
- (c) *The Riccati equation (7) has no algebraic solution over \mathcal{K} .*

Proof. From the Galois correspondence [19, Proposition 1.34 p. 25] applied to the above diagram (a) \iff (b), since PSL_2 is the only possible quotient of SL_2 by a finite group. The equivalence (a) \iff (c) is consequence of Proposition (1.2) (see [19, Exercise 1.36 page 28]). \square

We can now prove the main result of this article.

Theorem 3.3. *If the Riccati equation (7) has no algebraic solutions, then the Kummer \mathcal{D} -groupoid \mathcal{G} has no sub- \mathcal{D} -groupoids of order greater than 0.*

Proof. Assume that $\mathcal{H} \subseteq \mathcal{G}$ is a sub- \mathcal{D} -groupoid of order greater than 0. Then the order of \mathcal{H} is 1, 2 or 3. Let us examine those cases separately.

- a) Let us assume that \mathcal{H} is of order 3. The equation (8) of \mathcal{G} allows to express σ''' as a rational function of $\lambda, \sigma, \sigma', \sigma''$. Therefore \mathcal{G}_3 is the graph of a section from $\mathrm{Aut}_2(X)$ to $\mathrm{Aut}_3(X)$. Therefore, it is an irreducible surface. As \mathcal{H} is also an hypersurface of $\mathrm{Aut}_3(X)$ and it is contained in \mathcal{G}_3 then we have $\mathcal{H}_3 = \mathcal{G}_3$ and then $\mathcal{H} = \mathcal{G}$.
- b) Let us assume that \mathcal{H} is of order 2. Then $\mathrm{Lie}(\mathcal{H})$ is determined by a second order differential equation with coefficients in $\mathcal{K} = \mathbb{C}(X)$.

$$f'' + \alpha(\lambda)f' + \beta(\lambda)f = 0 \quad (11)$$

As $\mathrm{Lie}(\mathcal{H}) \subset \mathrm{Lie}(\mathcal{G})$ then, the 3-dimensional solution space of the equation (9) contains two linearly independent solutions of (11). Note that this 3-dimensional space is spanned by ψ_1^2 , ψ_2^2 and $\psi_1\psi_2$ where ψ_1 and ψ_2 are linearly independent solutions of (4). Inside this 3-dimensional space, the set of elements of the form ψ^2 with ψ a solution of (4) form a cone. In a complex 3-dimensional vector space every plane intersects a cone on 2 lines or a double line. Thus, there is a solution $f = \psi^2$ of (11) which is a square of a solution of (4). Replacing and using the original equation,

$$(\beta(\lambda) - R(\lambda))\psi^2 + 2\alpha(\lambda)\psi\psi' + 2\psi'^2 = 0$$

- b.i) If $\beta(\lambda) = R(\lambda)$, we have $2\psi'(\psi\alpha(\lambda) - \psi') = 0$. In that case, we also have that either ψ is constant or $\psi'/\psi = \alpha(\lambda)$ is a solution of the Riccati equation (7). Both cases contradict the hypothesis.
- b.ii) If $\beta(\lambda) \neq R(\lambda)$, there are algebraic functions of degree one or two such that:

$$(\beta(\lambda) - R)(\psi - \gamma_1(\lambda)\psi')(\psi - \gamma_2(\lambda)\psi') = 0$$

It implies that ψ satisfies a linear first order equation with a coefficient in an algebraic extension of \mathcal{K} , so that it is a Liouvillian function, which contradicts the hypothesis (by proposition 1.2, there is no Liouvillian solutions of (4)).

- c) Let us assume that \mathcal{H} is of order 1. Then $\text{Lie}(\mathcal{H}_1)$ is determined by a linear differential equation of order one whose solution space is contained in the solution space of (9). Let us consider then a solutions ψ of this equation inside the Picard-Vessiot extension $\mathcal{K}\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle$ of equation (9). We have a tower of Picard-Vessiot extensions,

$$\mathcal{K} \subseteq \mathcal{K}\langle f \rangle \subset \mathcal{K}\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle.$$

As $\mathcal{K} \subseteq \mathcal{K}\langle f \rangle$ is Picard-Vessiot then the group of automorphisms of $\mathcal{K}\langle\psi_1^2, \psi_2^2, \psi_1\psi_2\rangle$ fixing $\mathcal{K}\langle f \rangle$ is a normal subgroup of $\text{PSL}_2(\mathbb{C})$. As $\text{PSL}_2(\mathbb{C})$ it can be either the identity or $\text{PSL}_2(\mathbb{C})$.

- c.i) If it is the identity, then by the Galois correspondence the group of automorphisms of $\mathcal{K}\langle f \rangle$ over \mathcal{K} should be isomorphic to $\text{PSL}_2(\mathbb{C})$. But this is impossible as the dimension of the Galois group, which is 3, must coincide with the transcendence degree of the extension, which is 1.
- c.ii) If it is $\text{PSL}_2(\mathbb{C})$, then by Galois correspondence $\psi \in \mathcal{K}$, but f is a non-vanishing solution of (9) that does not have Liouvillian solutions.

Therefore \mathcal{G} does not have any sub- \mathcal{D} -groupoid of order greater than 0. \checkmark

3.3. Strong Minimality

The notion of strong minimality comes from model theory and in the particular case of differential equations generalizes and unifies some classical notions of irreducibility. We say that an algebraic differential equation of third order (1) is *strongly minimal* if and only if for every differential field \mathcal{L} (of which it is assumed that it includes the algebraic numbers) and every solution j , the transcendence degree $\text{tr.deg.}_{\mathcal{L}} \mathcal{L}\langle j \rangle$ is necessarily 0 or 3. That means no solution can satisfy a lower order equation unless it is an algebraic solution. A differential field \mathcal{L} for which there exists a solution j of (1) such that the transcendence degree of $\mathcal{L}\langle j \rangle$ over \mathcal{L} is 1 or 2 is called a *witness of non-strong minimality*.

The proof of strong minimality of the Schwarzian equation (1) with $R(\lambda) \in \mathbb{C}(\lambda)$ can be found in [5].

Proposition 3.4 (cf. [5] Th. 3.2). *Assume $R(\lambda) \in \mathbb{C}(\lambda)$. If the Riccati equation (7) has no algebraic solutions, then the equation (1) is strongly minimal.*

Furthermore, this criterion is a necessary and sufficient condition.

Proposition 3.5. *If the Riccati equation (7) has some algebraic solution, then the non-linear Schwarzian equation (1) is not strongly minimal.*

Proof. Suppose that the Riccati equation (7) has an algebraic solution $u(\lambda)$ in $\mathbb{C}(\lambda)^{\text{alg}}$. In that case, automatically, some solutions of the Schwarzian equation satisfy as well:

$$\frac{\lambda_{\tau\tau}}{(\lambda_{\tau})^2} = 2u(\lambda)$$

which is a second-order equation. Hence, $\mathbb{C}(\lambda)^{\text{alg}}$ witnesses the non-strong minimality of the equation. \square

Putting together these results with our main Theorem 3.3 and Proposition 1.2 we obtain the following.

Proposition 3.6. *Assume $R(\lambda) \in \mathbb{C}(\lambda)$. The following statements are equivalent:*

- (1) *Riccati equation (7) has no algebraic solutions.*
- (2) *Kummer groupoid \mathcal{G} , defined by the equation (8), has no proper sub- \mathcal{D} -groupoids of order greater than 0.*
- (3) *The linear equation (4) has Galois group $\text{SL}_2(\mathbb{C})$ over $\mathbb{C}(X)$.*
- (4) *The Schwarzian equation (1) is strongly minimal.*
- (5) *The linear-Schwarzian equation (2) has no Liouvillian solutions.*

Proof. The only point we still need to check is the following. If there is an algebraic solution $u \in \mathbb{C}(X)^{\text{alg}}$ of the Riccati equation (7), then \mathcal{G} has some non trivial pseudogroup of order greater than zero. In order to clarify this point, let us see how solutions of the Riccati equation (7) allow us to reduce the linear-Schwarzian equation (2). If $u(\lambda)$ is a solution of the Riccati equation, then let us look for a solution τ of the differential equation:

$$\frac{\tau_{\lambda\lambda}}{\tau_{\lambda}} = -2u(\lambda) \quad (12)$$

We compute the Schwarzian derivative of τ obtaining:

$$S_{\lambda}(\tau) = (-2u(\lambda))_{\tau} - \frac{1}{2}(-2u(\lambda))^2 = -2(u'(\lambda) + u(\lambda)^2) = R(\lambda),$$

so that any solution of (12) is also a solution of (2). Two any solutions of (12) with a common domain of definition are related by an affine transformation of $\overline{\mathbb{C}}$. Thus, (12) is the differential equation of an affine structure inside the projective structure determined by (2). The inverses of the affine charts satisfy the differential equation,

$$\frac{\lambda_{\tau\tau}}{\lambda_{\tau}^2} = 2u(\lambda). \quad (13)$$

If we look for symmetries φ , such that, for any solution λ of (13) the composition $\varphi \circ \lambda$ is also a solution, we arrive to the differential equation

$$2u(\varphi)\varphi_\lambda = 2u(\lambda) + \frac{\varphi_{\lambda\lambda}}{\varphi_\lambda}. \quad (14)$$

As u is, in general, a transcendent function, equation (14) does not define an Zariski closed subset of $\text{Aut}_2(X)$. However, if $u \in \mathbb{C}(X)^{\text{alg}}$ then equation (14) defines a \mathcal{D} -groupoid of order 2 contained in \mathcal{G} . \square

It is interesting how the criterion of strong-minimality for (1) coincides also with the minimality of the \mathcal{D} -groupoid of its symmetries. It remains unclear to us whether there is a direct relation between the minimality of the \mathcal{D} -groupoid and the strong minimality of the equation. It would be useful to know if the \mathcal{D} -groupoid can be used as a tool to detect strong minimality for some other differential equations.

Acknowledgments. We wish to express our gratitude to the anonymous referee, whose valuable suggestions and corrections significantly enhanced the quality of this paper. G. Casale is partially funded by CAPES-COFECUB project MA 932/19 and Centre Henri Lebesgue, program ANR-11-LABX-0020-0. D. Blázquez-Sanz and A. Arenas are partially funded by UNAL sede Medellín - Facultad de Ciencias project “Teoría de Galois diferencial y modelos de sistemas dinámicos algebraicos” (HERMES 60920).

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(Recibido en enero de 2024. Aceptado en diciembre de 2024)

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