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Ineffable sets and large cardinals

Conjuntos inefables y grandes cardinales

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Abstract. Suppose κ is a regular cardinal. We prove that if the set of H_{λ^+}-reflecting cardinals $\lambda < \kappa$ is ineffable, then κ is an H_{κ^+}-reflecting cardinal. Similarly, we also prove that if the set of Woodin cardinals/cardinals having the stationary reflection property below κ is ineffable, then κ is a Woodin cardinal/cardinal having the stationary reflection property.

Key words and phrases. Ineffable set, reflecting cardinal, Woodin cardinal, stationary reflection property.

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Resumen. Probamos que si el conjunto de cardinales λ bajo κ tales que λ es cardinal H_{λ^+}-reflejante es un subconjunto inefable de κ entonces κ resulta ser un cardinal H_{κ^+}-reflejante. De manera similar para la propiedades de ser cardinal de Woodin y la propiedad de reflexión estacionaria: si el conjunto de los cardinales λ bajo κ tales que λ es cardinal de Woodin (se satisface $RP(\lambda)$) es un subconjunto inefable de κ entonces κ es cardinal de Woodin (se tiene $RP(\kappa)$).

Palabras y frases clave. Conjunto inefable, cardinal reflejante, cardinal de Woodin, propiedad de reflexión estacionaria.

1. Introduction

We have proved in [1] that if the set of Ramsey cardinals, Rowbottom cardinals or Jónsson cardinals below κ is an ineffable subset of κ then κ turns out to be a Ramsey cardinal, Rowbottom or Jónsson cardinal respectively. Continuing with this idea, we extend the result for the property of being a Woodin cardinal, a reflecting cardinal and having the stationary reflection property.

2. Main results

Definition 2.1. Let $\kappa > \omega$ be a regular cardinal and $A \subseteq \kappa$. A is said to be an ineffable subset of κ if and only if for any sequence $\langle S_\zeta : \zeta \in A\rangle$ such that for all $\zeta \in A$, $S_\zeta \subseteq \zeta$ there exists $T \subseteq A$, T a stationary subset of κ such that for all $\zeta < \zeta' \in T$, $S_\zeta = \zeta \cap S_{\zeta'}$. In this case we say that the sequence $\langle S_\zeta : \zeta \in T\rangle$ is coherent. If $A = \kappa$, κ is said to be an ineffable cardinal.

The following definition is due to Miyamoto, [3]:

Definition 2.2. An inaccessible cardinal κ is said to be an H_{κ^+}-reflecting cardinal if and only if for any $E \subseteq \kappa$ and any formula φ, if there is some cardinal χ such that $H_\chi \models \varphi(E)$, then the set $\{\alpha < \kappa :$ There is a cardinal $\chi' < \kappa$ with $H_{\chi'} \models \varphi(E \cap \alpha)\}$ is a stationary subset of κ.

In [3] it is proved that κ is an H_{κ^+}-reflecting cardinal if and only if κ is a strongly unfoldable cardinal. Unfoldable cardinals were introduced by Villaveces in [4].

Theorem 2.3. *If $A = \{\lambda < \kappa : \lambda$ is an H_{λ^+}-reflecting cardinal$\}$ is an ineffable subset of κ then κ is an H_{κ^+}-reflecting cardinal.*

Proof. Let $E \subseteq \kappa$, φ be a formula and χ be a cardinal such that $H_\chi \models \varphi(E)$. Let ψ be the Σ_1-sentence "There is ρ a cardinal and there is a set y such that $y = H_\rho$ and $y \models \varphi(E)$". Since $H_{\kappa^+} \prec_1 V$ there is a cardinal $\rho \in H_{\kappa^+}$ and there exists $y \in H_{\kappa^+}$ such that $y = H_\rho$ and $y \models \varphi(E)$ in H_{κ^+}. So $H_{\kappa^+} \models \varphi(E)$ and therefore $\langle V_\kappa, \in, E\rangle \models \varphi(E)$. Hence the set:

$$C = \{\alpha < \kappa : \langle V_\alpha, \in, E \cap \alpha\rangle \prec \langle V_\kappa, \in, E\rangle\}$$

is a club subset of κ. So $A\cap C$ is an ineffable subset of κ and for every $\lambda \in A\cap C$, $\langle V_\lambda, \in, E\cap\lambda\rangle \models \varphi(E\cap\lambda)$. Then there is a cardinal χ_λ such that $H_{\chi_\lambda} \models \varphi(E\cap\lambda)$ for any $\lambda \in A \cap C$. Now since λ is an H_{λ^+}-reflecting cardinal the set

$$N_\lambda := \{\alpha < \lambda : \exists\eta < \lambda \text{ a cardinal, } H_\eta \models \varphi(E \cap \alpha)\}$$

is a stationary subset of λ for every $\lambda \in A \cap C$. So there exists $S \subseteq A \cap C$ a stationary subset of κ such that the sequence $\langle N_\lambda : \lambda \in S\rangle$ is coherent. Then the set $N = \bigcup_{\lambda\in S} N_\lambda$ is a stationary subset of κ such that

$$N \subseteq \{\alpha < \kappa : \exists\chi' < \kappa \text{ a cardinal, } H_{\chi'} \models \varphi(E \cap \alpha)\}.$$

(N is a stationary subset of κ: Given $D \subseteq \kappa$ a club subset of κ, then for $\lambda \in S \cap \overline{D}$, $D \cap \lambda$ is club in λ and hence $D \cap N_\lambda \neq \varnothing$). ☑

For the following theorem we use the fact that κ is a Woodin cardinal if and only if for any $B \subseteq V_\kappa$ the set

$$\{\alpha < \kappa : \forall\delta < \kappa \ (\alpha \text{ is } \delta\text{-strong for } B)\}$$

is stationary in κ (see [2]), where κ is γ-strong for B if and only if there is a $j : V \to M$ such that:

(1) $cp(j) = \kappa$ and $\gamma < j(\kappa)$,

(2) $V_\gamma \subseteq M$, and

(3) $B \cap V_\gamma = j(B) \cap V_\gamma$.

Theorem 2.4. *If $A = \{\delta < \kappa : \delta$ is a Woodin cardinal$\}$ is an ineffable subset of κ then κ is a Woodin cardinal.*

Proof. Let $B \subseteq V_\kappa$. We have to prove that the set $\{\delta < \kappa : \forall\lambda < \kappa(\delta$ is λ-strong for $B)\}$ is a stationary subset of κ. For $\mu \in A$, the set $S_\mu := \{\alpha < \mu : \forall\lambda < \mu\ (\alpha$ is λ-strong for $B \cap V_\mu)\}$ is a stationary subset of μ. Since A is an ineffable subset of κ, there exists $T \subseteq A$ a stationary subset of κ such that the sequence $\langle S_\mu : \mu \in T\rangle$ is coherent. Let $S = \bigcup_{\mu\in T} S_\mu$. Then S is a stationary subset of κ. We have to prove that $S \subseteq \{\delta < \kappa : \forall\lambda < \kappa(\delta$ is λ-strong for $B)\}$: Let $\delta \in S$. Then there exists $\mu \in T$ such that $\delta \in S_\mu$. We can suppose $\lambda < \mu$ since $S-(\lambda+1)$ is also a stationary subset of κ and the sequence $\langle S_\mu : \mu \in T\rangle$ is coherent. So δ is λ-strong for $B\cap V_\mu$ and there is $j : V \to M$ such that $cp(j) = \delta$ with $j(\delta) > \lambda$, $V_\lambda \subseteq M$ and $(B \cap V_\mu) \cap V_\lambda = B \cap V_\lambda = j(B \cap V_\mu) \cap V_\lambda = j(B) \cap j(V_\mu) \cap V_\lambda = j(B) \cap (V_{j(\mu)})^M \cap V_\lambda = j(B) \cap V_\lambda$. ☑

Definition 2.5. For $\lambda \geq \omega_2$ a regular cardinal, $RP(\lambda)$ is the statement: For every $S \subseteq [\lambda]^\omega$, S a stationary subset of $[\lambda]^\omega$ there exists $x \in [\lambda]^{\omega_1}$ such that $S \cap [x]^\omega$ is a stationary subset of $[x]^\omega$.

Definition 2.6. For $S \subseteq [\kappa]^\omega$ and $\lambda < \kappa$, $S|\lambda$ is the set $\{a \cap \lambda : a \in S\}$. If $S \subseteq [\kappa]^\omega$ is a stationary subset of $[\kappa]^\omega$ then $S|\lambda$ is a stationary subset of $[\lambda]^\omega$ (see [2]).

Theorem 2.7. *If $A = \{\lambda < \kappa : \lambda \geq \omega_2$ is a regular cardinal such that $RP(\lambda)\}$ is an ineffable subset of κ then $RP(\kappa)$.*

Proof. Let S be a stationary subset of $[\kappa]^\omega$. For every $\lambda \in A$, there exists $x_\lambda \in [\lambda]^{\omega_1}$ such that $(S|\lambda) \cap [x_\lambda]^\omega$ is a stationary subset of $[x_\lambda]^\omega$. Since A is an ineffable subset of κ, there is $T \subseteq A$ a stationary subset of κ such that the sequence $\langle x_\lambda : \lambda \in T\rangle$ is coherent, so there is $\lambda \in T$ such that $x := x_\lambda = x_{\lambda'}$ for every $\lambda' \geq \lambda$. Then $S \cap [x]^\omega$ is a stationary subset of $[x]^\omega$: Let C be a club subset of $[x]^\omega$ and let $\lambda_0 \in T$ be such that $\lambda_0 \geq \lambda$ and $x \subseteq \lambda_0$. Since $(S|\lambda_0) \cap [x]^\omega$ is a stationary subset of $[x]^\omega$ there exists $p_0 \in (S|\lambda_0) \cap C$ and $q_0 \in S$ such that $p_0 = q_0 \cap \lambda_0 = q_0 \cap x$. For some $\lambda_1 \in T$ with $\lambda_1 \geq \lambda_0$, $q_0 \subseteq \lambda_1$, so $q_0 \in S|\lambda_1$ and therefore there is $p_1 \in (S|\lambda_1) \cap C$ such that $q_0 \subseteq p_1 \subseteq x$. Hence $q_0 \cap x = q_0 = p_0 \in S$, so $p_0 \in S \cap C$. ☑

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