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On b-generalized derivations and commutativity of prime rings

Derivadas b-generalizadas y conmutatividad de anillos primos

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Abstract. Let $\mathscr{A}$ be a prime ring, $\mathscr{Z}(\mathscr{A})$ its center, $\mathscr{Q}$ its right Martindale quotient ring, $\mathscr{C}$ its extended centroid, ψ a non-zero b-generalized derivation of $\mathscr{A}$ with associated map ξ. In this article, we prove that: (i) If $[\psi(x), \psi(y)] = 0$ for all $x, y \in \mathscr{A}$, then $\mathscr{A}$ is either commutative or there exists $q \in \mathscr{Q}$ such that $\xi = \mathrm{ad}(q)$, $\psi(x) = -bxq$, and $qb = 0$. (ii) If $\psi(x) \circ \psi(y) = 0$ for all $x, y \in \mathscr{A}$, then $\mathscr{A}$ is either commutative with $\mathrm{char}(\mathscr{A}) = 2$ or there exists $q \in \mathscr{Q}$ such that $\psi(x) = -bxq$ and $qb = 0$. Additional results are established for cases involving $[\xi(x), \psi(x)] = 0$ or $\xi(x) \circ \psi(x) = 0$, where $\mathrm{char}(\mathscr{A}) \neq 2$. Furthermore, we give some examples that show the importance of the hypotheses of our theorems.

Key words and phrases. Prime ring, Martindale quotient ring, b-Generalized derivation.

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Resumen. Sea $\mathscr{A}$ un anillo primo, $\mathscr{Z}(\mathscr{A})$ su centro, $\mathscr{Q}$ su anillo de cocientes de Martindale por derecha, $\mathscr{C}$ su centroide extendido, ψ una derivada b-generalizada de $\mathscr{A}$ con mapa asociado ξ. En este artículo probamos los siguientes resultados: (i) Si $[\psi(x), \psi(y)] = 0$ para todo $x, y \in \mathscr{A}$, entonces o $\mathscr{A}$ es conmutativo o existe $q \in \mathscr{Q}$ tal que $\xi = \mathrm{ad}(q)$, $\psi(x) = -bxq$, y $qb = 0$. (ii) Si $\psi(x) \circ \psi(y) = 0$ para todo $x, y \in \mathscr{A}$, entonces o $\mathscr{A}$ es conmutativo con

$\text{char}(\mathscr{A}) = 2$ o existe $q \in \mathscr{Q}$ tal que $\psi(x) = -bxq$ y $qb = 0$. También se analizan los casos donde $[\xi(x), \psi(x)] = 0$ o $\xi(x) \circ \psi(x) = 0$, donde $\text{char}(\mathscr{A}) \neq 2$. Se incluyen ejemplos que ilustran la importancia de las hipótesis de los teoremas.

Palabras y frases clave. Anillos primo, anillo de cocientes de Martindale, derivadas b-generalizadas.

Introduction

Throughout this article, $\mathscr{A}$ denotes an associative ring with center $\mathscr{Z}(\mathscr{A})$. The commutator $xy - yx$ is denoted by $[x, y]$, and the anti-commutator $xy + yx$ by $x \circ y$.

A ring $\mathscr{A}$ is called prime if for any $x, y \in \mathscr{A}$, $x\mathscr{A}y = \{0\}$ implies $x = 0$ or $y = 0$ (where $x\mathscr{A}y$ represents the set of all elements of the form xay, with $a \in \mathscr{A}$). A ring is semi-prime if for any $x \in \mathscr{A}$, $x\mathscr{A}x = \{0\}$ implies $x = 0$. We denote the right Martindale quotient ring of $\mathscr{A}$ by $\mathscr{Q} = \mathscr{Q}_r(\mathscr{A})$.

A right Martindale quotient ring $\mathscr{Q}_r(\mathscr{A})$ of a prime ring $\mathscr{A}$ is a ring satisfying: (i) $\mathscr{A} \subseteq \mathscr{Q}_r(\mathscr{A})$; (ii) for every $q \in \mathscr{Q}_r(\mathscr{A})$, there exists a nonzero ideal I of $\mathscr{A}$ such that $qI \subseteq \mathscr{A}$; (iii) if $qI = 0$ for some $q \in \mathscr{Q}_r(\mathscr{A})$ and a nonzero ideal I of $\mathscr{A}$, then $q = 0$; (iv) if I is a nonzero ideal of $\mathscr{A}$ and $\phi : I \to \mathscr{A}$ is a right $\mathscr{A}$-module homomorphism, then there exists $q \in \mathscr{Q}_r(\mathscr{A})$ such that $\phi(x) = qx$ for all $x \in I$, [14].

The extended centroid of $\mathscr{A}$ is represented by $\mathscr{C}$, and the central closure of $\mathscr{A}$ is the ring $\mathscr{A}\mathscr{C}$. A prime ring $\mathscr{A}$ is centrally closed if $\mathscr{A} = \mathscr{A}\mathscr{C}$. In particular, the prime ring $\mathscr{Q}$ is centrally closed. It is known that $\mathscr{A} \subseteq \mathscr{A}\mathscr{C} \subseteq \mathscr{Q}$ [14]. The ring $\mathscr{Q}$ has a field as its center, namely $\mathscr{C}$, if and only if $\mathscr{A}$ is a prime ring. If $\mathscr{A}$ is a prime ring, then $\mathscr{A}\mathscr{C}$ and $\mathscr{Q}$ are also prime rings. For more details, see [14].

An additive map $\xi : \mathscr{A} \to \mathscr{A}$ is said to be a derivation of $\mathscr{A}$ if $\xi(xy) = \xi(x)y + x\xi(y)\ \forall\ x, y \in \mathscr{A}$. For $b \in \mathscr{A}$, we let $\text{ad}(b)$ denote the map $x \mapsto [b, x]$ for $x \in \mathscr{A}$. Clearly, $\text{ad}(b)$ is a derivation of $\mathscr{A}$, commonly known as the inner derivation of $\mathscr{A}$ induced by the element b. An additive map $\psi : \mathscr{A} \to \mathscr{A}$ is called a generalized derivation of $\mathscr{A}$ if there exists a derivation ξ of $\mathscr{A}$, such that $\psi(xy) = \psi(x)y + x\xi(y)\ \forall\ x, y \in \mathscr{A}$. For more studies about the previous concepts see [12, 15, 19, 27, 1, 8, 23, 28, 22, 21].

Following Kosan and Lee [17], let $\xi : \mathscr{A} \to \mathscr{Q}$ be an additive map and $b \in \mathscr{Q}$. An additive map $\psi : \mathscr{A} \to \mathscr{Q}$ is called a left b-generalized derivation with associated map ξ if $\psi(xy) = \psi(x)y + bx\xi(y)$ for all $x, y \in \mathscr{A}$. If $\mathscr{A}$ is prime, then ξ is a derivation of $\mathscr{A}$ [17]. We refer to ψ as a b-generalized derivation with associated pair (b, ξ). A generalized derivation with associated map ξ is a b-generalized derivation with pair $(1, \xi)$. The converse is generally not true (see [11]). The study of b-generalized derivations with nilpotent values has attracted considerable attention [29, 3, 4, 7].

A standard identity of degree m is a polynomial identity of the form $S_m(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(m)}$, where S_m is the symmetric group of degree m and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. A ring $\mathscr{A}$ satisfies a standard identity if $S_m(x_1, \ldots, x_m) = 0$ for all $x_1, \ldots, x_m \in \mathscr{A}$, [14].

In 2016, Liu [20] provided an early exploration of b-generalized derivations in the context of Engel conditions in prime rings. Liu demonstrated that if $[\psi(x^m), x^n]_k = 0$ for all x in a nonzero ideal I, with fixed positive integers m, n, k, where ψ is b-generalized derivation on $\mathscr{A}$, then $\psi(x) = \lambda x$ for all $x \in \mathscr{A}$, unless $\mathscr{A} \cong M_2(GF(2))$. This work extended previous findings on derivations and generalized σ-derivations, offering significant applications to skew derivations and structural analyses of semiprime rings. Building on this foundational work, in 2018, De Filippis and Wei [6] advanced the understanding of b-generalized skew derivations by investigating their effects on Lie ideals in non-commutative prime rings. For a nonzero b-generalized skew derivation ψ and a non-central Lie ideal L, they established two key results: (1) If $\operatorname{char}(\mathscr{A}) \neq 2, 3$, and $a[\psi(x), x]^n = 0$ for all $x \in L$, then $\psi(x)$ either takes the form λx, or $\mathscr{A}$ satisfies the standard identity of degree 4. (2) If $a[\psi(x), x]^n \in Z(\mathscr{A})$ for all $x \in \mathscr{A}$ and $\operatorname{char}(\mathscr{A}) = 0$ or $\operatorname{char}(\mathscr{A}) > n$, the same conclusions hold. These findings provided deeper structural insights into the constraints imposed by such derivations.

In 2022, Pehlivan and Alba? [24] expanded on earlier studies by analyzing b-generalized derivations as homomorphisms or anti-homomorphisms in prime rings. They defined these derivations as additive mappings $\psi : \mathscr{A} \to \mathscr{Q}$ satisfying $\psi(xy) = \psi(x)y + bx\xi(y)$ for all $x, y \in \mathscr{A}$, with $b \in \mathscr{Q}$ and ξ an associated derivation. Their results highlighted characterizations of these mappings when acting on noncentral Lie ideals and extended known results for generalized and skew derivations. A notable aspect was their exploration of mappings satisfying $\psi([x, y]) = \pm[x, y]$, implying that ψ acts either as the identity or its negation. Meanwhile, Prajapati and Gupta [25] turned their focus to b-generalized skew derivations in prime rings under specific centralizing conditions. They examined cases where the commutator $[p, \psi(x)x]$ is central for x in the image of a non-central multilinear polynomial. Their findings revealed that either p is central, or the square of the polynomial becomes central valued. Furthermore, ψ simplifies to the form $\psi(x) = qx$, where $[p, q]$ is central. This research showcased how centralizing constraints shape the structure of the ring and the derivation.

In 2023, Khan and Khan [16] contributed to the growing body of work on b-generalized derivations by analyzing their behavior in prime rings under certain algebraic conditions. They demonstrated that if a prime ring $\mathscr{A}$ admits a nonzero b-generalized derivation ψ such that $[\psi(x^m)x^n + x^n\psi(x^m), x^r]_k = 0$ for all x in a nonzero ideal I, with fixed integers m, n, r, k, then $\psi(x) = \lambda x$ for some $\lambda \in \mathscr{C}$, unless $\mathscr{A} \cong M_2(GF(2))$. This work generalized earlier results on derivations and identified exceptions in specific matrix rings.

Most recently, in 2024, Rania [26] examined power central-valued b-generalized skew derivations in prime rings. For a prime ring $\mathscr{A}$ of characteristic not equal to 2, they considered a nonzero b-generalized skew derivation ψ and a non-central Lie ideal L, where $(\psi(x)x)^n \in Z(\mathscr{A})$ for all $x \in L$. They showed that if $\mathrm{char}(\mathscr{A}) = 0$ or $\mathrm{char}(\mathscr{A}) = p \geq n+1$, then $\mathscr{A} \cong M_2(K)$, the 2×2 matrix ring over a field K. This research provided critical insights into the structural implications of power central-valued derivations.

In [17, Theorem 2.4] states that if $\mathscr{A}$ is a prime ring and ψ is a b-generalized derivation with associated map ξ such that $\psi(x)^n = 0 \ \forall \ x \in \mathscr{A}$, where n is a positive integer, then there exist $q \in \mathscr{Q}$ such that $\xi = \mathrm{ad}(q)$, $\psi(x) = -bxq$ for $x \in \mathscr{A}$, and $qb = 0$.

Hvala proved in [13, Proposition 4] that if ψ is a non-zero generalized derivation of a non-commutative prime ring satisfying $[\psi(x), \psi(y)] = 0 \ \forall \ x, y \in \mathscr{A}$, then there exist $c \in \mathscr{A}\mathscr{C} - \mathscr{C}$ and additive maps $\gamma, \phi : \mathscr{A} \to \mathscr{C}$ such that $\psi(x) = x \circ c$ and $\psi(x) = \gamma(x)c + \phi(x) \ \forall \ x \in \mathscr{A}$. Moreover, he had $c^2 \in \mathscr{C}$. In the same paper [13, Theorem 2], Hvala proved that if $\psi_1, \psi_2 : \mathscr{A} \to \mathscr{A}$ are non-zero generalized derivation of a non-commutative prime ring and $\mathrm{char}(\mathscr{A}) \neq 2$ satisfying $[\psi_1(x), \psi_2(x)] = 0 \ \forall \ x \in \mathscr{A}$, then there exists $\lambda \in \mathscr{C}$ such that $\psi_1(x) = \lambda\psi_2(x)$, $x \in \mathscr{A}$.

Motivated by the aforementioned results, this paper delves into the structure of prime rings using the framework of b-generalized derivations. We investigate cases where commutators or anti-commutators of these derivations satisfy specific conditions. In particular, we prove the following key results:

(1) If ψ is a b-generalized derivation of a prime ring $\mathscr{A}$ and satisfies $[\psi(x), \psi(y)] = 0$ for all $x, y \in \mathscr{A}$, then $\mathscr{A}$ is either commutative or there exists $0 \neq q \in Q$ such that $\psi(x) = -bxq$, $\xi = \mathrm{ad}(q)$, and $qb = 0$.

(2) If ψ satisfies $\psi(x) \circ \psi(y) = 0$ for all $x, y \in \mathscr{A}$, then $\mathscr{A}$ is commutative with $\mathrm{char}(\mathscr{A}) = 2$, or there exists $0 \neq q \in Q$ such that $\psi(x) = -bxq$ and $qb = 0$.

(3) For a non-zero b-generalized derivation ψ with an associated derivation ξ with $\mathrm{char}(\mathscr{A}) \neq 2$, if $[\xi(x), \psi(x)] = 0$ for all $x \in \mathscr{A}$, then either $\psi(x) = qx$ for some $q \in \mathscr{Q}$, or $\psi(x) = \lambda\xi(x)$ for some $\lambda \in \mathscr{C}$ and $b = \lambda$, or $\mathscr{A}$ is commutative.

(4) If $\xi(x) \circ \psi(x) = 0$ for all $x \in A$ with $\mathrm{char}(\mathscr{A}) \neq 2$, then $\psi(x) = qx$ for all $x \in \mathscr{A}$ and some $q \in \mathscr{Q}$, or $\psi(x) = \lambda\xi(x)$ for all $x \in \mathscr{A}$ and some $\lambda \in \mathscr{C}$ and $b = \lambda$.

These results not only extend the known classifications of prime rings but also highlight the nuanced behavior of b-generalized derivations. Furthermore, we present illustrative examples to demonstrate the indispensability of the hypotheses in these theorems, ensuring that the results cannot be generalized without the specified conditions.

1. Preliminaries

In the proof of our theorems, the following important results will be used:

Lemma 1.1. [9, Lemma 1.3.2] *Let $\mathscr{A}$ be a prime ring. Suppose that a_i, b_i are non-zero elements in $\mathscr{A}$ such that $\sum a_i x b_i = 0$ for all $x \in \mathscr{A}$. Then the $a_i's$ are linearly dependent over $\mathscr{C}$, and the $b_i's$ are linearly dependent over $\mathscr{C}$*

Since $\mathscr{Q}\mathscr{C} = \mathscr{Q}$ and $\mathscr{Q}(\mathscr{Q}) = \mathscr{Q}$, then Hvala's result [13, Lemma 2] can be presented in the following form.

Lemma 1.2. *Let $\mathscr{A}$ be a prime ring and $\psi : \mathscr{Q} \to \mathscr{Q}$ be an additive map satisfying $\psi(xy) = \psi(x)y \ \forall \ x, y \in \mathscr{A}$. Then there exists $q \in \mathscr{Q}$, $\psi(x) = qx \ \forall \ x \in \mathscr{A}$.*

Since $\mathscr{Q}\mathscr{C} = \mathscr{Q}$ and $\mathscr{Q}(\mathscr{Q}) = \mathscr{Q}$, then Bresar's result [5, Proposition 8] can be presented in the following form.

Lemma 1.3. *Let $\mathscr{A}$ be a prime ring. Suppose that*

$$\sum_{j=1}^{n} g_j(z) x a_j + \sum_{i=1}^{k} c_i z h_i(x) = 0$$

$\forall \ x, z \in \mathscr{Q}$, where $a_j, c_i \in \mathscr{Q}$ and $g_j : \mathscr{Q} \to \mathscr{Q}$, $h_i : \mathscr{Q} \to \mathscr{Q}$ are any maps. If the sets $\{a_1, \ldots, a_n\}$ and $\{c_1, \ldots, c_k\}$ are $\mathscr{C}$-independent, then there exist $q_{ij} \in \mathscr{Q}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$, such that

$$g_j(z) = -\sum_{i=1}^{k} c_i z q_{ij}, \quad h_i(x) = \sum_{j=1}^{n} q_{ij} x a_j,$$

$\forall \ x, z \in \mathscr{Q}$, $i = 1, \ldots, k$, $j = 1, \ldots, n$.

Lemma 1.4. [18, Theorem 2] *Let $\mathscr{A}$ be a prime ring with right Martindale quotient ring $\mathscr{Q}$. Then $\mathscr{A}$ and $\mathscr{Q}$ satisfy the same differential identities.*

Putting $\alpha = I$ in [6, Remark 1.9] (where I is an identity), we get

Lemma 1.5. *Let $\mathscr{A}$ be a prime ring, $b \in \mathscr{Q}$, $\xi : \mathscr{A} \to \mathscr{A}$ be an additive mapping of $\mathscr{A}$ and ψ be the b-generalized derivation of $\mathscr{A}$ with associated term (b, ξ). Then, ψ can be uniquely extended to $\mathscr{Q}$, and assumes the form $\psi(x) = ax + b\xi(x)$, where $a \in \mathscr{Q}$.*

2. The Main Results

I. N. Herstein [10] proved that if ξ is a derivation of a non-commutative prime ring $\mathscr{A}$ with char($\mathscr{A}$)$\neq 2$ satisfying $[\xi(x), \xi(y)] = 0 \ \forall\ x, y \in \mathscr{A}$ then $\xi = 0$. But this result does not hold for b-generalized derivations. This is demonstrated by the following example:

Example 2.1. $\mathscr{A} = M_2(\mathbb{F})$ the ring of 2×2 matrices over a field $\mathbb{F}$, and if we denote $b = e_{11}$, $q = e_{12}$, $\psi(x) = -bxq$ and $\xi = \mathrm{ad}(q) \ \forall\ x \in \mathscr{A}$ satisfying $[\psi(x), \psi(y)] = 0 \ \forall\ x, y \in \mathscr{A}$, but $\psi \neq 0$.

Now, we prove the following theorem by extending the result of I. N. Herstein [10] to b-generalized derivations.

Theorem 2.2. *Let $\mathscr{A}$ be a prime ring and let $\psi : \mathscr{A} \to \mathscr{Q}$ be a non-zero b-generalized derivation with associated map ξ. Suppose that $[\psi(x), \psi(y)] = 0 \ \forall\ x, y \in \mathscr{A}$, then either*

(1) $\mathscr{A}$ is commutative or

(2) there exists $0 \neq q \in \mathscr{Q}$ such that $\xi = ad(q)$, $\psi(x) = -bxq \ \forall\ x \in \mathscr{A}$, and $qb = 0$.

Proof. Suppose $\xi = 0$ or $b = 0$, then $\psi(xy) = \psi(x)y$ and hence by Lemma 1.2, $\psi(x) = qx \ \forall\ x \in \mathscr{A}$ and for some $0 \neq q \in \mathscr{Q}$. By hypothesis, we have

$$[\psi(x), \psi(y)] = 0 \tag{1}$$

$\forall\ x, y \in \mathscr{A}$. Replacing y by yt in (1), we have

$$\psi(y)[\psi(x), t] - by\xi(t)\psi(x) + \psi(x)by\xi(t) = 0 \tag{2}$$

$\forall\ x, y, t \in \mathscr{A}$. Now putting $\xi = 0$ or $b = 0$ in (2), we get $\psi(y)[\psi(x), t] = 0$. Taking t by tr in the last relation, we find that $\psi(y)t[\psi(x), r] = 0$ but $\psi \neq 0$ and so $[\psi(x), r] = 0$ hence $\psi(x) \in \mathscr{Z}(\mathscr{A})$. Replacing x by xy in the last relation, we arrive that $\psi(x) = 0$ or $y \in \mathscr{Z}(\mathscr{A})$. Thus $y \in \mathscr{Z}(\mathscr{A})$, that is $\mathscr{A}$ is commutative. From now onward, we will assume that

$$\xi \neq 0 \quad \text{and} \quad b \neq 0. \tag{3}$$

$\mathscr{A}$ and $\mathscr{Q}$ satisfy the same differential identities by Lemma 1.4, and ψ can be uniquely extended to $\mathscr{Q}$ by Lemma 1.5, and so Equation (2) becomes as

$$\psi(y)[\psi(x), t] - by\xi(t)\psi(x) + \psi(x)by\xi(t) = 0 \tag{4}$$

$\forall\ x, y, t \in \mathscr{Q}$. Substituting ybr for y in (4), we have

$$\psi(ybr)[\psi(x), t] - by(br\xi(t)\psi(x)) + \psi(x)bybr\xi(t) = 0$$

and by using (4) in the last relation, we get

$$\psi(ybr)[\psi(x),t] - by\psi(r)[\psi(x),t] - by\psi(x)br\xi(t)) + \psi(x)bybr\xi(t) = 0$$

hence

$$(\psi(ybr) - by\psi(r))[\psi(x),t] + [\psi(x),by]br\xi(t) = 0.$$

Since $\mathscr{A}$ is prime then by [17, Theorem 2.3], ξ is a derivation. Writing ts instead of t in the last expression, we find that

$$(\psi(ybr) - by\psi(r))t[\psi(x),s] + [\psi(x),by]brt\xi(s) = 0 \tag{5}$$

$\forall\ x,y,t,r,s \in \mathscr{Q}$. Since $\xi(s) \neq 0$, and by Lemma 1.1, we obtain following two cases

Case I: $\psi(ybr) - by\psi(r) = [\psi(x),by]br = 0$.

Case II: either, $\psi(ybr) - by\psi(r) \neq 0$ or $[\psi(x),by]br \neq 0$. Then by Lemma 1.1, both $\{\psi(ybr) - by\psi(r), [\psi(x),by]br\}$ and $\{[\psi(x),s],\xi(s)\}$ are $\mathscr{C}$-dependent. Thus, $\lambda_1(x,y,r)(\psi(ybr) - by\psi(r)) + \lambda_2(x,y,r)[\psi(x),by]br = 0$ for some $\lambda_1(x,y,r)$, $\lambda_2(x,y,r) \in \mathscr{C}$ not both zero. In Case I, we have

$$[\psi(x),by]bR = \{0\} \tag{6}$$

$\forall\ x,y \in \mathscr{Q}$. This implies that $[\psi(x),by]b = 0$, that is

$$\psi(x)byb - by\psi(x)b = 0.$$

Since $b \neq 0$ and by Lemma 1.1, $\{\psi(x)b, b\}$ are $\mathscr{C}$-dependent. That means, $\lambda_3(x)\psi(x)b + \lambda_4(x)b = 0$. for some $\lambda_3(x), \lambda_4(x) \in \mathscr{C}$ not both zero. We have

$$\lambda_3(x)\psi(x)b + \lambda_4(x)b = 0 \tag{7}$$

$\forall\ x \in \mathscr{Q}$. Assume that $\psi(x)b \neq 0$ and we have from (3) $b \neq 0$ and so $\lambda_3(x) \neq 0 \neq \lambda_4(x)$ because $\{\psi(x)b, b\}$ are $\mathscr{C}$-dependent. Thus, there exists $\lambda_4^{-1}(x) \in \mathscr{C}$. From (7) $b = -\lambda_4^{-1}(x)\lambda_3(x)\psi(x)b$ taking x by 0 in the last relation, we get $b = 0$, a contradiction. Thus,

$$\psi(x)b = 0 \tag{8}$$

$\forall\ x \in \mathscr{Q}$. Multiplying in the left in (8) by $\psi(y)$, we have $\psi(y)\psi(x)b = 0$. Replacing x by xt in the last expression and using (8), we get $\psi(y)\psi(x)tb = 0$ but $b \neq 0$ and so we conclude that

$$\psi(y)\psi(x) = 0 \tag{9}$$

$\forall\ x,y \in \mathscr{Q}$. By using (8) and (9) in (2), one can find that $\psi(y)t\psi(x) - by\xi(t)\psi(x) = 0$. Replacing y by z, x by r, and t by x in the last relation, we see that

$$\psi(z)x\psi(r) - bz\xi(x)\psi(r) = 0.$$

By using Lemma 1.3 in the last relation, we arrive to $g_1(z) = \psi(z)$, $a_1(r) = \psi(r)$, $h_1(x) = -\xi(x)\psi(r)$, and $c_1(r) = b$. Also we have $g_1(z) = -c_1(r)zq_{11}(r)$ and $h_1(x) = q_{11}(r)xa_1(r)$. Thus,

$$\psi(z) = -bzq_{11}(r) \quad \forall \quad r, z \in \mathscr{Q}. \tag{10}$$

$$\xi(x)\psi(r) = q_{11}(r)x\psi(r) \tag{11}$$

$\forall\ x, r, z \in \mathscr{Q}$. Taking r by s in (10), we have $bz(q_{11}(s) - q_{11}(r)) = 0$ but $b \neq 0$ and so $q_{11}(s) = q_{11}(r) = q_{11} = q \in \mathscr{Q}$. So from (10), we get

$$\psi(z) = -bzq \tag{12}$$

$\forall\ z \in \mathscr{Q}$. By using (12) in (11), we obtain $\xi(x)(-bzq) = qx(-bzq)$ this implies that $(\xi(x)b - qxb)zq = 0$ but $q \neq 0$ because of $\psi \neq 0$ and so $\xi(x)b = qxb$, replacing x by yx in the last relation, we get $yqxb + \xi(y)xb = qyxb$ this implies $(\xi(y) + [y, q])xb = 0$ hence $\xi(y) + [y, q] = 0$ that is $\xi(y) = [q, y]\ \forall\ y \in \mathscr{Q}$. From the last relation and (12), we have $\psi(z) = -bzq$ and $\xi(y) = [q, y]\ \forall\ z, y \in \mathscr{A}$ and from (8) and (12), we have $qb = 0$, as desired.

Before we start the proof of Case II, we will prove that if

$$\psi(ybr) - by\psi(r) = 0. \qquad \text{Then} \qquad [\psi(x), by]br = 0 \tag{13}$$

$\forall\ x, y, r \in \mathscr{Q}$. Suppose that $\psi(ybr) - by\psi(r) = 0$, and by using the last relation in (5), we obtain $[\psi(x), by]brt\xi(s) = 0$ but $\xi \neq 0$ and so $[\psi(x), by]br = 0$. Now, if we have Case II, then at least $\psi(ybr) - by\psi(r) \neq 0$ or $[\psi(x), by]br \neq 0$. If $\psi(ybr) - by\psi(r) = 0$, then from (13), we have $[\psi(x), by]br = 0$, a contradiction. So $\psi(ybr) - by\psi(r) \neq 0$. If $[\psi(x), by]br = 0$ same as in (6), as desired. Assume that $[\psi(x), by]br \neq 0$. Then from Case II, we have $\lambda_1(x, y, r) \neq 0 \neq \lambda_2(x, y, r)$ because $\{\psi(ybr) - by\psi(r), [\psi(x), by]br\}$ are $\mathscr{C}$-dependent, and hence there exist $\lambda_1^{-1}(x, y, r) \in \mathscr{C}$. From Case II, $\psi(ybr) - by\psi(r) = -\lambda_1^{-1}(x, y, r)\lambda_2(x, y, r)$ $[\psi(x), by]br$ replacing $x = 0$ in the last relation, we get $\psi(ybr) - by\psi(r) = 0$, a contradiction. ☑

Theorem 2.3. *Let $\mathscr{A}$ be a prime ring and let $\psi : \mathscr{A} \to \mathscr{Q}$ be a non-zero b-generalized derivation with associated map ξ. Suppose that $\psi(x) \circ \psi(y) = 0\ \forall\ x, y \in \mathscr{A}$. Then either*

(1) $\mathscr{A}$ is commutative and char($\mathscr{A}$)=2, or

(2) there exists $0 \neq q \in \mathscr{Q}$ such that $\xi = ad(q)$, $\psi(x) = -bxq\ \forall\ x \in \mathscr{A}$, and $qb = 0$.

Proof. In case $b = 0$ or $\xi = 0$ then using the same arguments as used in the beginning of the proof of Theorem 2.2 we get $\mathscr{A}$ is commutative. Now, by using the previous fact and our hypothesis, we get char($\mathscr{A}$)=2, as desired. Now, assume that $b \neq 0$ and $\xi \neq 0$. By hypothesis, we have

$$\psi(x) \circ \psi(y) = 0 \tag{14}$$

$\forall\, x, y \in \mathscr{A}$. Substituting yt for y in (14), we have

$$\psi(y)[\psi(x), t] - \psi(x)by\xi(t) - by\xi(t)\psi(x) = 0$$

$\forall\, x, y, t \in \mathscr{A}$. By Lemma 1.4 and Lemma 1.5, we find that

$$\psi(y)[\psi(x), t] - \psi(x)by\xi(t) - by\xi(t)\psi(x) = 0 \tag{15}$$

$\forall\, x, y, t \in \mathscr{Q}$. Taking y by ybr in (15), we get

$$\psi(ybr)[\psi(x), t] - \psi(x)bybr\xi(t) - by(br\xi(t)\psi(x)) = 0 \tag{16}$$

$\forall\, x, y, t, r \in \mathscr{Q}$. By using (15) in (16), we obtain

$$(\psi(ybr) - by\psi(r))[\psi(x), t] + [by, \psi(x)]br\xi(t) = 0$$

$\forall\, x, y, t, r \in \mathscr{Q}$. By [17, Theorem 2.3], then d is a derivation. Writing ts instead of t in the last relation, we have

$$(\psi(ybr) - by\psi(r))t[\psi(x), s] + [by, \psi(x)]brt\xi(s) = 0$$

$\forall\, x, y, t, r, s \in \mathscr{Q}$. Notice that the arguments given in the proof of Theorem 2.2 after Equation (5) still valid in the present situation and hence repeating the same process, we get the required result. ☑

Theorem 2.4. *Let $\mathscr{A}$ be a prime ring of char($\mathscr{A}$) $\neq 2$ and let $\psi : \mathscr{A} \to \mathscr{Q}$ be a non-zero b-generalized derivation with associated map ξ. Suppose that $[\xi(x), \psi(x)] = 0\ \forall\, x \in \mathscr{A}$. Then either*

(1) $\psi(x) = qx\ \forall\, x \in \mathscr{A}$ and some $0 \neq q \in \mathscr{Q}$ or

(2) Any b-generalized derivation ψ is a derivation and $\psi(x) = \lambda\xi(x)\ \forall\, x \in \mathscr{A}$ and some $0 \neq \lambda \in \mathscr{C}$ and $b = \lambda$ or

(3) $\mathscr{A}$ is commutative.

Proof. Suppose $\xi = 0$ or $b = 0$, then $\psi(xy) = \psi(x)y$ and hence by Lemma 1.2, $\psi(x) = qx\ \forall\, x \in \mathscr{A}$ and for some $0 \neq q \in \mathscr{Q}$. From now, we will assume that

$$\xi \neq 0 \quad \text{and} \quad b \neq 0. \tag{17}$$

By hypothesis, we have

$$[\xi(x), \psi(x)] = 0 \tag{18}$$

$\forall\ x \in \mathscr{A}$. By linearizing in (18), we have

$$[\xi(x), \psi(y)] + [\xi(y), \psi(x)] = 0 \tag{19}$$

$\forall\ x, y \in \mathscr{A}$. Replacing y by yt in (19) and since $\mathscr{A}$ is prime then by [17, Theorem 2.3], ξ is a derivation, we get

$$\psi(y)[\xi(x), t] + [\xi(x), by\xi(t)] + \xi(y)[t, \psi(x)] + y[\xi(t), \psi(x)] + [y, \psi(x)]\xi(t) = 0 \tag{20}$$

$\forall\ x, y, t \in \mathscr{A}$. Substituting ry for y in (20) then left multiplying (20) by r then subtracting them, we have

$$(\psi(ry) - r\psi(y))[\xi(x), t] - [b, r]y\xi(t)\xi(x) + [\xi(x)b - \psi(x), r]y\xi(t) + \xi(r)y[t, \psi(x)] = 0$$

$\forall\ x, y, t, r \in \mathscr{A}$. $\mathscr{A}$ and $\mathscr{Q}$ satisfy the same differential identities by Lemma 1.4, and by Lemma 1.5 ψ can be uniquely extended to $\mathscr{Q}$. Thus, we have

$$\begin{aligned}(\psi(ry) - r\psi(y))[\xi(x), t] - [b, r]y\xi(t)\xi(x) + [\xi(x)b - \psi(x), r]y\xi(t)\\ + \xi(r)y[t, \psi(x)] = 0\end{aligned} \tag{21}$$

$\forall\ x, y, t, r \in \mathscr{Q}$. Writing $y[b, s]k$ instead of y in (21), we obtain

$$\begin{aligned}(\psi(ry[b, s]k) - r\psi(y[b, s]k))[\xi(x), t] - [b, r]y([b, s]k\xi(t)\xi(x))\\ +[\xi(x)b - \psi(x), r]y[b, s]k\xi(t) + \xi(r)y[b, s]k[t, \psi(x)] = 0\end{aligned}$$

$\forall\ x, y, t, r, s, k \in \mathscr{Q}$. By using (21) in the last relation, we conclude that

$$\begin{aligned}&\{\psi(ry[b, s]k) - r\psi(y[b, s]k) - [b, r]y\psi(sk) + [b, r]ys\psi(k)\}[\xi(x), t]\\ &+\{[\xi(x)b - \psi(x), r]y[b, s] - [b, r]y[\xi(x)b - \psi(x), s]\}k\xi(t)\\ &+\{\xi(r)y[b, s] - [b, r]y\xi(s)\}k[t, \psi(x)] = 0\end{aligned}$$

$\forall\ x, y, t, r, s, k \in \mathscr{Q}$. Replacing t by tm in the last relation, we see that

$$\{\psi(ry[b, s]k) - r\psi(y[b, s]k) - [b, r]y\psi(sk) + [b, r]ys\psi(k)\}t[\xi(x), m] \tag{22}$$
$$+\{[\xi(x)b - \psi(x), r]y[b, s] - [b, r]y[\xi(x)b - \psi(x), s]\}kt\xi(m) \tag{23}$$
$$+\{\xi(r)y[b, s] - [b, r]y\xi(s)\}kt[m, \psi(x)] = 0$$

$\forall\ x, y, t, r, s, k, m \in \mathscr{Q}$. Since $\xi(m) \neq 0$, then we have following two cases:

Case I: $\psi(ry[b, s]k) - r\psi(y[b, s]k) - [b, r]y\psi(sk) + [b, r]ys\psi(k) = 0$, $([\xi(x)b - \psi(x), r]y[b, s] - [b, r]y[\xi(x)b - \psi(x), s])k = 0$ and $\{\xi(r)y[b, s] - [b, r]y\xi(s)\}k = 0$.

Case II: At least one among Case I is non-zero and since $\xi(m) \neq 0$, then by Lemma 1.1 both $\{\psi(ry[b, s]k) - r\psi(y[b, s]k) - [b, r]y\psi(sk) + [b, r]ys\psi(k), [\xi(x)b - \psi(x), r]y[b, s] - [b, r]y[\xi(x)b - \psi(x), s], (\xi(r)y[b, s] - [b, r]y\xi(s))k\}$ and $\{[\xi(x), m], \xi(m), [m, \psi(x)]\}$ are $\mathscr{C}$-dependent. Thus, we have $\lambda_1(x, m)[\xi(x), m]+$

$\lambda_2(x,m)\xi(m) + \lambda_3(x,m)[m,\psi(x)] = 0$ for some $\lambda_i(x,m) \in \mathscr{C}$ not all zero, $i = 1,2,3$. If we have Case I, then

$$\xi(r)y[b,s] - [b,r]y\xi(s) = 0 \tag{24}$$

$\forall\ y,r,s \in \mathscr{Q}$. Since $\xi(s) \neq 0$ and hence by Lemma 1.1, we have $\{\xi(r),[b,r]\}$ are linearly $\mathscr{C}$-dependent. Thus, we have

$$\lambda_4(r)\xi(r) + \lambda_5(r)[b,r] = 0 \tag{25}$$

for some $\lambda_4(r),\lambda_5(r) \in \mathscr{C}$ not both zero and $\forall\ r,s \in \mathscr{Q}$.

Subcase I: If $[b,r] = 0$, then

$$b \in \mathscr{C}. \tag{26}$$

By using (26) in (21), we see that

$$(\psi(ry) - r\psi(y))[\xi(x),t] + [\xi(x)b - \psi(x),r]y\xi(t) + \xi(r)y[t,\psi(x)] = 0.$$

Substituting ts for t in the last relation, we get

$$(\psi(ry) - r\psi(y))t[\xi(x),s] + [\xi(x)b - \psi(x),r]yt\xi(s) + \xi(r)yt[s,\psi(x)] = 0 \tag{27}$$

$\forall\ x,y,r,t,s \in \mathscr{Q}$. We have $\xi(r)y \neq 0$, otherwise we obtain a contradiction. Then by Lemma 1.1, both $\{\psi(ry) - r\psi(y),[\xi(x)b - \psi(x),r]y,\xi(r)y\}$ and $\{[\xi(x),s],\xi(s),[s,\psi(x)]\}$ are $\mathscr{C}$-dependent. Thus, we have

$$\lambda_6(x,s)[\xi(x),s] + \lambda_7(x,s)\xi(s) + \lambda_8(x,s)[s,\psi(x)] = 0 \tag{28}$$

for some $\lambda_i(x,s) \in \mathscr{C}$, not all zero, $i = 1,2,3$ and $\forall\ x,s \in \mathscr{Q}$. If $\lambda_7(x,s) \neq 0$, then there is $0 \neq \lambda_7^{-1}(x,s) \in \mathscr{C}$, and from (28), we obtain

$$\lambda_7^{-1}(x,s)\lambda_6(x,s)[\xi(x),s] + \xi(s) + \lambda_7^{-1}(x,s)\lambda_8(x,s)[s,\psi(x)] = 0.$$

Putting $x = 0$ in the last relation, we arrive to $\xi(s) = 0$, a contradiction. Thus

$$\lambda_7(x,s) = 0 \tag{29}$$

$\forall\ x,s \in \mathscr{Q}$. By using (29) in (28)

$$\lambda_6(x,s)[\xi(x),s] + \lambda_8(x,s)[s,\psi(x)] = 0 \tag{30}$$

$\forall\ x,s \in \mathscr{Q}$. If

$$[\xi(x),s] = 0 \tag{31}$$

$\forall\ x,s \in \mathscr{Q}$. Then $[\xi(x),x] = 0$ and by [2, Lemma 2.1], we get $\xi = 0$ or $\mathscr{Q}$ is commutative. If $\xi = 0$, we obtain a contradiction. If $\mathscr{Q}$ is commutative then $\mathscr{A}$ is commutative, as desired. Now, we will assume that

$$[\xi(x),s] \neq 0 \tag{32}$$

$\forall\ x, s \in \mathscr{Q}$. In (30) if

$$[s, \psi(x)] = 0 \tag{33}$$

$\forall\ x, s \in \mathscr{Q}$, then

$$\psi(x) \in \mathscr{C} \tag{34}$$

$\forall\ x \in \mathscr{Q}$. Writing xy instead of x in (34), we have $\psi(x)y + bx\xi(y) \in \mathscr{C}$ and by using (34) in the last relation, we get

$$[bx\xi(y), y] = 0 \tag{35}$$

$\forall\ x \in \mathscr{Q}$. By using (26) in the last relation, we obtain $b\mathscr{Q}[x\xi(y), y] = 0$ but $b \neq 0$ and so $[x\xi(y), y] = 0$. Taking x by rx in the last relation, we find that $[r, y]x\xi(y) = 0$ and since $\xi \neq 0$, then $[r, y] = 0$, hence $\mathscr{Q}$ is commutative and so $\mathscr{A}$ is commutative, as desired. Now, we will assume that

$$[s, \psi(x)] \neq 0 \tag{36}$$

$\forall\ x, s \in \mathscr{Q}$. From (32), (36) and (29), then $\lambda_6(x, s) \neq 0 \neq \lambda_8(x, s)$ and so there are $\lambda_6^{-1}(x, s), \lambda_8^{-1}(x, s) \in \mathscr{C}$. From (30), then $[\xi(x), s] = -\lambda_6^{-1}(x, s)\lambda_8(x, s)$ $[s, \psi(x)]$. Putting $\lambda_0(x, s) = -\lambda_6^{-1}(x, s)\lambda_8(x, s)$ in the last relation, we have $[\xi(x), s] = \lambda_0(x, s)[s, \psi(x)]$, by using the last relation in (27), we get

$$(\lambda_0(x, s)((\psi(ry) - r\psi(y)) + \xi(r)y)t[s, \psi(x)] + [\xi(x)b - \psi(x), r]yt\xi(s) = 0.$$

Since $\xi(s) \neq 0$, then we have following two cases :

(**P**) $\lambda_0(x, s)((\psi(ry) - r\psi(y)) + \xi(r)y)t = [\xi(x)b - \psi(x), r] = 0$.

((**Q**)) At least one of $\lambda_0(x, s)((\psi(ry)-r\psi(y))+\xi(r)y)t \neq 0$ or $[\xi(x)b-\psi(x), r] \neq 0$. Then by Lemma 1.1, both $\{\lambda_0(x, s)((\psi(ry) - r\psi(y)) + \xi(r)y)t, [\xi(x)b - \psi(x), r]\}$ and $\{[s, \psi(x)], \xi(s)\}$ are $\mathscr{C}$-dependent. Thus, we have $\lambda_9(x, s)[s, \psi(x)] + \lambda_{10}(x, s)\xi(s) = 0$. for some $\lambda_9(x, s), \lambda_{10}(x, s) \in \mathscr{C}$ not both zero. In (**Q**), we have $[s, \psi(x)] \neq 0 \neq \xi(s)$ from (36) and (17). So $\lambda_9(x, s) \neq 0 \neq \lambda_{10}(x, s)$ and hence there are $\lambda_9^{-1}(x, s), \lambda_{10}^{-1}(x, s) \in \mathscr{C}$ and so $\xi(s) = -\lambda_{10}^{-1}(x, s)\lambda_9(x, s)[s, \psi(x)]$. Putting $x = 0$ in the last relation, we get a contradiction. In (**P**), we have $[\xi(x)b - \psi(x), r] = 0$ by using (26) in the last relation, we obtain $[b\xi(x) - \psi(x), r] = 0$. Replacing x by xt in the last relation, we find that $(b\xi(x) - \psi(x))[t, r] = 0$. Taking t by ts in the last relation, we see that $(b\xi(x) - \psi(x))t[s, r] = 0$ and so $b\xi(x) - \psi(x) = 0$ or $[s, r] = 0$. If $[s, r] = 0$, then $\mathscr{Q}$ is commutative and so $\mathscr{A}$ is commutative, as desired. If $b\xi(x) - \psi(x) = 0$, then $\psi(x) = b\xi(x)$ using (26), we get $\psi(x) = \lambda\xi(x)\ \forall\ x \in \mathscr{Q}$ for some $0 \neq \lambda \in \mathscr{C}$. Thus $\psi(x) = \lambda\xi(x)\ \forall\ x \in \mathscr{A}$ for some $0 \neq \lambda \in \mathscr{C}$, as desired. By using $\psi(x) = \lambda\xi(x)$ in $\psi(x)y + x\psi(y)$, then

$$\psi(x)y + x\psi(y) = \lambda\xi(x)y + x\lambda\xi(y) = \lambda\xi(xy) = \psi(xy) \tag{37}$$

and so ψ is a derivation, as desired.

Subcase II: If

$$[b,r] \neq 0 \tag{38}$$

$\forall\ r \in \mathscr{Q}$. By using (38) and (17) in (25), we have $\lambda_4(r) \neq 0 \neq \lambda_5(r)$ and so there is $\lambda_4^{-1}(r) \in \mathscr{C}$. From (25), we get $\xi(r) = \lambda_4^{-1}(r)\lambda_5(r)[r,b]$. Putting $\lambda(r) = \lambda_4^{-1}(r)\lambda_5(r) \in \mathscr{C}$, that is

$$\xi(r) = \lambda(r)[r,b] \tag{39}$$

$\forall\ r \in \mathscr{Q}$. By using (39) in (24), we obtain $(\lambda(r) - \lambda(s))[r,b]y[b,s] = 0$ and by using (38) in the last relation, we see that $\lambda(r) = \lambda(s) = \lambda \in \mathscr{C}$. From (39), gives

$$\xi(r) = \lambda[r,b] \tag{40}$$

$\forall\ r \in \mathscr{Q}$. We have

$$\psi(x) = \psi(1x) = \psi(1)x + b\xi(x)$$

putting $\psi(1) = a$ in the last relation, we get $\psi(x) = ax + b\xi(x)$ by using (40) in the last relation, we find that

$$\psi(x) = ax + b\lambda[x,b] \tag{41}$$

$\forall\ x \in \mathscr{Q}$. By using (40) and (41) in (18), we obtain

$$[\lambda[x,b], ax + \lambda b[x,b]] = 0$$

and since $\lambda \neq 0$, then

$$[[x,b], ax + \lambda b[x,b]] = 0 \tag{42}$$

$\forall\ x \in \mathscr{Q}$. By linearizing (42), we see that

$$[[x,b], ay + \lambda b[y,b]] + [[y,b], ax + \lambda b[x,b]] = 0.$$

Substituting λ for y in the last relation, we arrive to $[[x,b], a\lambda] = 0$ and so

$$[[x,b],a] = 0 \tag{43}$$

$\forall\ x \in \mathscr{Q}$. Writing xb instead of x in (43), we get $[x,b][b,a] = 0$ taking x by rx in the last relation, we obtain $[r,b]x[b,a] = 0$ but from (38), then $[r,b] \neq 0$ and so

$$[b,a] = 0. \tag{44}$$

By using Jacobi identity in (43), gives

$$[[b,a],x]+[[a,x],b]=0.$$

By using (44) in the last relation, we find that $[[a,x],b]=0$. Putting x by xr in the last relation, we see that $2[a,x][r,b]=0$ and since char($\mathscr{A}$)$\neq 2$, then $[a,x][r,b]=0$. Replacing r by rs in the last relation, we have $[a,x]r[s,b]=0$ but from (38), then $[s,b]\neq 0$ and so $[a,x]=0$, hence

$$a\in\mathscr{C}. \tag{45}$$

By using (45) in (42), we find that

$$a[[x,b],x]+\lambda[[x,b],b][x,b]=0 \tag{46}$$

$\forall\ x\in\mathscr{Q}$. By linearizing (46), we conclude that

$$a[[x,b],y]+a[[y,b],x]+\lambda[[x,b],b][y,b]+\lambda[[y,b],b][x,b]=0 \tag{47}$$

$\forall\ x,y\in\mathscr{Q}$. Taking $y=b$ in (47), we get $a[[x,b],b]=0$ by using (40) in the last relation, we have $a\xi^2(x)=0$ by using (45) in the last relation, we obtain $a=0$ or $\xi^2(x)=0$. In case

$$\xi^2(x)=0 \tag{48}$$

$\forall\ x\in\mathscr{Q}$. Substituting xy for x in the last relation, we see that $2\xi(x)\xi(y)=0$ and so $\xi(x)\xi(y)=0$, putting y by ry in the last relation, we find that $\xi(x)r\xi(y)=0$, hence $\xi=0$, a contradiction. If $a=0$ by using the last relation in (46), we get

$$[[x,b],b][x,b]=0 \tag{49}$$

$\forall\ x\in\mathscr{Q}$. Since $a=0$, then from (41) and (40), we arrive to $\psi(x)=b\xi(x)$ by using the last relation in (18), we have $[\xi(x),b\xi(x)]=0$ and so

$$[\xi(x),b]\xi(x)=0 \tag{50}$$

$\forall\ x\in\mathscr{Q}$. By linearizing (50), we see that

$$[\xi(x),b]\xi(y)+[\xi(y),b]\xi(x)=0 \tag{51}$$

$\forall\ x,y\in\mathscr{Q}$. Writing $y\xi(x)$ instead of y in (51) and using (50), we conclude that

$$[\xi(x),b]y\xi^2(x)+y[\xi^2(x),b]\xi(x)+[y,b]\xi^2(x)\xi(x)=0.$$

Taking y by ry in the last relation then left multiplying the last relation then subtract them, we get

$$[[\xi(x),b],r]y\xi^2(x)+[r,b]y\xi^2(x)\xi(x)=0 \tag{52}$$

$\forall\ x, y, r \in \mathscr{Q}$. Since $\xi^2(x) \neq 0$, otherwise we obtain a contradiction same as we did in (48) and $[r, b] \neq 0$. Then by Lemma 1.1, both $\{[[\xi(x), b], r], [r, b]\}$ and $\{\xi^2(x), \xi^2(x)\xi(x)\}$ are $\mathscr{C}$-dependent. Thus, we have

$$\lambda_1(x, r)[[\xi(x), b], r] + \lambda_2(x, r)[r, b] = 0 \tag{53}$$

$\forall\ x, r \in \mathscr{Q}$. and $\lambda_1(x, r), \lambda_2(x, r) \in \mathscr{C}$ not both zero. We have $[r, b] \neq 0$ from (38). If $[[\xi(x), b], r] \neq 0$, then $\lambda_1(x, r) \neq 0 \neq \lambda_2(x, r)$ and so there is $\lambda_2^{-1}(x, r) \in \mathscr{C}$ hence $[r, b] = -\lambda_2^{-1}(x, r)\lambda_1(x, r)[[\xi(x), b], r]$. Putting $x = 0$ in the last relation, we get $[r, b] = 0$, a contradiction. So $[[\xi(x), b], r] = 0$, that is $[\xi(x), b] \in \mathscr{C}$, by using (40) in the last relation, we find that $\lambda[[x, b], b] \in \mathscr{C}$ this implies that $[[x, b], b] \in \mathscr{C}$, by using the last relation in (49), we obtain $[[x, b], b]\mathscr{Q}[x, b] = 0$ but $[x, b] \neq 0$ thus $[[x, b], b] = 0$. Replacing x by xy in the last relation, we see that $2[x, b][y, b] = 0$ hence $[x, b][y, b] = 0$, taking y by yr in the last relation, we get $[x, b]y[r, b] = 0$, and so $b \in \mathscr{C}$, a contradiction with (38).

Case II: We have

$$\lambda_1(x, m)[\xi(x), m] + \lambda_2(x, m)\xi(m) + \lambda_3(x, m)[m, \psi(x)] = 0 \tag{54}$$

$\forall\ x, m \in \mathscr{Q}$. If $[\xi(x), m] = 0$ same as in (31), we get $\mathscr{Q}$ is commutative and hence, $\mathscr{A}$ is commutative , as desired. Now, we will assume that

$$[\xi(x), m] \neq 0 \tag{55}$$

$\forall\ x, m \in \mathscr{Q}$. If $[m, \psi(x)] = 0$ same as in (33) we can get (35), that is

$$[bx\xi(y), y] = 0 \tag{56}$$

$\forall\, x, y \in \mathscr{Q}$. Substituting bx for x in (56), we have $[b, y]bx\xi(y) = 0$ but $\xi \neq 0$ and so $[b, y]b = 0$, taking y by ry in the last relation, we obtain $[b, r]yb = 0$, then $[b, r] = 0$ or $b = 0$. If $b = 0$, we obtain a contradiction with (17). If $[b, r] = 0$, then $b \in \mathscr{C}$, by using the last relation in (56), we see that $b\mathscr{Q}[x\xi(y), y] = 0$ but $b \neq 0$ and so $[x\xi(y), y] = 0$, putting x by rx in the last relation, $[r, y]x\xi(y) = 0$ and since $\xi \neq 0$, then $[r, y] = 0$, that is, $\mathscr{Q}$ is commutative hence $\mathscr{A}$ is commutative, as desired. Now, if

$$[m, \psi(x)] \neq 0 \tag{57}$$

$\forall\, x, m \in \mathscr{Q}$. Now, if $\lambda_1(x, m) = 0$ in (54), then $\lambda_2(x, m) \neq 0 \neq \lambda_3(x, m)$ because of (17) and (57), also (54) is a $\mathscr{C}$-dependent. Thus there is $\lambda_2^{-1}(x, m) \in \mathscr{C}$ and so from (54), we have $\xi(m) = -\lambda_2^{-1}(x, m)\lambda_3(x, m)[m, \psi(x)]$. Taking $x = 0$ in the last relation, we found that $\xi(m) = 0$, a contradiction. So

$$\lambda_1(x, m) \neq 0 \tag{58}$$

$\forall\ x, m \in \mathscr{Q}$. Similarly, if $\lambda_3(x,m) = 0$, then we have a contradiction (using (55) and (58) then replace x by 0 in (54)) and so

$$\lambda_3(x,m) \neq 0 \tag{59}$$

$\forall\ x, m \in \mathscr{Q}$. If $\lambda_2(x,m) \neq 0$, and from (54), then

$$\lambda_2^{-1}(x,m)\lambda_1(x,m)[\xi(x),m] + \xi(m) + \lambda_2^{-1}(x,m)\lambda_3(x,m)[m,\psi(x)].$$

Putting $x = 0$ in the last relation, we obtain $\xi(m) = 0$, a contradiction. Hence $\lambda_2(x,m) = 0$, by using the last relation in (54), we see that

$$\lambda_1(x,m)[\xi(x),m] + \lambda_3(x,m)[m,\psi(x)] = 0.$$

By using (55),(57), (58) and (59) in the last relation, we conclude that there is $\lambda_3^{-1}(x,m) \in \mathscr{C}$ and so $[m,\psi(x)] = -\lambda_3^{-1}(x,m)\lambda_1(x,m)[\xi(x),m]$, putting $0 \neq \lambda(x,m) = -\lambda_3^{-1}(x,m)\lambda_1(x,m) \in \mathscr{C}$ hence

$$[m,\psi(x)] = \lambda(x,m)[\xi(x),m] \tag{60}$$

$\forall\ x, m \in \mathscr{Q}$. By using (60) in (22), we get

$$\begin{aligned}
&\{\psi(ry[b,s]k) - r\psi(y[b,s]k) - [b,r]y\psi(sk) + [b,r]ys\psi(k)\}t[\xi(x),m] \\
&+\{[\xi(x)b - \psi(x),r]y[b,s] - [b,r]y[\xi(x)b - \psi(x),s]\}kt\xi(m) \\
&+\{\xi(r)y[b,s] - [b,r]y\xi(s)\}kt\lambda(x,m)[\xi(x),m] = 0
\end{aligned}$$

$\forall\ x, y, t, r, s, k, m \in \mathscr{Q}$. Hence,

$$\begin{aligned}
&\{\{\psi(ry[b,s]k) - r\psi(y[b,s]k) - [b,r]y\psi(sk) + [b,r]ys\psi(k)\} \\
&+\{\xi(r)y[b,s] - [b,r]y\xi(s)\}k\lambda(x,m)\}t[\xi(x),m] \\
&+\{[\xi(x)b - \psi(x),r]y[b,s] - [b,r]y[\xi(x)b - \psi(x),s]\}kt\xi(m)
\end{aligned}$$

$\forall\ x, y, t, r, s, k, m \in \mathscr{Q}$. Since $\xi(m) \neq 0$ using by Lemma 1.1, we have following two cases

Case A:

$$\begin{aligned}
&(\{\psi(ry[b,s]k) - r\psi(y[b,s]k) - [b,r]y\psi(sk) + [b,r]ys\psi(k)\} \\
&+\{\xi(r)y[b,s] - [b,r]y\xi(s)\}k\lambda(x,m)) \\
&=\{[\xi(x)b - \psi(x),r]y[b,s] - [b,r]y[\xi(x)b - \psi(x),s]\}k = 0.
\end{aligned}$$

Case B:

$$\begin{aligned}
&(\{\psi(ry[b,s]k) - r\psi(y[b,s]k) - [b,r]y\psi(sk) + [b,r]ys\psi(k)\} \\
&+\{\xi(r)y[b,s] - [b,r]y\xi(s)\}k\lambda(x,m)) \neq 0
\end{aligned}$$

or

$$\{[\xi(x)b-\psi(x),r]y[b,s]-[b,r]y[\xi(x)b-\psi(x),s]\}k\neq 0.$$

In Case B, since $\xi(m)\neq 0$, by Lemma 1.1, we get

$\lambda_{11}(x,m)[\xi(x),m]+\lambda_{12}(x,m)\xi(m)=0$ for some $\lambda_{11}(x,m),\lambda_{12}(x,m)\in\mathscr{C}$ not both zero.

We have $[\xi(x),m]\neq 0\neq\xi(m)$ from (17) and (55) and so there is $\lambda_{12}^{-1}(x,m)\in\mathscr{C}$ and so $\xi(m)=-\lambda_{12}^{-1}(x,m)\lambda_{11}(x,m)[\xi(x),m]$, putting $x=0$ in the last relation, we get $\xi(m)=0$, a contradiction. In Case A, we have

$$\begin{aligned}&\{\psi(ry[b,s]k)-r\psi(y[b,s]k)-[b,r]y\psi(sk)+[b,r]ys\psi(k)\}\\&+\{\xi(r)y[b,s]-[b,r]y\xi(s)\}k\lambda(x,m)=0\end{aligned}\tag{61}$$

$\forall\ x,y,r,s,k,m\in\mathscr{Q}$.

Replacing m with t and x with l in (61), we obtain the relation

$$\{\xi(r)y[b,s]-[b,r]y\xi(s)\}k(\lambda(x,m)-\lambda(l,t))=0.$$

This implies that either

$$\xi(r)y[b,s]-[b,r]y\xi(s)=0,$$

which is similar to (24), or

$$\lambda(x,m)-\lambda(l,t)=0$$

for all $x,m,l,t\in\mathscr{Q}$.

In the case where $\lambda(x,m)-\lambda(l,t)=0$, we can fix $l,t\in\mathscr{Q}$ and denote $\lambda=\lambda(l,t)$. It follows that $0\neq\lambda=\lambda(x,m)\in\mathscr{C}$. Using (60), we then deduce that

$$[m,\psi(x)]=\lambda[\xi(x),m],$$

which leads to

$$[\psi(x)-\lambda\xi(x),m]=0\tag{62}$$

for all $x,m\in\mathscr{Q}$. Substituting xy for x in (62), we have

$$(\psi(x)-\lambda\xi(x))[y,m]+[(b-\lambda)x\xi(y),m]=0\tag{63}$$

$\forall\ x,y,m\in\mathscr{Q}$. Putting $m=y$ in (63), we get

$$[(b-\lambda)x\xi(y),y]=0\tag{64}$$

$\forall\ x,y\in\mathscr{Q}$. Replacing x by $(b-\lambda)x$ in (64), we see that $[(b-\lambda),y](b-\lambda)x\xi(y)=0$ but $\xi\neq 0$, then $[(b-\lambda),y](b-\lambda)=0$ and so $[b,y](b-\lambda)=0$. Writing ry

instead of y in the last relation, we obtain $[b,r]y(b-\lambda)=0$ hence $[b,r]=0$ or $b-\lambda=0$. If $b-\lambda=0$, then $b=\lambda$ by using the last relation in (63), we find that $(\psi(x)-\lambda\xi(x))[y,m]=0$, replacing y by ry in the last relation, we see that $(\psi(x)-\lambda\xi(x))r[y,m]=0$ and so $\psi(x)-\lambda\xi(x)=0$ or $[y,m]=0$. If $[y,m]=0$, then $\mathscr{Q}$ is commutative hence $\mathscr{A}$ is commutative, as desired. If $\psi(x)-\lambda\xi(x)=0$, then $\psi(x)=\lambda\xi(x)$, $\forall\ x\in\mathscr{Q}$ and from (37), then ψ is a derivation and so $\psi(x)=\lambda\xi(x)$, $\forall\ x\in\mathscr{A}$, as desired. Now, if $[b,r]=0$, then $b\in\mathscr{C}$, by using the last relation in (64), we conclude that $(b-\lambda)\mathscr{Q}[x\xi(y),y]=(0)$ and so $b-\lambda=0$ or $[x\xi(y),y]=0$. If $b-\lambda=0$, then $b=\lambda$ same above. If $[x\xi(y),y]=0$. Replacing x by rx in the last relation, we see that $[r,y]x\xi(y)=0$ but $\xi\neq 0$ and so $[r,y]=0$, then $\mathscr{Q}$ is commutative and hence $\mathscr{A}$ is commutative, as desired. ☑

Corollary 2.5. *Let $\mathscr{A}$ be a prime ring with char$(\mathscr{A})\neq 2$ and let $\psi:\mathscr{A}\to\mathscr{Q}$ be a non-zero b-generalized derivation with associated map ξ. Suppose that any one of the following holds:*

(1) $[\xi(x),\psi(y)]=0$

(2) $[\xi(x),\psi(y)]\pm[x,y]=0$

(3) $[\xi(x),\psi(y)]\pm\psi([x,y])=0$

(4) $[\xi(x),\psi(y)]\pm\xi([x,y])=0$

(5) $[\xi(x),\psi(y)]\pm[\xi(x),\xi(y)]=0$

(6) $[\xi(x),\psi(y)]\pm[\psi(x),\psi(y)]=0$

(7) $[\xi(x),\psi(y)]+[\xi(y),\psi(x)]=0\ \forall\ x,y\in\mathscr{A}$.

Then one of the following conclusions holds:

(1) $\psi(x)=qx\ \forall\ x\in\mathscr{A}$ *and some* $0\neq q\in\mathscr{Q}$

(2) Any b-generalized derivation ψ is a derivation and $\psi(x)=\lambda\xi(x)\ \forall\ x\in\mathscr{A}$ and some $0\neq\lambda\in\mathscr{C}$ and $b=\lambda$

(3) $\mathscr{A}$ is commutative.

Proof. Putting $y=x$ in from (1) to (7) and using Theorem 2.4, we get the required conclusions. ☑

Theorem 2.6. *Let $\mathscr{A}$ be a prime ring with char$(\mathscr{A})\neq 2$ and let $\psi:\mathscr{A}\to\mathscr{Q}$ be a non-zero b-generalized derivation with associated map ξ. Suppose that $\xi(x)\circ\psi(x)=0\ \forall\ x\in\mathscr{A}$. Then either*

(1) $\psi(x)=qx\ \forall\ x\in\mathscr{A}$ *and some* $0\neq q\in\mathscr{Q}$ *or*

(2) Any b-generalized derivation ψ is a derivation and $\psi(x) = \lambda\xi(x)\ \forall\ x \in \mathscr{A}$ and some $0 \neq \lambda \in \mathscr{C}$ and $b = \lambda$.

Proof. In case, $b = 0$ or $\xi = 0$ using the same techniques as used in Theorem 2.4, we have $\psi(x) = qx\ \forall\ x \in \mathscr{A}$ and for some $0 \neq q \in \mathscr{Q}$. By our hypothesis, we have

$$\xi(x) \circ \psi(x) = 0 \tag{65}$$

$\forall\ x \in \mathscr{A}$. By linearizing (65), we have

$$\xi(x) \circ \psi(y) + \xi(y) \circ \psi(x) = 0 \tag{66}$$

$\forall\ x, y \in \mathscr{A}$. By [17, Theorem 2.3], d is a derivation. Substituting yt for y in (66), we get

$$\psi(y)[\xi(x), t] - by(\xi(x) \circ \xi(t)) - [\xi(x), by]\xi(t) - \xi(y)[t, \psi(x)] - y(\xi(t) \circ \psi(x)) + [y, \psi(x)]\xi(t) = 0$$

$\forall\ x, y, t \in \mathscr{A}$. Replacing y by ry in the last relation then left multiplying the last relation by r then subtract them, we see that

$$(\psi(ry) - r\psi(y))[\xi(x), t] - [b, r]y\xi(t)\xi(x) + [r, \xi(x)b + \psi(x)]y\xi(t) - \xi(r)y[t, \psi(x)] = 0$$

$\forall\ x, y, t, r \in \mathscr{A}$. By Lemma 1.4, $\mathscr{A}$ and $\mathscr{Q}$ satisfy the same differential identities and by Lemma 1.5, then ψ can be uniquely extended to $\mathscr{Q}$ and so

$$(\psi(ry) - r\psi(y))[\xi(x), t]v - [b, r]y\xi(t)\xi(x) + [r, \xi(x)b + \psi(x)]y\xi(t) - \xi(r)y[t, \psi(x)] = 0$$

$\forall\ x, y, t, r \in \mathscr{Q}$. The arguments presented in the proof of Theorem 2.4 after Equation (21) still true in the current context, thus we may acquire the requisite conclusion by repeating the process. ☑

Remark 2.7. *In Corollary 2.5, we can replace the conditions $[\xi(x), \psi(y)] = 0$ and $[\xi(y), \psi(x)] = 0$ with $\xi(x) \circ \psi(y) = 0$ and $\xi(y) \circ \psi(x) = 0$, respectively, by using Theorem 2.6. Then, one of the following conclusions holds:*

(1) $\psi(x) = qx$ for all $x \in \mathscr{A}$ and some $0 \neq q \in \mathscr{Q}$.

(2) Any b-generalized derivation ψ is a derivation, and $\psi(x) = \lambda\xi(x)$ for all $x \in \mathscr{A}$ and some $0 \neq \lambda \in \mathscr{C}$, with $b = \lambda$.

The following examples show that in all of our theorems, the requirement primeness of $\mathscr{A}$ cannot be neglected.

Example 2.8. Consider the ring $\mathscr{A} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$ and $char(\mathscr{A}) \neq 2$. Let $\psi : \mathscr{A} \to \mathscr{Q}$ be a non-zero b-generalized derivation associated with a map ξ such that $\psi \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} 0 & -y \\ 0 & 0 \end{pmatrix} = \xi \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $[\psi(\mathscr{X}), \psi(\mathscr{Y})] = 0$ and $\psi(\mathscr{X}) \circ \psi(\mathscr{Y}) = 0$ in Theorems 2.2 and 2.3 $\forall$ $\mathscr{X}, \mathscr{Y} \in \mathscr{A}$. Now, there is $\mathscr{T} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ such that $\psi(\mathscr{T}) \neq \lambda \mathscr{T}$ $\forall$ $\lambda \in \mathscr{C}$, $\psi(\mathscr{T}) \neq -b\mathscr{T}q$ and $qb \neq 0$ $\forall$ $0 \neq q = \begin{pmatrix} q_1 & q_2 \\ q_3 & q_3 \end{pmatrix} \in \mathscr{Q}$. Also $\mathscr{A}$ is non-commutative. Note that $\mathscr{A}$ is not prime because $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \mathscr{A} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. But $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq 0 \neq \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Example 2.9. Consider $\mathscr{A}, \psi, \xi, \mathscr{T}, q$ same is in Example 2.8 with $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $[\xi(\mathscr{X}), \psi(\mathscr{X})] = 0$ and $\xi(\mathscr{X}) \circ \psi(\mathscr{X}) = 0$ in Theorems 2.4 and 2.6 $\forall$ $\mathscr{X} \in \mathscr{A}$, and there is $\mathscr{T} \in \mathscr{A}$ ($\mathscr{T}$ same is in Example 2.8) such that $\psi(\mathscr{T}) \neq q\mathscr{T}$ $\forall$ $q \in \mathscr{Q}$ and $\psi(\mathscr{X}) = \lambda \xi(\mathscr{X})$ where $\lambda = 1$ but $b \neq \lambda$, because $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b \neq b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Note that $char(R) \neq 2$, $\mathscr{A}$ is non-commutative and it is not prime.

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