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Inertial Halpern-type method for solving monotone variational inequality and fixed point problems in Banach spaces

Método inercial de tipo Halpern para resolver desigualdades variacionales monótonas y problemas de punto fijo en espacios de Banach

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Abstract. In this paper, we introduce inertial Tseng's method and Halpern-type algorithm for solving monotone variational inequality and fixed point problems in 2-uniformly convex and 2-uniformly smooth real Banach spaces. We establish strong convergence of our proposed method under some assumptions on parameters without knowledge of the operator norm. Finally, we give numerical experiments to illustrate the efficiency of our main result.

Key words and phrases. Inertial method, Halpern Tseng's extradient subgradient method, monotone variational inequality problem, demigeneralized mapping, Strong convergence, Banach space.

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Resumen. En este artículo aplicamos el método de Tseng y algoritmos de tipo Halpern para resolver problema de desigualdad variacional monótona y problemas de punto fijo en espacios reales de Banach 2-uniformemente convexos y 2-uniformemente suaves. Probamos la convergencia fuerte del método propuesto bajo hipótesis sobre los parámetros que no dependen de la norma del operador. Finalmente presentamos ejemplos numéricos que ilustran nuestros resultados.

Palabras y frases clave. método inercial, método de Halpern Tseng de subgradiente, problema de desigualdad variacional monótona, convergencia fuerte, espacio de Banach.

1. Introduction

Let E be a real Banach space with its dual E^*. Let C be a nonempty closed convex subset of E and $\langle \cdot, \cdot \rangle$ denote the duality pairing between E and E^*. The variational inequality problem (VIP) with respect to F is a problem of finding $u \in C$ such that

$$\langle u - v, Fu \rangle \leq 0, \quad \forall\, v \in C, \tag{1}$$

where F is an operator from C into E^*.We denote by $VI(C, F)$ the set of solutions to the variational inequality problem (1). The theory of VIP plays vital role as a suitable tool for studying many real-life problems arising in game theory, engineering, economics, optimization theory, network distribution, image processing, and generalized Nash equilibrium (see [6, 9, 11, 12, 24]). It is well known in Hilbert space H that u solves (1) if and only if it solves the following fixed point equation: Find $u \in C$ such that for all $\lambda > 0$,

$$u = P_C(I - \lambda F)(u). \tag{2}$$

Several iterative algorithms for finding the approximate solutions of the VIP (1) have been introduced and analyzed by many authors in the setting of Hilbert space H, among which is the well-known classical projected gradient method that generates a sequence $\{x_n\}$ iteratively by

$$x_0 \in C, \quad x_{n+1} = P_C(x_n - \lambda F(x_n)),$$

where P_C is the metric projection of H onto C and λ is a positive real number. Another notable method for solving the VIP (1) which requires two projections per iteration onto the feasible set C is the extragradient method of Korpelevich [13]. This method generates a sequence $\{x_n\}$ by

$$x_0 \in C,\ y_n = P_C(x_n - \lambda F(x_n)), \quad x_{n+1} = P_C(x_n - \lambda F(y_n)), \tag{3}$$

where $\lambda \in (0, \frac{1}{L})$ and $L > 0$ is a Lipschitz constant of F. Due to the double projections, the implementation of the method (3) is observed to be complicated and computationally costly when C is not a simple convex set. To overcome this setback, Censor et al. [32, 31] introduced the subgradient extragradient method

(SEM) by replacing the second projection onto the whole feasible set C with a constructible half-space T_n. In particular, the SEM generates a sequence $\{x_n\}$ iteratively by

$$\begin{cases} x_0 \in C; \lambda > 0, \\ y_n = P_C(x_n - \lambda F(x_n)), \\ T_n = \{x \in H : \langle x_n - Fx_n - y_n, x - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda F(y_n)), \;\; n \geq 0. \end{cases} \tag{4}$$

They proved that the sequence $\{x_n\}$ in (4) converges weakly to a solution of VIP (1) provided that L is the Lipschitz constant of F and the stepsize λ satisfies $\lambda \in (0, \frac{1}{L})$. Jolaoso et al. [15] successfully employed the inertial technique to accelerate the convergence process of the subgradient extragradient algorithm for variational inequality and common fixed point problems in Hilbert spaces. Recently, the extragradient method and SEM have been constructively modified and developed by many authors (see Shehu [22], Shehu et al. [33], Solodov et al. [23], Tseng [27] and references therein).

In another development, the problem of finding a common element in the solution set of VIP and the fixed point problem (FPP) of certain nonlinear mappings is very vital in convex and nonlinear analysis. This is due to the fact that the constraints of some optimization problems can be modeled as VIP and FPP. In 2018, Chidume and Nnakwe [7] introduced a modified subgradient extragradient method for finding common elements in the solution sets of VIP and FPP in the 2-uniformly convex real Banach space E. The method iteratively generates a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C; \lambda \in (0, \frac{1}{L}) \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda A(x_n)); \\ T_n = \{x \in E : \langle Jx_n - \lambda Ax_n - Jy_n, x - y_n \rangle \leq 0\}; \\ x_{n+1} = J^{-1}(\beta Jx_n + (1-\beta) JS\Pi_{T_n} J^{-1}(Jx_n - \lambda A(y_n)), \quad n \geq 0, \end{cases} \tag{5}$$

where Π_C is the generalized projection of E onto C and J is the normalized duality mapping of E into 2^{E^*} and $\beta \in (0,1)$. They proved that the sequence generated by the algorithm converges weakly to a common element in the solution set of the VIP and the fixed point of a relatively nonexpansive mapping S. To establish strong convergence, the Subgradient Extragradient Method (SEM) is combined with the Halpern method by Liu [16] in Banach spaces. Liu [16] iteratively defined a sequence $\{x_n\}$ by

$$\begin{cases} x_0 \in C; \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A(x_n)); \\ T_n = \{x \in E : \langle Jx_n - \lambda_n Ax_n - Jy_n, x - y_n \rangle \leq 0\}; \\ w_n = \Pi_{T_n} J^{-1}(Jx_n - \lambda_n A(y_n)); \\ x_{n+1} = J^{-1}(\alpha_n Jx_0 + (1-\alpha_n) Jw_n; \quad n \geq 0, \end{cases} \tag{6}$$

where $\{\alpha_n\} \subset [0,1]$ is a sequence satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\lambda_n\} \subset (0,\infty)$ and $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \frac{1}{L}$. For more similar methods of finding a common solution for VIP and FPP in Hilbert and Banach spaces readers may see, for example, [14, 15].

In order to speed up the convergence process of algorithms for solving variational inequality and fixed point problems, Khan et al. [1] introduced a modified inertial Halpern-type subgradient extragradient method for approximating a common solution of the monotone variational inequality problem and the fixed point of demigeneralized mapping in a 2-uniformly convex and uniformly smooth real Banach space. Since then, many researchers have studied variational inequality problems in Banach spaces using the modified inertial method (see [10, 5, 20], and the references therein).

Motivated and inspired by the results of the authors mentioned above, it is our aim in this paper to introduce and analyze the normal inertial method with Halpern-type Tsen's Subgradient Extragradient Algorithm for approximation of a common element in sets of solutions of the monotone variational inequality problem and the fixed point problem for the demigeneralized mappings in the setting of uniformly smooth and 2-uniformly convex real Banach spaces. Under some assumptions imposed on the parameters, we prove that our proposed method converges strongly to some common solution of the problems under study. Finally, we present numerical experiments to show the efficiency of the algorithmic method constructed.

2. Preliminaries

In this section, we present some preliminary definitions and concepts which are needed for the establishment of the main result of this paper. Let E be a real Banach space with dual E^*, let $S_E(x) := \{x \in E : ||x|| = 1\}$ denote the unit sphere of E. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x, x^* \rangle$. Also, we denote the strong (resp. weak) convergence of a sequence $\{x_n\} \subset E$ to a point $x \in E$ by $x_n \to x$ (resp. $x_n \rightharpoonup x$). A Banach space E is said to be *smooth* if for each $x, y \in S_E$, $\lim_{t\to 0} \frac{||x+ty||-||x||}{t}$ exists. If for all $x, y \in S_E$ with $x \neq y$, for any $\lambda \in (0,1)$, $||\lambda x + (1-\lambda)y|| < 1$, then E is called *strictly convex.* The space E is said to be *uniformly convex* if for any $\epsilon \in (0,2]$ there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in S_E$, $||x-y|| \geq \epsilon$, we have $\frac{||x+y||}{2} \leq 1 - \delta$. The *modulus of convexity* of E is the function $\delta_E : [0,2] \to [0,1]$ defined for all $\epsilon \in [0,2]$ by

$$\delta_E(\epsilon) = \begin{cases} \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = 1, \|y\| = 1, \|x-y\| \geq \epsilon\right\}, \text{ if } 0 < \epsilon \leq 2; \\ 0 \text{ if } \epsilon = 0. \end{cases}$$

In terms of modulus of convexity, the space E is said to be uniformly convex if and only if for all $\epsilon \in (0,2]$, we have that $\delta_E(\epsilon) > 0$; and for $p \in (1,+\infty)$, the space E is said to be p-uniformly convex if and only if there exists a constant $c_p > 0$ such that $\epsilon \in (0,2]$, $\delta_E(\epsilon) \geq c_p\epsilon^p$. It is obvious that every p-uniformly convex real normed space is uniformly convex.

The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ for all $\tau > 0$ defined by

$$\rho_E(\tau) = \sup\Big\{\frac{||x+\tau y|| + ||x-\tau y||}{2} - 1 : x, y \in S_E\Big\}.$$

The space E is called *uniformly smooth* if $\lim\limits_{\tau\to 0}\frac{\rho_E(\tau)}{\tau} = 0$; and E is called *q-uniformly smooth* if there exits a positive real number C_q such that for any $\tau > 0$, $\rho_E(\tau) \leq C_q\tau^q$. Hence, every q-uniformly smooth Banach space is uniformly smooth. We know that the space L_p, ℓ_p and W_p^m for $1 \leq p < 2$ are 2-uniformly convex and uniformly smooth (see [29] for more details).

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) := \{f \in E^* : \langle x, f\rangle = ||f||^2 = ||x||^2\}.$$

It is known that J has the following properties (for more details see [8, 21, 25]):

(a) If E is smooth, then J is single-valued.

(b) If E is uniformly smooth, then J is norm to norm uniformly continuous on bounded subset of E.

(c) If E is uniformly smooth, then the dual space E^* is uniformly convex; and if E is and uniformly convex, then the dual space E^* is uniformly smooth. Furthermore, J and J^{-1} are both uniformly continuous on bounded subsets of E and E^*, respectively.

(d) If E is a reflexive, strictly convex and smooth Banach space, then J^{-1} (the duality mapping from E^* into E) is single-valued, one to one and onto.

Let $\phi : E \times E \to [0,\infty)$ denote the Lyapunov functional in sense of Alber [2] defined $\forall\ x, y \in E$ by

$$\phi(x,y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2. \tag{7}$$

The functional ϕ satisfies the following properties (see [19]): $\forall\ x, y, z \in E$

(P1) $(||x|| - ||y||)^2 \leq \phi(x,y) \leq (||x|| + ||y||)^2$, (P2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle$,

(P3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq ||x||||Jx - Jy|| + ||y - x||||y||$,;

(P4) $\phi(z, J^{-1}(\alpha Jx + (1-\alpha)Jy)) \leq \alpha\phi(z,x) + (1-\alpha)\phi(z,y)$, where $\alpha \in (0,1)$ and $x, y \in E$.

Now, we introduce another functional $V : E \times E^* \to [0, \infty)$ by [2], which is a mild modification and have a relationship with Lyapunov functional in (7) as follows: for all $x \in E$ and $x^* \in E^*$

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2 \tag{8}$$

From the Definition of ϕ in (7), we get for all $x \in E$ and $x^* \in E^*$

$$V(x, x^*) = \phi(x, J^{-1}(x^*)). \tag{9}$$

For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous, convex function from E^* into $\mathbb{R}$.

Let E a be reflexive, strictly convex and smooth Banach space and C a nonempty closed and convex subset of E. Then by [2], for each $x \in E$, there exists a unique element $u \in C$ (denoted by $\Pi_C x$) such that

$$\phi(u, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$, defined by $\Pi_C x = u$ is called the generalized projection operator (see [3]), which have the following important characteristic.

Lemma 2.1 ([4])**.** *Let C be a nonempty, closed and convex subset of a smooth Banach space E, then $u = \Pi_C x$ if and only if*

$$\langle u - w, Jx - Ju\rangle \geq 0, \quad \forall w \in C.$$

Lemma 2.2. (see [18]) *Let E be a reflexive, strictly convex and smooth Banach space and C be a nonempty closed and convex subset of E. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall\ y \in C.$$

Lemma 2.3. [2] *Let E be a reflexive, strictly convex and smooth Banach space and let V be as in (8). Then, for all $x \in E$ and $x^*, y^* \in E^*$*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*).$$

Lemma 2.4 ([19])**.** *Let E be a uniformly smooth real Banach space and $r > 0$. Then there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to [0, \infty]$ such that $g(0) = 0$ and*

$$\phi(u, J^{-1}(tJv + (1-t)Jw)) \leq t\phi(u,v) + (1-t)\phi(u,w) - t(1-t)g(||Jv - Jw||)$$

for all $t \in [0,1]$, $u \in E$ and $v, w \in B_r := \{z \in E : ||z|| \leq r\}$.

Lemma 2.5. [25] *Let C be a nonempty, closed and convex subset of X and $F : C \to X^*$ be monotone and continuous mapping. For any $y \in C$, we have*

$$y \in VI(C, F) \;\Leftrightarrow\; \langle F(z), z - y\rangle \geq 0 \;\; \forall\, z \in C.$$

Lemma 2.6 ([29])**.** *Let E be a uniformly convex real Banach space and $r > 0$. Let $B_r(0) := \{x \in E : ||x|| \leq r\}$. Then, there exists a continuous strictly increasing convex function $g : [0, 2r] \to \mathbb{R}$, such that $g(0) = 0$ and*

$$||\beta y + (1-\beta)z||^2 \leq \beta||y||^2 + (1-\beta)||y||^2 - \beta(1-\beta)g(||y - z||), \tag{10}$$

for all $y, z \in B_r$ and $\beta \in [0, 1]$.

Lemma 2.7 ([18])**.** *Let E be a uniformly convex and smooth real Banach space and $\{u_n\}$ and $\{v_n\}$ be two sequences in E. If $\lim_{n\to\infty} \phi(u_n, v_n) = 0$ and either $\{u_n\}$ or $\{v_n\}$ is bounded, then $\lim_{n\to\infty} ||u_n - v_n|| = 0$.*

Lemma 2.8 ([29])**.** *Let E be a 2-uniformly convex and smooth real Banach space. Then, there exists a positive real-valued constant α such that*

$$\alpha||x - y||^2 \leq \phi(x, y), \qquad \forall\, x, y \in E.$$

Lemma 2.9 ([29])**.** *Let E be a 2-uniformly smooth real Banach space, then there exists $s_0 > 0$ such that for all $x, y \in E$, the following holds*

$$||x + y||^2 \leq ||x||^2 + 2\langle x, Jy\rangle + 2s_0^2||y||^2.$$

Definition 2.10. Let C be a nonempty subset of E and let $T : C \to E$ be a mapping. A point $x^* \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to x^* and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. The set of asymptotic fixed points of T is denoted by $\tilde{\mathcal{F}}(T)$.

A mapping $T : C \to E$ is said to be:

(i) relatively nonexapnsive (see [18]) if $\mathcal{F}(T) \neq \varnothing$, $\mathcal{F}(T) = \tilde{\mathcal{F}}(T)$ (where $\mathcal{F}(T) = \{x \in C : Tx = x\}$), and

$$\forall\, x \in C,\; p \in \mathcal{F}(T), \quad \phi(p, Tx) \leq \phi(p, x).$$

(ii) firmly nonexpansive type if for all $x, y \in C$, we have

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \leq \phi(Tx, y) + \phi(Ty, x).$$

Recently, Takahashi et al. [28] introduced a new class of mappings as follows: Let E be a smooth Banch space and let J be the duality mapping on E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. Then a mapping $U : C \to E$ with $\mathcal{F}(U) \neq \varnothing$ is called $(\eta, s)-$demigeneralized if, for any $x \in C$ and $q \in \mathcal{F}(U)$, we have that

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux) + s\phi(Ux, x); \tag{11}$$

in particular, an $(\eta, 0)-$demigeneralized mapping satisfies

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux). \tag{12}$$

Lemma 2.11. (Takahashi et al., [28]) *Let E be a smooth Banach space and C be a nonempty closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$ and s be a real number with $s \in [0, \infty)$. Let U be an $(\eta, s)-$demigeneralized mapping of C into E. Then*

(i) $\mathcal{F}(U)$ is closed and convex.

(ii) For any $\alpha \in [0, 1)$, let $T = J^{-1}(\alpha J + (1 - \alpha)JU)$, where J is the duality mapping on E. If U is $(\eta, 0)-$demigeneralized mapping, then $T : C \to E$ is also $(\eta, 0)-$demigeneralized mapping.

Lemma 2.12. [1] *Let E be a smooth Banach space and C a nonempty closed and convex subset of E. Let $k \in (-\infty, 0]$ and $T : C \to E$ be a $(k, 0)-$demigeneralized mapping with $\mathcal{F}(T) \neq \varnothing$. Let λ be a real number in $(0, 1]$ and define $S = J^{-1}((1 - \lambda)J + \lambda JT)$, where J is the duality mapping on E. Then*

(i) $\mathcal{F}(T) = \mathcal{F}(S)$,

(ii) S is relatively-nonexpansive mapping of C into E.

Lemma 2.13 ([30])**.** *If $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the following inequality:*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, n \geq 0,$$

where, (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$ and $\sum \gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

Definition 2.14. A self-mapping T on a Banach space is said to be demiclosed at y if for any sequence $\{x_n\}$ which converges weakly to x, and the sequence $\{Tx_n\}$ converges strongly to y, then $T(x) = y$. In particular, if $y = 0$, then T is demiclosed at 0.

Lemma 2.15 ([17])**.** *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$.*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

3. Results

In this section we introduce inertial Halpern technique with Tseng's extragradient method for monotone variational inequality problem. First, we make the following assumptions:

Assumption.

(A1) C is a nonempty closed and convex subset of 2-uniformly smooth and 2-convex real Banach space E with dual E^*.

(A2) $F : C \to E^*$ is a monotone and L-Lipschitz continuous mapping with $L > 0$.

(A3) The mapping F is weakly sequentially continuous, i.e., for each sequence $\{x_n\} \subset C$, we have $F(x_n) \rightharpoonup F(x)$ whenever $x_n \rightharpoonup x$ as $n \to \infty$.

(A4) $T : E \to E$ be a $\tau-$demigeneralized mapping with $\tau \in (-\infty, 1)$ such that T is demiclosed at zero and $\mathcal{F}(T) \neq \varnothing$.

(A5) $\{\alpha_n\} \subset (0, 1)$ is a sequence with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$; $\{\gamma_n\}$ a positive sequence in $(0, \alpha/2)$ such that $\gamma_n = \circ(\alpha_n)$, where α is defined in Lemma 2.8; $\theta > 0$ $\delta \in \big(0, \frac{1}{s_0\sqrt{2}}\big)$, where $s_0 > 0$ is as in Lemma 2.9.

(A6) J is a normalized duality mapping on E and $\Gamma := V(C, F) \cap \mathcal{F}(T)$ is nonempty set.

We now state and prove the following theorem:

Theorem 3.1. *Suppose conditions (A1) to (A6) are satisfied. Let $\{x_n\}_{n=1}^{\infty}$ be sequence generated by Algorithm 3, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some point in Γ.*

Proof. Observe that from the definition of $\{\lambda_n\}$, we have that $\forall\ n \in \mathbb{N} \cup \{0\}$, $\lambda_{n+1} \leq \lambda_n$. So, the sequence $\{\lambda_n\}$ is monotone nonincreasing. Thus, $\forall\ n \in \mathbb{N}\cup\{0\}$, $\lambda_n \leq \lambda_0$. Moreover, for all $n \in \mathbb{N}\cup\{0\}$, $\min\big\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\big\} \leq \lambda_n$. To see this, we make use of mathematical induction. Observe that $\min\big\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\big\} \leq \lambda_0$. Assuming for some $k \geq 0$ that $\min\big\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\big\} \leq \lambda_k$, we show that $\min\big\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\big\} \leq \lambda_{k+1}$. Using the fact that F is L-Lipschitz mapping, we observe that if $||F(z_k) - F(y_k)|| \neq 0$, then

$$\frac{\delta\sqrt{\alpha}||z_k - y_k||}{||F(z_k) - F(y_k)||} \geq \frac{\delta\sqrt{\alpha}||z_k - y_k||}{L||z_k - y_k||} = \frac{\delta\sqrt{\alpha}}{L} \geq \min\Big\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\Big\}.$$

Initialization: Choose $x_0, x_1 \in C$ to be arbitrary.
Iterative Steps: Calculate x_{n+1} as follows:

Step 1. Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose r_n such that $0 \leq r_n \leq \bar{r_n}$, where

$$\bar{r}_n = \begin{cases} \min\{\theta, \frac{\gamma_n}{\|J(x_n)-J(x_{n-1})\|}\}, & if \quad x_n \neq x_{n-1}, \\ \\ \theta, & otherwise \end{cases} \tag{13}$$

Step 2. Compute

$$\begin{cases} z_n = J^{-1}[J(x_n) + r_n(J(x_n) - J(x_{n-1}))]; \\ y_n = \Pi_C[J^{-1}(J(z_n) - \lambda_n F(z_n))]; \end{cases} \tag{14}$$

where the step size λ_n is chosen as follows: for some $\lambda_0 > 0$, for all $n \in \mathbb{N}$,

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \frac{\delta\sqrt{\alpha}||z_n - y_n||}{\|F(z_n)-F(y_n)\|}\right\}, & if \quad F(z_n) - F(y_n) \neq 0, \\ \\ \lambda_n, & otherwise \end{cases} \tag{15}$$

If $x_n = z_n = y_n$, then stop for x_n is a solution. Otherwise,

Step 3. Compute

$$\begin{cases} w_n = J^{-1}(J(y_n) + \lambda_n(F(z_n) - F(y_n))); \\ v_n = J^{-1}((1-\mu_n)J(w_n) + \mu_n J(Tw_n)); \\ x_{n+1} = J^{-1}(\alpha_n J(u) + \beta_n J(x_n) + \delta_n J(v_n)) \ n \in \mathbb{N}. \end{cases} \tag{16}$$

where $0 < k_0 < \beta_n, \delta_n < 1$, for some constant k_0, $\alpha_n + \beta_n + \delta_n = 1$, and $0 < \xi_0 < \mu_n < 1$, for some k_0, ξ_0. Set $n := n + 1$ and return to **Step 1**.

So, using the fact that $\lambda_k \geq \min\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\}$, we obtain

$$\lambda_{k+1} = \min\left\{\lambda_k, \frac{\delta\sqrt{\alpha}||z_k - y_k||}{||F(z_k) - F(y_k)||}\right\} \geq \min\left\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\right\}.$$

If on the other hand $||F(z_k) - F(y_k)|| = 0$, then it is easy to see from (15) that

$$\lambda_{k+1} = \lambda_k = \lambda_0 \geq \min\left\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\right\}.$$

Hence, for all $n \in \mathbb{N} \cup \{0\}$, $\min\{\lambda_0, \frac{\delta\sqrt{\alpha}}{L}\} \leq \lambda_n \leq \lambda_0$. This together with the monotonicity of the sequence $\{\lambda_n\}$ implies that $\lim\limits_{n\to\infty} \lambda_n$ exists.

We now show that the sequence $\{x_n\}$ is bounded. To see this, let $p \in \Gamma$, then from (P2) (see page 3) we get,

$$\phi(p, x_n) = \phi(z_n, x_n) + \phi(p, z_n) + 2\langle z_n - p, Jx_n - Jz_n\rangle.$$

This implies that

$$\phi(p, z_n) = \phi(p, x_n) - \phi(z_n, x_n) + 2\langle z_n - p, Jz_n - Jx_n\rangle. \tag{17}$$

So, using (17), Lemma 2.8 and the fact that for all $a, b \in \mathbb{R}$, $2ab \leq a^2 + b^2$, we get that

$$\begin{aligned}
\phi(p, z_n) &= \phi(p, x_n) - \phi(z_n, x_n) + 2\langle z_n - p, Jz_n - Jx_n\rangle \\
&\leq \phi(p, x_n) - \phi(z_n, x_n) + 2||z_n - p||||Jz_n - Jx_n|| \\
&= \phi(p, x_n) - \phi(z_n, x_n) + 2r_n||z_n - p||||Jx_n - Jx_{n-1}|| \\
&\leq \phi(p, x_n) - \phi(z_n, x_n) + r_n\Big[||z_n - p||^2 + 1\Big]||Jx_n - Jx_{n-1}|| \\
&\leq \phi(p, x_n) - v\phi(z_n, x_n) + r_n\big[2(||z_n - x_n||^2 + ||x_n - p||^2) + 1\big]||Jx_n - Jx_{n-1}|| \\
&\leq \phi(p, x_n) - \phi(z_n, x_n) + \frac{2\gamma_n}{\alpha}(\phi(z_n, x_n) + \phi(p, x_n)) + \gamma_n \\
&= \left(1 + \frac{2\gamma_n}{\alpha}\right)\phi(p, x_n) - \left(1 - \frac{2\gamma_n}{\alpha}\right)\phi(z_n, x_n) + \gamma_n.
\end{aligned} \tag{18}$$

Also, using Lemma 2.9, we obtain that

$$\begin{aligned}
\phi(p, w_n) &= \phi(p, J^{-1}(J(y_n) + \lambda_n(F(z_n) - F(y_n)))) \\
&= ||p||^2 - 2\langle p, J(y_n) + \lambda_n(F(z_n) - F(y_n))\rangle + ||w_n||^2 \\
&= ||p||^2 - 2\langle p, J(y_n)\rangle - 2\lambda_n\langle p, F(z_n) - F(y_n)\rangle + ||w_n||^2 \\
&= ||p||^2 + ||y_n||^2 - ||y_n||^2 - 2\langle p, J(y_n)\rangle - 2\lambda_n\langle p, F(z_n) - F(y_n)\rangle + ||w_n||^2 \\
&= \phi(p, y_n) + ||w_n||^2 - ||y_n||^2 - 2\lambda_n\langle p, F(z_n) - F(y_n)\rangle \\
&= \phi(p, y_n) + ||J^{-1}(J(y_n) + \lambda_n(F(z_n) - F(y_n))||^2 - ||y_n||^2 \\
&\quad - 2\lambda_n\langle p, F(z_n) - F(y_n)\rangle \\
&\le \phi(p, y_n) + ||y_n||^2 + 2\lambda_n\langle y_n, F(z_n) - F(y_n)\rangle + 2s_0^2\lambda_n^2||F(z_n) - F(y_n)||^2 \\
&- ||y_n||^2 - 2\lambda_n\langle p, F(z_n) - F(y_n)\rangle \\
&= \phi(p, y_n) + 2\lambda_n\langle y_n - p, F(z_n) - F(y_n)\rangle + 2s_0^2\lambda_n^2||F(z_n) - F(y_n)||^2.
\end{aligned} \tag{19}$$

Since $y_n = \Pi_C\left[J^{-1}(J(z_n) - \lambda_n F(z_n))\right]$, then setting $u_n = J^{-1}(J(z_n) - \lambda_n F(z_n))$, we obtain from Lemma 2.1 that

$$\langle y_n - p, Jy_n - Ju_n\rangle \le 0$$

thus

$$\langle y_n - p, Jy_n - Jz_n\rangle \le -\lambda_n\langle y_n - p, F(z_n)\rangle. \tag{20}$$

From (P2), (15), (19), (20) and Lemma 2.8, we obtain that

$$\begin{aligned}
\phi(p, w_n) &\le \phi(p, z_n) - \phi(y_n, z_n) + 2\langle y_n - p, Jy_n - Jz_n\rangle + 2\lambda_n\langle y_n - p, F(z_n) - F(y_n)\rangle \\
&+ 2s^2\lambda_n^2||F(z_n) - F(y_n)||^2 \\
&\le \phi(p, z_n) - \phi(y_n, z_n) - 2\lambda_n\langle y_n - p, F(z_n)\rangle + 2\lambda_n\langle y_n - p, F(z_n) - F(y_n)\rangle \\
&+ 2s^2\lambda_n^2||F(z_n) - F(y_n)||^2 \\
&= \phi(p, z_n) - \phi(y_n, z_n) - 2\lambda_n\langle y_n - p, F(y_n)\rangle + 2s_0^2\lambda_n^2||F(z_n) - F(y_n)||^2 \\
&\le \phi(p, z_n) - \phi(y_n, z_n) - 2\lambda_n\langle y_n - p, F(y_n) - F(p)\rangle - 2\lambda_n\langle y_n - p, F(p)\rangle \\
&\quad + 2s_0^2\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\alpha||z_n - y_n||^2 \\
&\le \phi(p, z_n) - \Big(1 - 2s_0^2\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, z_n) - 2\lambda_n\langle y_n - p, F(y_n) - F(p)\rangle \\
&\quad - 2\lambda_n\langle y_n - p, F(p)\rangle.
\end{aligned} \tag{21}$$

Since p is in $VI(C, F)$ and F is monotone, then we obtain from (21) that

$$\phi(p, w_n) \le \phi(p, z_n) - \Big(1 - 2s_0^2\delta^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n, z_n). \tag{22}$$

From (8), (9), Lemma 2.6 and Lemma 2.12, we get that

$$
\begin{aligned}
\phi(p,x_{n+1}) &= \phi(p, J^{-1}(\alpha_n J(u) + \beta_n Jx_n + \delta_n Jv_n)) \\
&= V(p, \alpha_n J(u) + \beta_n Jx_n + \delta_n Jv_n) \\
&= ||\alpha_n J(u) + \beta_n Jx_n + \delta_n Jv_n||^2 - 2\langle p, \alpha_n J(u) + \beta_n Jx_n + \delta_n Jv_n\rangle + ||p||^2 \\
&\leq \alpha_n ||u||^2 + \beta_n ||x_n||^2 + \delta_n ||v_n||^2 - \beta_n\delta_n g(||x_n - v_n||) - 2\alpha_n \langle p, J(u)\rangle \\
&- 2\beta_n \langle p, Jx_n\rangle - 2\delta_n \langle p, Jv_n\rangle + ||p||^2 \\
&= \alpha_n \phi(p,u) + \beta_n \phi(p,x_n) + \delta_n \phi(p,v_n) - \beta_n\delta_n g(||x_n - v_n||) \\
&= \alpha_n \phi(p,u) + \beta_n \phi(p,x_n) + \delta_n \phi(p, J^{-1}((1-\mu_n)Jw_n + \mu_n JTw_n)) \\
&- \beta_n\delta_n g(||x_n - v_n||^2) \\
&\leq \alpha_n \phi(p,u) + \beta_n \phi(p,x_n) + \delta_n \phi(p,w_n) - \beta_n\delta_n g(||x_n - v_n||). \quad (23)
\end{aligned}
$$

Thus, using (22) and (18) in (23), we get that

$$
\begin{aligned}
\phi(p,x_{n+1}) &= \alpha_n \phi(p,u) + \beta_n \phi(p,x_n) + \delta_n \Big[\big(1 + \frac{2\gamma_n}{\alpha}\big)\phi(p,x_n) \\
&- \big(1 - \frac{2\gamma_n}{\alpha}\big)\phi(z_n,x_n) + \gamma_n \\
&- \Big(1 - 2s_0^2\delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\phi(y_n,z_n)\Big] - \beta_n\delta_n g(||x_n - v_n||) \\
&\leq \alpha_n \phi(p,u) + \big[\beta_n + \delta_n + \frac{2\gamma_n}{\alpha}\big]\phi(p,x_n) - \big(1 - \frac{2\gamma_n}{\alpha}\big)\delta_n \phi(z_n,x_n) \\
&+ \delta_n\gamma_n - \Big(1 - 2s_0^2\delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\delta_n \phi(y_n,z_n) - \beta_n\delta_n g(||x_n - v_n||) \\
&= \alpha_n \phi(p,u) + \big[(1-\alpha_n) + \frac{2\gamma_n\delta_n}{\alpha}\big]\phi(p,x_n) - \big(1 - \frac{2\gamma_n}{\alpha}\big)\delta_n \phi(z_n,x_n) \\
&+ \delta_n\gamma_n - \Big(1 - 2s_0^2\delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\delta_n \phi(y_n,z_n) - \beta_n\delta_n g(||x_n - v_n||). \quad (24)
\end{aligned}
$$

Taking $\tau_0 \in (0, \frac{\alpha}{2})$ and from (A6), there exists a natural number N_0 such that for all $n \geq N_0$, $\frac{2\gamma_n}{\alpha} < \alpha_n \tau_0$. Moreover, since $\lim_{n\to\infty} \lambda_n$ exists, then $\lim_{n\to\infty} \frac{\lambda_n^2}{\lambda_{n+1}^2} = 1$, so that $\lim_{n\to\infty} \Big(1 - 2s_0^2\delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big) = 1 - 2s_0^2\delta^2 > 0$ (since $\delta \in (0, \frac{1}{s_0\sqrt{2}})$). Therefore, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $\Big(1 - 2s_0^2\delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big) > 0$.

So, $\forall\, n \geq N_2 := \max\{N_0, N_1\}$ we obtain from (24) that

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \alpha_n \phi(p, u) + \big[1 - \alpha_n(1-\tau_0)\big]\phi(p, x_n) + \frac{\alpha_n \tau_0 \alpha}{2} \\
&\quad - \big(1 - \frac{2\gamma_n}{\alpha}\big)\delta_n \phi(z_n, x_n) \\
&\quad - \Big(1 - 2s_0^2\delta^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\delta_n \phi(y_n, z_n) - \beta_n \delta_n g(||x_n - v_n||) \qquad (25) \\
&\leq \big[1 - \alpha_n(1-\tau_0)\big]\phi(p, x_n) + \alpha_n(1-\tau_0)\Big[\frac{2\phi(p,u) + \tau_0\alpha}{2(1-\tau)}\Big]
\end{aligned}$$

It is thus easy to see by mathematical induction that for all $n \geq N_2$,

$$\phi(p, x_n) \leq \max\big\{\phi(p, x_N), \frac{2\phi(p,u) + \tau_0\alpha}{2(1-\tau_0)}\big\}.$$

Therefore, for fixed $p \in \Gamma$, $\{\phi(p, x_n)\}$ is bounded and from Lemma 2.8, we obtain that $\{x_n\}$ is bounded. Thus, $\{v_n\}$, $\{y_n\}$, $\{w_n\}$ and $\{Tw_n\}$ are all bounded.

Next, for $p^* = \Pi_\Gamma u \in \Gamma$, we establish that sequence $\{\phi(p^*, x_n)\}_{n=1}^\infty$ converges strongly to zero. Two cases arise.

Case 1: Suppose that the sequence $\{\phi(p^*, x_n)\}_{n=1}^\infty$ is non-increasing sequence of real numbers, then since $\{\phi(p^*, x_n)\}_{n=1}^\infty$ is bounded, $\lim_{n\to\infty}\phi(p^*, x_n)$ exists. Thus, $\lim_{n\to\infty}\Big(\phi(p^*, x_n) - \phi(p^*, x_{n+1})\Big) = 0$. We first show that

$$\limsup_{n\to\infty}\langle x_{n+1} - p^*, Ju - Jp^*\rangle \leq 0.$$

Now, since $\{x_n\}$ is bounded, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n - p^*, Ju - Jp^*\rangle = \lim_{k\to\infty}\langle x_{n_k} - p^*, Ju - Jp^*\rangle.$$

Observe that from (25), we get

$$\begin{aligned}
0 \leq \Big(1 - \frac{2\gamma_{n_k}}{\alpha}\Big)\delta_{n_k}\phi(z_{n_k}, x_{n_k}) &+ \Big(1 - 2s_0^2\delta^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\Big)\delta_{n_k}\phi(y_{n_k}, z_{n_k}) \\
&+ \beta_{n_k}\delta_{n_k} g(||x_{n_k} - v_{n_k}||) \\
&\leq \big(\phi(p^*, x_{n_k}) - \phi(p^*, x_{n_k+1})\big) \\
&+ \alpha_{n_k}\big[(1-\tau)\phi(p^*, x_{n_k}) + \phi(p^*, u) + \frac{\tau\alpha}{2}\big].
\end{aligned} \qquad (26)$$

Taking lim sup on both sides of (26), we obtain

$$\begin{aligned}
\limsup_{k\to\infty}\Big(\Big(1 - \frac{2\gamma_{n_k}}{\alpha}\Big)\delta_{n_k}\phi(z_{n_k}, x_{n_k}) + \Big(1 - 2s_0^2\delta^2\frac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\Big)\delta_{n_k}\phi(y_{n_k}, z_{n_k}) \\
+ \beta_{n_k}\delta_{n_k} g(||x_{n_k} - v_{n_k}||)\Big) \leq 0 \qquad (27)
\end{aligned}$$

Since $\lim_{k\to\infty}\big(1-\frac{2\gamma_{n_k}}{\alpha}\big)>0$, $\lim_{k\to\infty}\Big(1-2s_0^2\delta^2\dfrac{\lambda_{n_k}^2}{\lambda_{n_k+1}^2}\Big)>0$ and $\beta_{n_k},\delta_{n_k}>k_0>0\ \forall\, k\in\mathbb{N}$, then (27) implies that

$$\lim_{k\to\infty}\phi(z_{n_k},x_{n_k})=\lim_{k\to\infty}\phi(y_{n_k},z_{n_k})=\lim_{k\to\infty}g(||x_{n_k}-v_{n_k}||)=0.$$

So, from Lemma 2.7 and the property of g in Lemma 2.6, it follows that

$$\lim_{k\to\infty}||x_{n_k}-z_{n_k}||=\lim_{k\to\infty}||z_{n_k}-y_{n_k}||=\lim_{k\to\infty}||x_{n_k}-v_{n_k}||=0. \tag{28}$$

Also,

$$||x_{n_k}-y_{n_k}||\le||x_{n_k}-z_{n_k}||+||z_{n_k}-y_{n_k}||. \tag{29}$$

So, using (28) in (29) gives

$$\lim_{k\to\infty}||x_{n_k}-y_{n_k}||=0. \tag{30}$$

Since J is norm-to-norm uniformly continuous on bounded sets, we obtain that

$$\lim_{k\to\infty}||Jx_{n_k}-Jz_{n_k}||=\lim_{k\to\infty}||Jz_{n_k}-Jy_{n_k}||=\lim_{k\to\infty}||Jx_{n_k}-Jv_{n_k}||=0. \tag{31}$$

By the definition of $\{w_n\}$, we get that

$$||Jw_{n_k}-Jy_{n_k}||=\lambda_{n_k}||F(z_{n_k})-F(y_{n_k})||\le\lambda_{n_k}L||z_{n_k}-y_{n_k}||. \tag{32}$$

Thus, using (28) in (32), we obtain that

$$\lim_{k\to\infty}||Jw_{n_k}-Jy_{n_k}||=0. \tag{33}$$

Furthermore,

$$||Jv_{n_k}-Jw_{n_k}||\le||Jv_{n_k}-Jx_{n_k}||+||Jx_{n_k}-Jz_{n_k}||+||Jz_{n_k}-Jy_{n_k}||+||Jy_{n_k}-Jw_{n_k}||. \tag{34}$$

Therefore, using (31) and (33) in (34) we obtain that

$$\lim_{k\to\infty}||Jv_{n_k}-Jw_{n_k}||=0. \tag{35}$$

Observe further that

$$||Jw_{n_k}-Jx_{n_k}||\le||Jw_{n_k}-Jv_{n_k}||+||Jv_{n_k}-Jw_{n_k}||. \tag{36}$$

Using (31) and (35) in (36), we obatain that

$$\lim_{k\to\infty}||Jw_{n_k}-Jx_{n_k}||=0. \tag{37}$$

By definition of $\{v_n\}$ and the fact that T is demi-generalized mapping, we get that

$$\begin{aligned}\langle Jw_{n_k} - Jv_{n_k}, w_{n_k} - p\rangle &= \langle Jw_{n_k} - [(1-\mu_n)Jw_{n_k} + \mu_n JTw_{n_k}, w_{n_k} - p\rangle \\ &= \mu_n\langle Jw_{n_k} - JTw_{n_k}, w_{n_k} - p\rangle \\ &\geq \mu_n\frac{(1-\tau)}{2}\phi(Tw_{n_k}, w_{n_k}). \end{aligned} \tag{38}$$

Since $\tau < 1$ and $\mu_n >\geq \xi_0 > 0$, then $\mu_n(1-\tau_0) > \xi_0(1-\tau_0) > 0$. Thus, (38) and (35) give that

$$\begin{aligned}\phi(Tw_{n_k}, w_{n_k}) &\leq \frac{1}{\mu_n(1-\tau)}||Jw_{n_k} - Jv_{n_k}||||w_{n_k} - p|| \\ &\leq \frac{1}{\xi_0(1-\tau)}||Jw_{n_k} - Jv_{n_k}||||w_{n_k} - p|| \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Using Lemma 2.6 , we get that

$$\lim_{k\to\infty}||w_{n_k} - Tw_{n_k}|| = 0. \tag{39}$$

Since $\{x_{n_k}\}$ is bounded in E and since E is reflexive real Banach space, then there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\})$ such that $\{x_{n_{k_j}}\}$ converges weakly to x^* in E. Using (37) and the fact that J^{-1} is norm-to-norm uniformly continuous on bounded sets, then

$$\lim_{j\to\infty}||w_{n_{k_j}} - x_{n_{k_j}}|| = 0.$$

Thus, $w_{n_{k_j}} \rightharpoonup x^*$ in E. Since T is demiclosed a zero, then with the use of (39) we get that $x^* \in F(T)$.

Next, we show that $x^* \in VI(C, F)$. From (28) and (30), we have $\lim_{j\to\infty}||x_{n_{k_j}} - y_{n_{k_j}}|| = \lim_{j\to\infty}||z_{n_{k_j}} - y_{n_{k_j}}|| = 0$. Thus, it is easy to see that $z_{n_{k_j}} \rightharpoonup x^*$ and $y_{n_{k_j}} \rightharpoonup x^*$ as $j \to \infty$. Since $y_{n_{k_j}} = \Pi_C(J^{-1}(J(z_{n_{k_j}}) - \lambda_{n_{k_j}}F(z_{n_{k_j}}))$, then by Lemma 2.1, we get that

$$0 \leq \langle z - y_{n_{k_j}}, J(y_{n_{k_j}}) - J(z_{n_{k_j}}) + \lambda_{n_{k_j}}F(z_{n_{k_j}})\rangle \quad \forall z \in C.$$

This implies (using boundedness of $\{y_{n_{k_j}}\}$, $\{F(z_{n_{k_j}})\}$ and the fact that for all $n \in \mathbb{N}$, $\lambda_n \leq \lambda_0$) that for some constant $M_0 > 0$,

$$\begin{aligned}0 &\leq \langle z - y_{n_{k_j}}, J(y_{n_{k_j}}) - J(z_{n_{k_j}})\rangle + \lambda_{n_{k_j}}\langle z - y_{n_{k_j}}, F(z_{n_{k_j}})\rangle \\ &= \langle z - y_{n_{k_j}}, J(y_{n_{k_j}}) - J(z_{n_{k_j}})\rangle + \lambda_{n_{k_j}}\langle z - z_{n_{k_j}}, F(z_{n_{k_j}})\rangle + \lambda_{n_{k_j}}\langle z_{n_{k_j}} - y_{n_{k_j}}, F(z_{n_{k_j}})\rangle \\ &\leq \left(||J(y_{n_{k_j}}) - J(z_{n_{k_j}})|| + ||z_{n_{k_j}} - y_{n_{k_j}}||\right)\max\{M_0, \lambda_0 M_0\} + \lambda_{n_{k_j}}\langle z - z_{n_{k_j}}, F(z_{n_{k_j}})\rangle.\end{aligned} \tag{40}$$

Since F is monotone, we obtain that

$$\begin{aligned}\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(z_{n_{k_j}})\rangle &= \lambda_{n_{k_j}}\langle z-x^*,F(z_{n_{k_j}})-F(x^*)\rangle\\ &+\lambda_{n_{k_j}}\langle x^*-z_{n_{k_j}},F(z_{n_{k_j}})-F(x^*)\rangle\\ &+\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(x^*)\rangle\\ &= \lambda_{n_{k_j}}\langle z-x^*,F(z_{n_{k_j}})-F(x^*)\rangle\\ &-\lambda_{n_{k_j}}\langle z_{n_{k_j}}-x^*,F(z_{n_{k_j}})-F(x^*)\rangle\\ &+\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(x^*)\rangle\\ &\leq \lambda_{n_{k_j}}\langle z-x^*,F(z_{n_{k_j}})-F(x^*)\rangle\\ &+\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(x^*)\rangle\\ &\leq \lambda_0|\langle z-x^*,F(z_{n_{k_j}})-F(x^*)\rangle|\\ &+\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(x^*)\rangle. \end{aligned} \tag{41}$$

Using (41) in (40), we obtain that

$$\begin{aligned}0\leq &\big(||J(y_{n_{k_j}})-J(z_{n_{k_j}})||+||z_{n_{k_j}}-y_{n_{k_j}}||\big)\max\{M_0,\lambda_0M_0\}\\ &+\lambda_0|\langle z-x^*,F(z_{n_{k_j}})-F(x^*)\rangle|+\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(x^*)\rangle\end{aligned} \tag{42}$$

Since $\lim\limits_{n\to\infty}\lambda_n$ exists, $\lim\limits_{j\to\infty}||z_{n_{k_j}}-y_{n_{k_j}}||=0$, $\lim\limits_{j\to\infty}||J(y_{n_{k_j}})-J(z_{n_{k_j}})||=0$, $z_{n_{k_j}}\rightharpoonup x^*$ as $j\to\infty$, and from the fact that F is weakly sequentially continuous, $\lim\limits_{j\to\infty}\langle z-x^*,F(z_{n_{k_j}})-F(x^*)\rangle=0$, then we obtain from (42) that

$$0\leq \lim_{j\to\infty}\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(x^*)\rangle. \tag{43}$$

Furthermore, since

$$0<\min\big\{\lambda_0,\frac{\delta\sqrt{\alpha}}{L}\big\}\leq\lambda_{n_{k_j}}\leq\lambda_0, \tag{44}$$

we obtain that $\lim\limits_{n\to\infty}\lambda_n>0$. So, we obtain from (43) that,

$$\begin{aligned}0&\leq \lim_{j\to\infty}\lambda_{n_{k_j}}\langle z-z_{n_{k_j}},F(x^*)\rangle\\ &=\big(\lim_{j\to\infty}\lambda_{n_{k_j}}\big)\lim_{j\to\infty}\langle z-z_{n_{k_j}},F(x^*)\rangle.\end{aligned} \tag{45}$$

So, using (44) and (45), we easily obtain that

$$0\leq \lim_{j\to\infty}\langle z-z_{n_{k_j}},F(x^*)\rangle. \tag{46}$$

But

$$\lim_{j\to\infty}\langle z-z_{n_{k_j}},F(x^*)\rangle=\langle z-x^*,F(x^*)\rangle. \tag{47}$$

Combining (46) and (47), we obtain that

$$\langle z - x^*, F(x^*)\rangle \geq 0, \quad \forall z \in C.$$

Therefore, using Lemma 2.5, we get $x^* \in VI(C,F)$. Thus, $x^* \in VI(C,F) \cap F(T)$; and since $p^* = \Pi_\Gamma u$, then by Lemma 2.1, we get that

$$\langle x^* - p^*, Ju - Jp^*\rangle \leq 0.$$

But

$$\begin{aligned}\limsup_{n\to\infty}\langle x_n - p^*, Ju - Jp^*\rangle &= \lim_{k\to\infty}\langle x_{n_k} - p^*, Ju - Jp^*\rangle = \lim_{j\to\infty}\langle x_{n_{k_j}} - p^*, Ju - Jp^*\rangle \\ &= \langle x^* - p, Ju - Jp^*\rangle \leq 0.\end{aligned}$$

Also, from (28) and definition of x_{n+1} and (28), we get (using the fact that $\lim\limits_{n\to\infty} \alpha_n = 0$) that

$$\begin{aligned}||Jx_{n+1} - Jx_n|| &\leq \alpha_n||Ju - Jx_n|| + \delta_n||Jv_n - Jx_n|| \\ &\leq \alpha_n||Ju - Jx_n|| + ||Jv_n - Jx_n|| \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Since J^{-1} is norm-to-norm uniformly continuous on a set, then

$$\lim_{n\to\infty}||x_{n+1} - x_n|| = 0. \tag{48}$$

So,

$$\begin{aligned}\limsup_{n\to\infty}\langle x_{n+1} - p^*, Ju - Jp^*\rangle &\leq \limsup_{n\to\infty}\langle x_{n+1} - x_n, Ju - Jp^*\rangle \\ &+ \limsup_{n\to\infty}\langle x_n - p^*, Ju - Jp^*\rangle \\ &\leq \limsup_{n\to\infty}||x_{n+1} - x_n||||Ju - Jp^*|| \\ &+ \limsup_{n\to\infty}\langle x_n - p^*, Ju - Jp^*\rangle \leq 0. \end{aligned} \tag{49}$$

Moreover, from (9), (18), (20), (22),(39) and Lemma 2.3, we get that

$$\begin{aligned}\phi(p^*, x_{n+1}) &= \phi(p^*, J^{-1}(\alpha_n J(u) + \beta_n Jx_n + \delta_n Jv_n)) \\ &= V(p^*, \alpha_n J(u) + \beta_n Jx_n + \delta_n Jv_n) \\ &\leq V(p^*, \alpha_n J(u) + \beta_n Jx_n + \delta_n Jv_n - \alpha_n(Ju - Jp^*)) \\ &+ 2\alpha_n\langle x_{n+1} - p^*, Ju - Jp^*\rangle \\ &= V(p^*, \alpha_n Jp^* + \beta_n Jx_n + \delta_n Jv_n) + 2\alpha_n\langle x_{n+1} - p^*, Ju - Jp^*\rangle \\ &\leq \alpha_n\phi(p^*, p^*) + \beta_n\phi(p^*, x_n) + \delta_n\phi(p^*, v_n) \\ &+ 2\alpha_n\langle x_{n+1} - p^*, Ju - Jp^*\rangle \\ &\leq (1 - \alpha_n)\phi(p^*, x_n) + 2\alpha_n\langle x_{n+1} - p^*, Ju - Jp^*\rangle.\end{aligned}$$

Therefore,

$$\phi(p^*, x_{n+1}) \leq (1-\alpha_n)\phi(p^*, x_n) + 2\alpha_n\langle x_{n+1} - p^*, Ju - Jp^*\rangle. \tag{50}$$

It follows from (49), (50) and Lemma 2.13 that $\phi(p^*, x_n) \to 0$ as $n \to \infty$ and by Lemma 2.8, we obtain that $||x_n - p^*|| \to 0$ as $n \to \infty$. Hence $x_n \to p^* = \Pi_\Gamma u \in \Gamma$.

Case 2. Suppose that $\{\phi(p^*, x_n)\}_{n=1}^{\infty}$ is a non-decreasing sequence real numbers. Then, by Lemma 2.15, we set $\Upsilon_n := \phi(p^*, x_n)$ and let $r : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some n_0 large enough) by

$$r(n) := \max\{k \in \mathbb{N} : k \leq n, \Upsilon_k \leq \Upsilon_{k+1}\}.$$

Then, $\{r_n\}$ is a non-decreasing sequence of $\mathbb{N}$ such that $r(n) \to \infty$ as $n \to \infty$; and

$$0 \leq \Upsilon_{r(n)} \leq \Upsilon_{r(n)+1}, \quad \forall\, n \geq n_0$$

which means that $\phi(p^*, x_{r(n)}) \leq \phi(p^*, x_{r(n)+1})$, for all $n \geq n_0$. Since $\{\phi(p^*, x_{r(n)})\}$ is bounded, then $\lim\limits_{n\to\infty} \phi(p^*, x_{r(n)})$ exists. Thus, following the same line of action as in Case 1, we obtain that $\lim\limits_{n\to\infty} \phi(p^*, x_{r(n)}) = 0$, which implies that

$$\lim_{n\to\infty} \phi(p^*, x_{r(n)+1}) = 0.$$

Therefore, by Lemma 2.15, we get that

$$\phi(p^*, x_n) \leq \phi(p^*, x_{r(n)+1}) \to 0 \text{ as } n \to \infty$$

which gives that $\lim\limits_{n\to\infty} \phi(p^*, x_n) = 0$, so that by Lemma 2.8 $\lim\limits_{n\to\infty} ||p^* - x_n|| = 0$. So, we obtain again that $x_n \to p^* := \Pi_\Gamma u$. Hence, the sequence $\{x_n\}$ convergs strongly to $p^* = \Pi_\Gamma u \in \Gamma$. This completes the proof. □ ☑

4. Numerical Example

This section illustrates the performance of our Algorithm 3 with the support of numerical experiments. Furthermore, we compare our iterative techniques with the methods of Censor et al. [32, 31] (Algorithm (4)), Chidume and Nnakwe [7] (Algorithm (5)), and Liu [16](Algorithm (6))

Example 4.1. Let $E = (l_2(\mathbb{R}), ||.||_{l_2})$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),$ $x_i \in \mathbb{R} : \sum\limits_{i=1}^{\infty} |x_i|^2 < \infty\}$ and for all $x \in l_2(\mathbb{R})$, $||x||_{l_2} := \left(\sum\limits_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$.

Let $C = \{x \in l_2(\mathbb{R}) : |x_i| \leq \frac{1}{i},\ i = 1, 2, 3, \dots\}$. Thus, we have an explicit formula for P_C. Now, define the operator $F : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$F(x) = \left(||x|| + \frac{1}{||x|| + \alpha}\right) x,$$

for some $\alpha > 0$. Then, F is monotone on $l_2(\mathbb{R})$ (see [26]). In this experiment, the stopping criterion is $\varepsilon = 10^{-4}$, and the starting points are selected as follows:

Case 1: Take $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$.

Case 2: Take $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$ and $x_0 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

Case 3: Take $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$ and $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$.

Case 4: Take $x_1 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$ and $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$.

The numerical results are reported in Table 1 and Figure 1.

Table 3. Numerical results for Example 4.1 with $\varepsilon = 10^{-4}$.

Cases		Algorithm (3)	Algorithm (6)	Algorithm (5)	Algorithm (4)
1	Iter. CPU	132 0.0333	1506 0.0839	1869 0.0996	1868 0.1002
2	Iter. CPU	137 0.0396	1548 0.0878	1862 0.1047	1860 0.1045
3	Iter. CPU	141 0.0329	1539 0.0824	1813 0.0985	1811 0,0881
4	Iter. CPU	132 0.0349	1168 0.0669	1500 0.0808	1499 0.0711

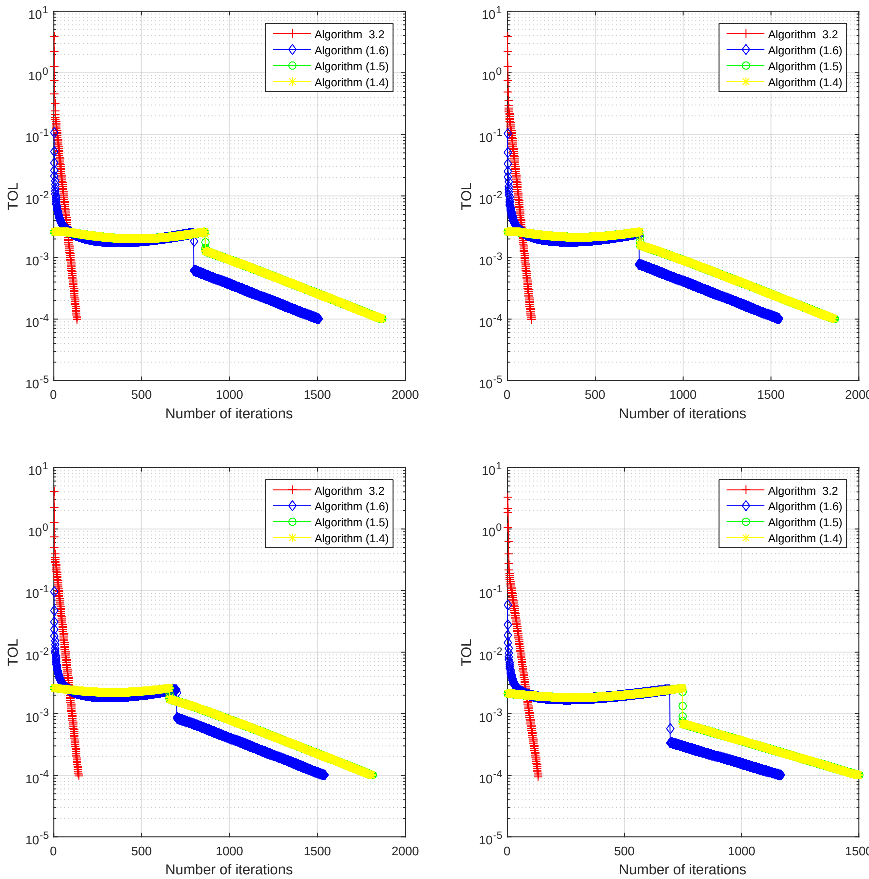

FIGURE 1. The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 4.1: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

5. Conclusion

We introduced an algorithm with Halpern-type and inertial Tseng's method for finding the common solution of the monotone variational inequality problem and the fixed point problem of demigeneralized mapping in a 2-uniformly convex and uniformly 2-smooth real Banach space. We proved that the sequence generated by our proposed iterative technique converges strongly to a solution of the problems under study. Furthermore, a numerical example was constructed to show that the iterative algorithm introduced and studied has a greater convergence rate.

Future Work: This research opens several avenues for further exploration:

- Extend the proposed algorithm to other types of Banach spaces, such as p-uniformly convex and uniformly p-smooth spaces, and focus on enhancing the algorithm's efficiency. Additionally, developing adaptive strategies for selecting parameters or incorporating advanced acceleration techniques can significantly reduce the computational cost and increase the algorithm's practicality for large-scale problems.
- Investigating stochastic variants of the proposed algorithm could be another fruitful direction, especially in the context of solving problems with uncertainty or noise in the data. This could involve incorporating stochastic approximation techniques or exploring the algorithm's performance in probabilistic settings.
- Given the growing importance of optimization algorithms in machine learning, applying our method to common machine learning tasks, such as training neural networks or solving support vector machine problems, could be highly impactful. This would also provide a practical validation of the algorithm's effectiveness in high-dimensional data spaces.

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