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Some new Hadamard-type inequalities via fractional integral operators

Algunas nuevas desigualdades tipo Hadamard a través de operadores integrales fraccionarios

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Abstract. The article presents new inequalities of Hadamard-type which are obtained using fractional integral operators belonging to a function whose third-order derivative is convex. The proposed Hadamard-type inequalities have the potential for application in various areas where it is required to estimate the properties of functions with a convex third-order derivative. Examples of functions are given based on a comparative analysis of the estimates of the upper bounds of the Hadamard-type inequalities obtained using the classical and extended Hölder inequalities. Finally, applications to special functions are provided.

Key words and phrases. Convex functions, Hadamard's inequality, Hölder's inequality, fractional calculus, special functions.

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Resumen. El artículo presenta nuevas desigualdades de tipo Hadamard que se obtienen utilizando operadores integrales fraccionarios pertenecientes a una función cuya derivada de tercer orden es convexa. Las desigualdades de tipo Hadamard propuestas tienen potencial de aplicación en diversas áreas donde se requiere estimar las propiedades de funciones con una derivada convexa de tercer orden. Se dan ejemplos de funciones basados en un análisis comparativo de las estimaciones de los límites superiores de las desigualdades de tipo Hadamard obtenidas utilizando las desigualdades Hölder clásica y extendida. Finalmente, se proporcionan aplicaciones a funciones especiales.

Palabras y frases clave. Funciones convexas, desigualdad de Hadamard, desigualdad de Hölder, cálculo fraccional, funciones especiales.

1. Introduction

The definition of convexity is well-known in the literature:

Definition 1.1. Function $\Psi : [\varrho_1, \varrho_2] \to \mathbb{R}$, is said to be convex, if we have

$$\Psi\left(\sigma\theta_1 + (1-\sigma)\theta_2\right) \le \sigma\Psi(\theta_1) + (1-\sigma)\Psi(\theta_2)$$

for all $\theta_1, \theta_2 \in [\varrho_1, \varrho_2]$ and $\sigma \in [0,1]$.

There are many significant inequalities established for the class of convex functions, but the Hermite–Hadamard inequality, also known as Hadamard's inequality, is one of the most significant. The study of Hadamard-type inequalities is a hot topic in mathematical analysis and has a wide range of applications in various fields, including optimization, mathematical physics, and statistics.

Let $I \subseteq \mathbb{R}$ be an interval, $\Psi : I \to \mathbb{R}$ be a convex function and let $\varrho_1, \varrho_2 \in I$ with $\varrho_1 < \varrho_2$. The double inequality

$$\Psi\left(\frac{\varrho_1 + \varrho_2}{2}\right) \le \frac{1}{\varrho_2 - \varrho_1}\int_{\varrho_1}^{\varrho_2} \Psi(\sigma)d\sigma \le \frac{\Psi(\varrho_1) + \Psi(\varrho_2)}{2}$$

is called the Hermite–Hadamard inequality. The above inequality is reversed if Ψ is concave.

Recently, new Hermite–Hadamard and other types of integral inequalities have been introduced using fractional operators. These inequalities generalize classical ones and could have potential applications in various areas of mathematics and its associated fields (see [1, 3, 5, 10, 16, 18, 19] and the references therein). These extensions offer improved estimates for integral operators and give more accurate bounds on function values.

One key feature of the new Hadamard-type inequalities is their connection with fractional integral operators. Fractional integrals generalize the classical ones and account for the particularities of functions exhibiting fractal properties. By incorporating fractional integrals into Hadamard-type inequalities, new possibilities arise for studying complex mathematical models.

The classical definition of the Riemann–Liouville fractional integrals is the following (see e.g. [21]):

Definition 1.2. Let $\Psi \in L[\varrho_1, \varrho_2]$. The Riemann–Liouville integrals $J^{\alpha}_{\varrho_1^+}\Psi$ and $J^{\alpha}_{\varrho_2^-}\Psi$ of order $\alpha > 0$ with $\varrho_1 \ge 0$ are defined by

$$J^{\alpha}_{\varrho_1^+}\Psi(\theta) = \frac{1}{\Gamma(\alpha)}\int_{\varrho_1}^{\theta} (\theta - \sigma)^{\alpha-1}\Psi(\sigma)\,d\sigma, \qquad \theta > \varrho_1,$$

and

$$J_{\varrho_2-}^{\alpha}\Psi(\theta)=\frac{1}{\Gamma(\alpha)}\int_{\theta}^{\varrho_2}(\sigma-\theta)^{\alpha-1}\Psi(\sigma)\,d\sigma,\qquad \theta<\varrho_2,$$

respectively. Note that for $\alpha=1$, the fractional integrals simplify to classical integrals of Ψ.

In the vast majority of studies dealing with the theory of integral inequalities, two classical inequalities are employed: Hölder's inequality and the power mean inequality.

Theorem 1.3 ([17], Hölder's inequality). *Let $\zeta>1$ and $\frac{1}{\zeta}+\frac{1}{\xi}=1$. If Ψ and $\mathfrak{g}$ are real functions defined on $[\varrho_1,\varrho_2]$ and if $|\Psi|^{\zeta},|\mathfrak{g}|^{\xi}\in L[\varrho_1,\varrho_2]$, then*

$$\int_{\varrho_1}^{\varrho_2}|\Psi(\sigma)\,\mathfrak{g}(\sigma)|\,d\sigma\le\left(\int_{\varrho_1}^{\varrho_2}|\Psi(\sigma)|^{\zeta}\,d\sigma\right)^{\frac{1}{\zeta}}\left(\int_{\varrho_1}^{\varrho_2}|\mathfrak{g}(\sigma)|^{\xi}\,d\sigma\right)^{\frac{1}{\xi}}.\tag{1}$$

Equality occurs if and only if $C_1|\Psi(\sigma)|^{\zeta}=C_2|\mathfrak{g}(\sigma)|^{\xi}$ holds almost everywhere, where C_1 and C_2 are constants, not both of them zero.

Theorem 1.4 ([12], improved Hölder inequality). *Let Ψ and $\mathfrak{g}$ be real functions defined on $[\varrho_1,\varrho_2]$. If $|\Psi|^{\zeta},|\mathfrak{g}|^{\xi}\in L[\varrho_1,\varrho_2]$ with $\frac{1}{\zeta}+\frac{1}{\xi}=1$ and $\zeta>1$, then*

$$\begin{aligned}&\int_{\varrho_1}^{\varrho_2}|\Psi(\sigma)\mathfrak{g}(\sigma)|\,d\sigma\\&\quad\le\frac{1}{\varrho_2-\varrho_1}\Bigg\{\left(\int_{\varrho_1}^{\varrho_2}(\varrho_2-\sigma)|\Psi(\sigma)|^{\zeta}d\sigma\right)^{\frac{1}{\zeta}}\left(\int_{\varrho_1}^{\varrho_2}(\varrho_2-\sigma)|\mathfrak{g}(\sigma)|^{\xi}d\sigma\right)^{\frac{1}{\xi}}\\&\qquad+\left(\int_{\varrho_1}^{\varrho_2}(\sigma-\varrho_1)|\Psi(\sigma)|^{\zeta}d\sigma\right)^{\frac{1}{\zeta}}\left(\int_{\varrho_1}^{\varrho_2}(\sigma-\varrho_1)|\mathfrak{g}(\sigma)|^{\xi}d\sigma\right)^{\frac{1}{\xi}}\Bigg\}.\end{aligned}\tag{2}$$

Theorem 1.5 ([17], power mean inequality). *Let $\xi\ge1$ and $\frac{1}{\zeta}+\frac{1}{\xi}=1$. If Ψ and $\mathfrak{g}$ are real functions defined on $[\varrho_1,\varrho_2]$ and if $|\Psi|,|\Psi||\mathfrak{g}|^{\xi}\in L[\varrho_1,\varrho_2]$, then*

$$\int_{\varrho_1}^{\varrho_2}|\Psi(\sigma)\,\mathfrak{g}(\sigma)|\,d\sigma\le\left(\int_{\varrho_1}^{\varrho_2}|\Psi(\sigma)|\,d\sigma\right)^{1-\frac{1}{\xi}}\left(\int_{\varrho_1}^{\varrho_2}|\Psi(\sigma)|\,|\mathfrak{g}(\sigma)|^{\xi}\,d\sigma\right)^{\frac{1}{\xi}}.\tag{3}$$

Theorem 1.6 ([13], improved power mean integral inequality). *Let $\xi\ge1$ and let Ψ and $\mathfrak{g}$ be real functions defined on $[\varrho_1,\varrho_2]$. If $|\Psi|,|\Psi||\mathfrak{g}|^{\xi}\in L[\varrho_1,\varrho_2]$, then*

$$\begin{aligned}
&\int_{\varrho_1}^{\varrho_2} |\Psi(\sigma)\mathfrak{g}(\sigma)| d\sigma \\
&\quad \le \frac{1}{\varrho_2-\varrho_1}\left\{ \left(\int_{\varrho_1}^{\varrho_2} (\varrho_2-\sigma)|\Psi(\sigma)| d\sigma\right)^{1-\frac{1}{\xi}} \left(\int_{\varrho_1}^{\varrho_2} (\varrho_2-\sigma)|\Psi(\sigma)||\mathfrak{g}(\sigma)|^{\xi} d\sigma\right)^{\frac{1}{\xi}} \right. \\
&\qquad \left. + \left(\int_{\varrho_1}^{\varrho_2} (\sigma-\varrho_1)|\Psi(\sigma)| d\sigma\right)^{1-\frac{1}{\xi}} \left(\int_{\varrho_1}^{\varrho_2} (\sigma-\varrho_1)|\Psi(\sigma)||\mathfrak{g}(\sigma)|^{\xi} d\sigma\right)^{\frac{1}{\xi}} \right\}. \qquad (4)
\end{aligned}$$

The upcoming paragraph summarizes some recent studies pertaining to the article's subject, specifically focusing on functions whose third derivative belongs to a particular convex class.

In [2], the author derived parameterized integral inequalities of Hadamard and Simpson types for concave and r-convex functions. For certain values of r, these inequalities were established using dedicated computational tools. Some new inequalities of Hadamard-type for s-convex functions were presented by the authors in [7, 20]. For extended s-convex functions, Simpson-type inequalities were obtain by Chun and Qi in [9]. In [26], Wu et al. obtained new parameterized inequalities of the Hadamard-type for quasi-convex functions, moreover, an application to special means of real numbers was also given there. In [8], Chun and Qi discovered some new Hermite–Hadamard type integral inequalities for the classical convex functions, while in [24], Shuang et al. did for (α, m)-convex functions. Recently, some fractional estimations are provided pertaining convex mappings in [11] by Hezenci et al., while in [14], Li and Du obtained one for (α, m)-GA-convex functions, centering Simpson-type integral inequalities. Many other fractal-fractional variants for differentiable convex mappings can be observed in [6, 4, 22, 23].

The aim of this article is to introduce new variations of Hadamard-type inequalities by incorporating fractional integral operators, specifically tailored for functions with convex third-order derivatives. The derivation of these results relies on the fundamental properties of convexity and the application of Hölder's inequality, alongside its extended forms. Furthermore, a comparative analysis of the estimates of the proven Hadamard-type inequalities is provided, accompanied by relevant applications.

2. Main results

Lemma 2.1. *Let $\Psi : I \subset \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on I° (I° is the interior of I), where $\varrho_1, \varrho_2 \in I$ with $\varrho_1 < \varrho_2$. If $\Psi''' \in L[\varrho_1, \varrho_2]$, then we have*

$$
\begin{aligned}
&\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}\\
&-\frac{\Gamma(\alpha+2)}{4(\varrho_2-\varrho_1)^{\alpha}}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J^{\alpha+1}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha+1}_{\varrho_1^+}\Psi(\varrho_2)\right]-\left[J^{\alpha}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha}_{\varrho_1^+}\Psi(\varrho_2)\right]\right\}\\
&=\frac{\varrho_2-\varrho_1}{4\left(\alpha+2\right)}\left[\Psi'(\varrho_2)-\Psi'(\varrho_1)\right]+\frac{\left(\varrho_2-\varrho_1\right)^3}{4\left(\alpha+2\right)}\left(I_1-I_2\right), \qquad (5)
\end{aligned}
$$

where $\alpha>0$,

$$
\begin{aligned}
I_1&=\int_0^1\sigma^{\alpha+2}(1-\sigma)\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)d\sigma,\\
I_2&=\int_0^1\sigma^{\alpha+2}(1-\sigma)\Psi'''(\sigma\varrho_2+(1-\sigma)\varrho_1)d\sigma.
\end{aligned}
$$

Proof. By integrating both integrals by parts three times, we get for the first integral

$$
\begin{aligned}
I_1=&\int_0^1\sigma^{\alpha+2}(1-\sigma)\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)d\sigma\\
=&\frac{\Psi'(\varrho_1)}{\left(\varrho_1-\varrho_2\right)^2}-\frac{2\left(\alpha+2\right)}{\left(\varrho_1-\varrho_2\right)^3}\Psi(\varrho_1)-\frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)}{\left(\varrho_1-\varrho_2\right)^3}\int_0^1\sigma^{\alpha-1}\Psi(\sigma\varrho_1+(1-\sigma)\varrho_2)d\sigma\\
&+\frac{\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)}{\left(\varrho_1-\varrho_2\right)^3}\int_0^1\sigma^{\alpha}\Psi(\sigma\varrho_1+(1-\sigma)\varrho_2)d\sigma.
\end{aligned}
$$

After changing variables by setting $\sigma\varrho_1+(1-\sigma)\varrho_2=\mathfrak{z}$, we obtain

$$I_1 = \frac{\Psi'(\varrho_1)}{(\varrho_2-\varrho_1)^2} + \frac{2(\alpha+2)}{(\varrho_2-\varrho_1)^3}\Psi(\varrho_1) + \frac{\alpha(\alpha+1)(\alpha+2)}{(\varrho_2-\varrho_1)^3}\int_{\varrho_2}^{\varrho_1}\left(\frac{\varrho_2-\mathfrak{z}}{\varrho_2-\varrho_1}\right)^{\alpha-1}$$

$$\Psi(\mathfrak{z})\left(\frac{-d\mathfrak{z}}{\varrho_2-\varrho_1}\right) - \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{(\varrho_2-\varrho_1)^3}\int_{\varrho_2}^{\varrho_1}\left(\frac{\varrho_2-\mathfrak{z}}{\varrho_2-\varrho_1}\right)^{\alpha}\Psi(\mathfrak{z})\left(\frac{-d\mathfrak{z}}{\varrho_2-\varrho_1}\right)$$

$$= \frac{\Psi'(\varrho_1)}{(\varrho_2-\varrho_1)^2} + \frac{2(\alpha+2)}{(\varrho_2-\varrho_1)^3}\Psi(\varrho_1) + \frac{\alpha(\alpha+1)(\alpha+2)}{(\varrho_2-\varrho_1)^{\alpha+3}}\int_{\varrho_1}^{\varrho_2}(\varrho_2-\mathfrak{z})^{\alpha-1}\Psi(\mathfrak{z})d\mathfrak{z}$$

$$- \frac{(\alpha+1)(\alpha+2)(\alpha+3)}{(\varrho_2-\varrho_1)^{\alpha+4}}\int_{\varrho_1}^{\varrho_2}(\varrho_2-\mathfrak{z})^{\alpha}\Psi(\mathfrak{z})d\mathfrak{z}$$

$$= \frac{\Psi'(\varrho_1)}{(\varrho_2-\varrho_1)^2} + \frac{2(\alpha+2)}{(\varrho_2-\varrho_1)^3}\Psi(\varrho_1) + \frac{\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha)}{(\varrho_2-\varrho_1)^{\alpha+3}}J^{\alpha}_{\varrho_1^+}\Psi(\varrho_2)$$
$$- \frac{(\alpha+1)(\alpha+2)(\alpha+3)\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha+4}}J^{\alpha+1}_{\varrho_1^+}\Psi(\varrho_2).$$

Similarly, for the second integral, we have

$$I_2 = \int_0^1 \sigma^{2+\alpha}(1-\sigma)\Psi'''(\sigma\varrho_2+(1-\sigma)\varrho_1)d\sigma$$

$$= \frac{\Psi'(\varrho_2)}{(\varrho_2-\varrho_1)^2} - \frac{2(\alpha+2)}{(\varrho_2-\varrho_1)^3}\Psi(\varrho_2) - \frac{(\alpha+2)(\alpha+1)\alpha}{(\varrho_2-\varrho_1)^3}\int_0^1\sigma^{\alpha-1}\Psi(\sigma\varrho_2+(1-\sigma)\varrho_1)d\sigma$$

$$+ \frac{(\alpha+3)(\alpha+2)(\alpha+1)}{(\varrho_2-\varrho_1)^3}\int_0^1\sigma^{\alpha}\Psi(\sigma\varrho_2+(1-\sigma)\varrho_1)d\sigma,$$

and after the change of variables $\sigma\varrho_2+(1-\sigma)\varrho_1=\mathfrak{z}$, we get

$$I_2 = \frac{\Psi'(\varrho_2)}{(\varrho_2-\varrho_1)^2} - \frac{2(\alpha+2)}{(\varrho_2-\varrho_1)^3}\Psi(\varrho_2) - \frac{\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha)}{(\varrho_2-\varrho_1)^{\alpha+3}}J^{\alpha}_{\varrho_2^-}\Psi(\varrho_1)$$
$$+ \frac{(\alpha+1)(\alpha+2)(\alpha+3)\Gamma(\alpha+1)}{(\varrho_2-\varrho_1)^{\alpha+4}}J^{\alpha+1}_{\varrho_2-}\Psi(\varrho_1).$$

By subtracting the last equalities, we obtain

$$\begin{aligned} I_1 - I_2 = &- \frac{\Psi'(\varrho_2) - \Psi'(\varrho_1)}{(\varrho_2 - \varrho_1)^2} + \frac{2(\alpha+2)}{(\varrho_2 - \varrho_1)^3}\left[\Psi(\varrho_2) + \Psi(\varrho_1)\right] \\ &- \frac{(\alpha+1)(\alpha+2)(\alpha+3)\Gamma(\alpha+1)}{(\varrho_2 - \varrho_1)^{\alpha+4}}\left[J^{\alpha+1}_{\varrho_2^-}\Psi(\varrho_1) + J^{\alpha+1}_{\varrho_1^+}\Psi(\varrho_2)\right] \\ &+ \frac{\alpha(\alpha+1)(\alpha+2)\Gamma(\alpha)}{(\varrho_2 - \varrho_1)^{\alpha+3}}\left[J^{\alpha}_{\varrho_2^-}\Psi(\varrho_1) + J^{\alpha}_{\varrho_1^+}\Psi(\varrho_2)\right] \\ = &- \frac{\Psi'(\varrho_2) - \Psi'(\varrho_1)}{(\varrho_2 - \varrho_1)^2} + \frac{2(\alpha+2)}{(\varrho_2 - \varrho_1)^3}\left[\Psi(\varrho_2) + \Psi(\varrho_1)\right] \\ &- \frac{\Gamma(\alpha+4)}{(\varrho_2 - \varrho_1)^{\alpha+4}}\left[J^{\alpha+1}_{\varrho_2^-}\Psi(\varrho_1) + J^{\alpha+1}_{\varrho_1^+}\Psi(\varrho_2)\right] \\ &+ \frac{\Gamma(\alpha+3)}{(\varrho_2 - \varrho_1)^{\alpha+3}}\left[J^{\alpha}_{\varrho_2^-}\Psi(\varrho_1) + J^{\alpha}_{\varrho_1^+}\Psi(\varrho_2)\right]. \end{aligned}$$

Finally, we multiply both parts of the last equality by the expression $\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)}$ and simplify to obtain the following:

$$\begin{aligned} &\frac{(\varrho_2 - \varrho_1)^3}{4(\alpha+2)}(I_1 - I_2) \\ = &- \frac{\varrho_2 - \varrho_1}{4(\alpha+2)}\left[\Psi'(\varrho_2) - \Psi'(\varrho_1)\right] + \frac{\Psi(\varrho_2) + \Psi(\varrho_1)}{2} \\ &- \frac{\Gamma(\alpha+2)}{4(\varrho_2-\varrho_1)^{\alpha}}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J^{\alpha+1}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha+1}_{\varrho_1^+}\Psi(\varrho_2)\right] - \left[J^{\alpha}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha}_{\varrho_1^+}\Psi(\varrho_2)\right]\right\}. \end{aligned}$$

The proof is complete. ☑

Corollary 2.2. *For* $\alpha = 1$, (5) *yields*

$$\begin{aligned} &\frac{\Psi(\varrho_2) + \Psi(\varrho_1)}{2} - \frac{1}{\varrho_2 - \varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma \\ &= \frac{\varrho_2 - \varrho_1}{12}\left[\Psi'(\varrho_2) - \Psi'(\varrho_1)\right] + \frac{(\varrho_2 - \varrho_1)^3}{12}(I_1 - I_2), \end{aligned}$$

where

$$I_1 = \int_0^1 \sigma^3(1-\sigma)\Psi'''(\sigma\varrho_1 + (1-\sigma)\varrho_2)d\sigma,$$

$$I_2 = \int_0^1 \sigma^3(1-\sigma)\Psi'''(\sigma\varrho_2 + (1-\sigma)\varrho_1)d\sigma.$$

Theorem 2.3. *Let $\Psi : I \to \mathbb{R}$, $\Psi \in C^3(I^\circ)$ and $\Psi''' \in L[\varrho_1, \varrho_2]$, where $\varrho_1, \varrho_2 \in I$. If $|\Psi'''|$ is a convex function, then for all $\alpha > 0$, the following inequality holds:*

$$\begin{aligned}&\Bigg|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}\\&-\frac{\Gamma(\alpha+2)}{4\left(\varrho_2-\varrho_1\right)^{\alpha}}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J^{\alpha+1}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha+1}_{\varrho_1^+}\Psi(\varrho_2)\right]-\left[J^{\alpha}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha}_{\varrho_1^+}\Psi(\varrho_2)\right]\right\}\Bigg|\\&\quad\le\frac{\varrho_2-\varrho_1}{4\left(\alpha+2\right)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|+\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)(\alpha+3)(\alpha+4)}\{|\Psi'''(\varrho_1)|+|\Psi'''(\varrho_2)|\}.\end{aligned} \tag{6}$$

***Proof*.** From Lemma 2.1, taking into account that $|\Psi'''|$ is convex, we get

$$\begin{aligned}&\Bigg|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}\\&-\frac{\Gamma(\alpha+2)}{4\left(\varrho_2-\varrho_1\right)^{\alpha}}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J^{\alpha+1}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha+1}_{\varrho_1^+}\Psi(\varrho_2)\right]-\left[J^{\alpha}_{\varrho_2^-}\Psi(\varrho_1)+J^{\alpha}_{\varrho_1^+}\Psi(\varrho_2)\right]\right\}\Bigg|\\&\quad\le\frac{\varrho_2-\varrho_1}{4\left(\alpha+2\right)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|\\&\qquad+\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)}\left[\int_0^1\sigma^{\alpha+2}(1-\sigma)|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|d\sigma\right.\\&\qquad\qquad\left.+\int_0^1\sigma^{\alpha+2}(1-\sigma)|\Psi'''(\sigma\varrho_2+(1-\sigma)\varrho_1)|d\sigma\right]\\&\quad\le\frac{\varrho_2-\varrho_1}{4\left(\alpha+2\right)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|\\&\qquad+\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)}\left[\int_0^1\sigma^{\alpha+2}(1-\sigma)[\sigma|\Psi'''(\varrho_1)|+(1-\sigma)|\Psi'''(\varrho_2)|]d\sigma\right.\\&\qquad\qquad\left.+\int_0^1\sigma^{\alpha+2}(1-\sigma)[\sigma|\Psi'''(\varrho_2)|+(1-\sigma)|\Psi'''(\varrho_1)|]d\sigma\right],\end{aligned}$$

and by calculating the integrals, we obtain the upper bound

$$\frac{\varrho_2-\varrho_1}{4(\alpha+2)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|+\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)(\alpha+3)(\alpha+4)}\left\{|\Psi'''(\varrho_1)|+|\Psi'''(\varrho_2)|\right\}.$$

☑

Corollary 2.4. *If we choose $\alpha = 1$, then from (6), we obtain*

$$\begin{aligned}&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right|\\&\quad\le\frac{\varrho_2-\varrho_1}{12}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|+\frac{(\varrho_2-\varrho_1)^3}{240}\left[|\Psi'''(\varrho_1)|+|\Psi'''(\varrho_2)|\right].\end{aligned}$$

Theorem 2.5. *Let $\Psi : I \to \mathbb{R}$, $\Psi \in C^3(I^\circ)$ and $\Psi''' \in L[\varrho_1, \varrho_2]$, where $\varrho_1, \varrho_2 \in I$. If $|\Psi'''|^\xi$ is a convex function, then for all $\alpha > 0, \xi > 1$ and $\frac{1}{\zeta}+\frac{1}{\xi} = 1$, the following inequality holds:*

$$\begin{aligned}
&\left| \frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2} \right. \\
&\left. -\frac{\Gamma(\alpha+2)}{4(\varrho_2-\varrho_1)^\alpha}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J_{\varrho_2^-}^{\alpha+1}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha+1}\Psi(\varrho_2)\right]-\left[J_{\varrho_2^-}^{\alpha}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha}\Psi(\varrho_2)\right]\right\}\right| \\
&\le \frac{\varrho_2-\varrho_1}{4(\alpha+2)}\left[\Psi'(\varrho_2)-\Psi'(\varrho_1)\right] \\
&\quad + \frac{(\varrho_2-\varrho_1)^3}{2(\alpha+2)}\left(B(\alpha\zeta+2\zeta+1,\zeta+1)\right)^{\frac{1}{\zeta}}\left[\frac{|\Psi'''(\varrho_1)|^\xi+|\Psi'''(\varrho_2)|^\xi}{2}\right]^{\frac{1}{\xi}}, \qquad (7)
\end{aligned}$$

where $B(.,.)$ is the Euler Beta function.

Proof. From Lemma 2.1, taking the properties of modulus into account, we obtain

$$\begin{aligned}
&\left| \frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2} \right. \\
&\left. -\frac{\Gamma(\alpha+2)}{4(\varrho_2-\varrho_1)^\alpha}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J_{\varrho_2^-}^{\alpha+1}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha+1}\Psi(\varrho_2)\right]-\left[J_{\varrho_2^-}^{\alpha}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha}\Psi(\varrho_2)\right]\right\}\right| \\
&\le \frac{\varrho_2-\varrho_1}{4(\alpha+2)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right| + \frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)}\left(|I_1|+|I_2|\right). \qquad (8)
\end{aligned}$$

By using the Hölder's inequality (1) and considering that $|\Psi'''|^\xi$ is convex, for $|I_1|$, we have

$$|I_1| \le \left(\int_0^1 \sigma^{(\alpha+2)\zeta}|1-\sigma|^\zeta d\sigma\right)^{\frac{1}{\zeta}}\left(\int_0^1 |\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^\xi d\sigma\right)^{\frac{1}{\xi}}.$$

Let us calculate the integrals:

$$\int_0^1 \sigma^{\alpha\zeta+2\zeta}(1-\sigma)^\zeta d\sigma = B(\alpha\zeta+2\zeta+1,\zeta+1),$$

and

$$\int_0^1 \left[\sigma|\Psi'''(\varrho_1)|^\xi+(1-\sigma)|\Psi'''(\varrho_2)|^\xi\right]d\sigma = \frac{|\Psi'''(\varrho_1)|^\xi+|\Psi'''(\varrho_2)|^\xi}{2}.$$

Thus, for first integral, we get

$$|I_1| \le \left[B(\alpha\zeta+2\zeta+1,\zeta+1)\right]^{\frac{1}{\zeta}}\left[\frac{|\Psi'''(\varrho_1)|^\xi+|\Psi'''(\varrho_2)|^\xi}{2}\right]^{\frac{1}{\xi}}.$$

Analogously, for I_2, we can write

$$|I_2| \leq \left[B\left(\alpha\zeta+2\zeta+1,\zeta+1\right)\right]^{\frac{1}{\zeta}}\left[\frac{\left|\Psi'''(\varrho_1)\right|^{\xi}+\left|\Psi'''(\varrho_2)\right|^{\xi}}{2}\right]^{\frac{1}{\xi}}.$$

By summing $|I_1|$, $|I_2|$ and taking (8) into account, we get (7). ☑

Corollary 2.6. *If we choose $\alpha = 1$, then from (7), we obtain*

$$\begin{aligned}\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right| &\leq \frac{\varrho_2-\varrho_1}{12}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right| \\ &+\frac{(\varrho_2-\varrho_1)^3}{6}\left(B\left(3\zeta+1,\zeta+1\right)\right)^{\frac{1}{\zeta}}\left[\frac{\left|\Psi'''(\varrho_1)\right|^{\xi}+\left|\Psi'''(\varrho_2)\right|^{\xi}}{2}\right]^{\frac{1}{\xi}}. \qquad (9)\end{aligned}$$

Theorem 2.7. *Let $\Psi : I \to \mathbb{R}$, $\Psi \in C^3(I^\circ)$ and $\Psi''' \in L\left[\varrho_1, \varrho_2\right]$, where $\varrho_1, \varrho_2 \in I$. If $\left|\Psi'''\right|^{\xi}$ is a convex function, then for all $\alpha > 0, \xi > 1$ and $\frac{1}{\zeta}+\frac{1}{\xi}=1$ the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}\right. \\ &\left.-\frac{\Gamma(\alpha+2)}{4\left(\varrho_2-\varrho_1\right)^{\alpha}}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J_{\varrho_2^-}^{\alpha+1}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha+1}\Psi(\varrho_2)\right]-\left[J_{\varrho_2^-}^{\alpha}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha}\Psi(\varrho_2)\right]\right\}\right| \\ &\quad\leq \frac{\varrho_2-\varrho_1}{4\left(\alpha+2\right)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|+\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)} \\ &\qquad\times\left[\left(B\left(\alpha\zeta+2\zeta+1,\zeta+2\right)\right)^{\frac{1}{\zeta}}+\left(B\left(\alpha\zeta+2\zeta+2,\zeta+1\right)\right)^{\frac{1}{\zeta}}\right]\left(\mathbf{M}_1+\mathbf{M}_2\right), \qquad (10)\end{aligned}$$

where $B(.,.)$ is the Euler Beta function and

$$\mathbf{M_1}=\left(\frac{\left|\Psi'''(\varrho_1)\right|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{6}\right)^{\frac{1}{\xi}}, \quad \mathbf{M_2}=\left(\frac{2\left|\Psi'''(\varrho_1)\right|^{\xi}+\left|\Psi'''(\varrho_2)\right|^{\xi}}{6}\right)^{\frac{1}{\xi}}.$$

Proof. By using the improved Hölder inequality (2) for $|I_1|$ from (8), we get

$$
\begin{aligned}
|I_1| &\leq \int_0^1 |\sigma^{\alpha+2}(1-\sigma)||\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|d\sigma \\
&\leq \left(\int_0^1 (1-\sigma)|\sigma^{\alpha+2}(1-\sigma)|^{\zeta}d\sigma\right)^{\frac{1}{\zeta}}\left(\int_0^1 (1-\sigma)|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma\right)^{\frac{1}{\xi}} \\
&\quad + \left(\int_0^1 \sigma|\sigma^{\alpha+2}(1-\sigma)|^{\zeta}d\sigma\right)^{\frac{1}{\zeta}}\left(\int_0^1 \sigma|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma\right)^{\frac{1}{\xi}} \\
&= \left(\int_0^1 \sigma^{\alpha\zeta+2\zeta}(1-\sigma)^{1+\zeta}d\sigma\right)^{\frac{1}{\zeta}}\left(\int_0^1 (1-\sigma)|\Psi'''(\sigma\varrho_1+(1-\sigma))\varrho_2|^{\xi}d\sigma\right)^{\frac{1}{\xi}} \\
&\quad + \left(\int_0^1 \sigma^{1+\alpha\zeta+2\zeta}(1-\sigma)^{\zeta}d\sigma\right)^{\frac{1}{\zeta}}\left(\int_0^1 \sigma|\Psi'''(\sigma\varrho_1+(1-\sigma))\varrho_2|^{\xi}d\sigma\right)^{\frac{1}{\xi}}.
\end{aligned}
$$

Here

$$
\begin{aligned}
&\int_0^1 \sigma^{\alpha\zeta+2\zeta}(1-\sigma)^{1+\zeta}d\sigma = B\left(\alpha\zeta+2\zeta,+1,\zeta+2\right), \\
&\int_0^1 \sigma^{1+\alpha\zeta+2\zeta}(1-\sigma)^{\zeta}d\sigma = B\left(\alpha\zeta+2\zeta+2,\zeta+1\right),
\end{aligned}
$$

and by using the definition of convexity,

$$
\begin{aligned}
&\int_0^1 (1-\sigma)|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2|^{\xi}d\sigma \\
&\leq |\Psi'''(\varrho_1)|^{\xi}\int_0^1 \sigma(1-\sigma)d\sigma + |\Psi'''(\varrho_2)|^{\xi}\int_0^1 (1-\sigma)^2 d\sigma = \frac{|\Psi'''(\varrho_1)|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{6}
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \sigma|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma \\
&\leq |\Psi'''(\varrho_1)|^{\xi}\int_0^1 \sigma^2 d\sigma + |\Psi'''(\varrho_2)|^{\xi}\int_0^1 \sigma\left(1-\sigma\right)d\sigma = \frac{2\left|\Psi'''(\varrho_1)\right|^{\xi}+|\Psi'''(\varrho_2)|^{\xi}}{6}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|I_1| &\leq \left(B\left(\alpha\zeta+2\zeta+1,\zeta+2\right)\right)^{\frac{1}{\zeta}}\cdot\left(\frac{|\Psi'''(\varrho_1)|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{6}\right)^{\frac{1}{\xi}} \\
&\quad + \left(B\left(\alpha\zeta+2\zeta+2,\zeta+1\right)\right)^{\frac{1}{\zeta}}\cdot\left(\frac{2\left|\Psi'''(\varrho_1)\right|^{\xi}+|\Psi'''(\varrho_2)|^{\xi}}{6}\right)^{\frac{1}{\xi}}.
\end{aligned}
$$

Similarly, for $|I_2|$ from (8), we get

$$|I_2| \le \left(B\left(\alpha\zeta+2\zeta+1,\zeta+2\right)\right)^{\frac{1}{\zeta}} \cdot \left(\frac{2\left|\Psi'''(\varrho_1)\right|^{\xi}+\left|\Psi'''(\varrho_2)\right|^{\xi}}{6}\right)^{\frac{1}{\xi}}$$
$$+\left(B\left(\alpha\zeta+2\zeta+2,\zeta+1\right)\right)^{\frac{1}{\zeta}} \cdot \left(\frac{\left|\Psi'''(\varrho_1)\right|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{6}\right)^{\frac{1}{\xi}}.$$

Summing the integrals yields

$$|I_1|+|I_2| \le \left(B\left(\alpha\zeta+2\zeta+1,\zeta+2\right)\right)^{\frac{1}{\zeta}}\left(\mathbf{M_1}+\mathbf{M_2}\right)$$
$$+\left(B\left(\alpha\zeta+2\zeta+2,\zeta+1\right)\right)^{\frac{1}{\zeta}}\left(\mathbf{M_2}+\mathbf{M_1}\right).$$

Considering the last inequality, we deduce (10) from (8). The proof is complete. ☑

Corollary 2.8. *By selecting $\alpha=1$, inequality (10) yields*

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right| \le \frac{\varrho_2-\varrho_1}{12}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|$$
$$+\frac{(\varrho_2-\varrho_1)^3}{12}\left[\left(B\left(3\zeta+1,\zeta+2\right)\right)^{\frac{1}{\zeta}}+\left(B\left(3\zeta+2,\zeta+1\right)\right)^{\frac{1}{\zeta}}\right]\left(\mathbf{M_1}+\mathbf{M_2}\right), \tag{11}$$

where $\mathbf{M_1}$ and $\mathbf{M_2}$ are defined above.

Theorem 2.9. *Let $\Psi : I \to \mathbb{R}$, $\Psi \in C^3(I^\circ)$ and $\Psi''' \in L\left[\varrho_1,\varrho_2\right]$, where $\varrho_1,\varrho_2 \in I$. If $|\Psi'''|^{\xi}$ is a convex function on $[\varrho_1,\varrho_2]$, then for all $\alpha>0, \xi \ge 1$, the following inequality holds:*

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}\right.$$
$$\left.-\frac{\Gamma(\alpha+2)}{4\left(\varrho_2-\varrho_1\right)^{\alpha}}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J_{\varrho_2^-}^{\alpha+1}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha+1}\Psi(\varrho_2)\right]-\left[J_{\varrho_2^-}^{\alpha}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha}\Psi(\varrho_2)\right]\right\}\right|$$
$$\le \frac{\varrho_2-\varrho_1}{4\left(\alpha+2\right)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|+\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)(\alpha+3)(\alpha+4)}\cdot\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right), \tag{12}$$

where

$$\mathbf{\Phi_1}=\left[\frac{(\alpha+3)\left|\Psi'''(\varrho_1)\right|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{\alpha+5}\right]^{\frac{1}{\xi}},$$
$$\mathbf{\Phi_2}=\left[\frac{2\left|\Psi'''(\varrho_1)\right|^{\xi}+(\alpha+3)\left|\Psi'''(\varrho_2)\right|^{\xi}}{\alpha+5}\right]^{\frac{1}{\xi}}.$$

Proof. By using the power mean inequality (3) and considering that $|\Psi'''|^{\xi}$ is convex, we get for $|I_2|$ from (8)

$$\begin{aligned}
|I_1| &\leq \int_0^1 |\sigma^{\alpha+2}(1-\sigma)||\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|d\sigma \\
&\leq \left(\int_0^1 |\sigma^{\alpha+2}(1-\sigma)|d\sigma\right)^{1-\frac{1}{\xi}} \left(\int_0^1 \sigma^{\alpha+2}(1-\sigma)\left|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)\right|^{\xi} d\sigma\right)^{\frac{1}{\xi}} \\
&\leq \left(\int_0^1 |\sigma^{\alpha+2}(1-\sigma)|d\sigma\right)^{1-\frac{1}{\xi}} \\
&\quad\times \left(\int_0^1 \sigma^{\alpha+2}(1-\sigma)\left[\sigma\left|\Psi'''(\varrho_1)\right|^{\xi}+(1-\sigma)\left|\Psi'''(\varrho_2)\right|^{\xi}\right] d\sigma\right)^{\frac{1}{\xi}}.
\end{aligned}$$

Let us calculate the integrals:

$$\int_0^1 \left|\sigma^{\alpha+2}(1-\sigma)\right| d\sigma = \frac{1}{(\alpha+3)(\alpha+4)}$$

and

$$\begin{aligned}
&\int_0^1 \sigma^{\alpha+2}(1-\sigma)\left[\sigma\left|\Psi'''(\varrho_1)\right|^{\xi}+(1-\sigma)\left|\Psi'''(\varrho_2)\right|^{\xi}\right] d\sigma \\
&\quad= \left|\Psi'''(\varrho_1)\right|^{\xi}\int_0^1 \sigma^{\alpha+3}(1-\sigma)d\sigma + \left|\Psi'''(\varrho_2)\right|^{\xi}\int_0^1 \sigma^{\alpha+2}(1-\sigma)^2 d\sigma \\
&\quad= \frac{(\alpha+3)\left|\Psi'''(\varrho_1)\right|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{(\alpha+3)(\alpha+4)(\alpha+5)}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
|I_1| &\leq \left(\frac{1}{(\alpha+3)(\alpha+4)}\right)^{1-\frac{1}{\xi}} \left[\frac{(\alpha+3)\left|\Psi'''(\varrho_1)\right|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{(\alpha+3)(\alpha+4)(\alpha+5)}\right]^{\frac{1}{\xi}} \\
&= \frac{1}{(\alpha+3)(\alpha+4)}\left[\frac{(\alpha+3)\left|\Psi'''(\varrho_1)\right|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{\alpha+5}\right]^{\frac{1}{\xi}}.
\end{aligned}$$

Similarly, for $|I_2|$ from (8), we get

$$|I_2| \leq \frac{1}{(\alpha+3)(\alpha+4)}\left[\frac{(\alpha+3)\left|\Psi'''(\varrho_2)\right|^{\xi}+2\left|\Psi'''(\varrho_1)\right|^{\xi}}{\alpha+5}\right]^{\frac{1}{\xi}}.$$

By summing the upper estimations for $|I_1|$ and $|I_2|$ and considering notations $\mathbf{\Phi_1}$ and $\mathbf{\Phi_2}$, we get

$$|I_1|+|I_2| \leq \frac{1}{(\alpha+3)(\alpha+4)}\cdot\left(\mathbf{\Phi_1}+\mathbf{\Phi_2}\right).$$

Thus, from (8), we deduce (12). The proof is concluded. ☑

Corollary 2.10. *If we choose* $\alpha = 1$*, then from* (12)*, we obtain*

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right| \leq \frac{\varrho_2-\varrho_1}{12}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|$$
$$+\frac{(\varrho_2-\varrho_1)^3}{240}\left\{\left[\frac{2\left|\Psi'''(\varrho_1)\right|^{\xi}+\left|\Psi'''(\varrho_2)\right|^{\xi}}{3}\right]^{\frac{1}{\xi}}+\left[\frac{\left|\Psi'''(\varrho_1)\right|^{\xi}+2\left|\Psi'''(\varrho_2)\right|^{\xi}}{3}\right]^{\frac{1}{\xi}}\right\}. \tag{13}$$

Theorem 2.11. *Let* $\Psi : I \to \mathbb{R}$*,* $\Psi \in C^3(I^\circ)$ *and* $\Psi''' \in L[\varrho_1, \varrho_2]$*, where* $\varrho_1, \varrho_2 \in I$*. If* $|\Psi'''|^{\xi}$ *is a convex function on* $[\varrho_1, \varrho_2]$*, then for all* $\alpha > 0, \xi \geq 1$*, the following inequality holds:*

$$\Bigg|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}$$
$$-\frac{\Gamma(\alpha+2)}{4\left(\varrho_2-\varrho_1\right)^{\alpha}}\left\{\frac{\alpha+3}{\varrho_2-\varrho_1}\left[J_{\varrho_2^-}^{\alpha+1}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha+1}\Psi(\varrho_2)\right]-\left[J_{\varrho_2^-}^{\alpha}\Psi(\varrho_1)+J_{\varrho_1^+}^{\alpha}\Psi(\varrho_2)\right]\right\}\Bigg|$$
$$\leq \frac{\varrho_2-\varrho_1}{4\left(\alpha+2\right)}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right|$$
$$+\frac{(\varrho_2-\varrho_1)^3}{4(\alpha+2)}\left[B\left(\alpha+3,3\right)^{1-\frac{1}{\xi}}\left(\mathbf{P_1}+\mathbf{P_3}\right)+B\left(\alpha+4,2\right)^{1-\frac{1}{\xi}}\left(\mathbf{P_2}+\mathbf{P_4}\right)\right], \tag{14}$$

where $B(.,.)$ *is the Euler Beta function and*

$$\mathbf{P_1} = \left[\left|\Psi'''(\varrho_1)\right|^{\xi}B\left(\alpha+4,3\right)+\left|\Psi'''(\varrho_2)\right|^{\xi}B\left(\alpha+3,4\right)\right]^{\frac{1}{\xi}},$$
$$\mathbf{P_2} = \left[\left|\Psi'''(\varrho_1)\right|^{\xi}B\left(\alpha+5,2\right)+\left|\Psi'''(\varrho_2)\right|^{\xi}B\left(\alpha+4,3\right)\right]^{\frac{1}{\xi}},$$
$$\mathbf{P_3} = \left[\left|\Psi'''(\varrho_2)\right|^{\xi}B\left(\alpha+4,3\right)+\left|\Psi'''(\varrho_1)\right|^{\xi}B\left(\alpha+3,4\right)\right]^{\frac{1}{\xi}},$$
$$\mathbf{P_4} = \left[\left|\Psi'''(\varrho_2)\right|^{\xi}B\left(\alpha+5,2\right)+\left|\Psi'''(\varrho_1)\right|^{\xi}B\left(\alpha+4,3\right)\right]^{\frac{1}{\xi}}.$$

Proof. By using the improved power mean inequality (4) and considering the convexity of $|\Psi'''|^{\xi}$, we have

$$
\begin{aligned}
|I_1| &\leq \int_0^1 \left|\sigma^{\alpha+2}(1-\sigma)\right| |\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|d\sigma \\
&\leq \left(\int_0^1 (1-\sigma)|\sigma^{\alpha+2}(1-\sigma)|d\sigma\right)^{1-\frac{1}{\xi}} \\
&\quad \times \left(\int_0^1 (1-\sigma)|\sigma^{\alpha+2}(1-\sigma)||\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma\right)^{\frac{1}{\xi}} \\
&\quad + \left(\int_0^1 \sigma|\sigma^{\alpha+2}(1-\sigma)|d\sigma\right)^{1-\frac{1}{\xi}} \left(\int_0^1 \sigma|\sigma^{\alpha+2}(1-\sigma)||\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma\right)^{\frac{1}{\xi}} \\
&= \left(\int_0^1 \sigma^{\alpha+2}(1-\sigma)^2 d\sigma\right)^{1-\frac{1}{\xi}} \left(\int_0^1 (\sigma^{\alpha+2}(1-\sigma)^2|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma\right)^{\frac{1}{\xi}} \\
&\quad + \left(\int_0^1 \sigma^{\alpha+3}(1-\sigma)d\sigma\right)^{1-\frac{1}{\xi}} \left(\int_0^1 \sigma^{\alpha+3}(\lambda-\sigma)|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma\right)^{\frac{1}{\xi}}.
\end{aligned}
$$

Here

$$
\begin{aligned}
\int_0^1 \sigma^{\alpha+2}(1-\sigma)^2 d\sigma &= B(\alpha+3,3), \\
\int_0^1 \sigma^{\alpha+3}(1-\sigma)d\sigma &= B(\alpha+4,2),
\end{aligned}
$$

and by using the definition of convexity, we get

$$
\begin{aligned}
&\int_0^1 \sigma^{\alpha+2}(1-\sigma)^2|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma \\
&\quad \leq |\Psi'''(\varrho_1)|^{\xi} \int_0^1 \sigma^{\alpha+3}(1-\sigma)^2 d\sigma + |\Psi'''(\varrho_2)|^{\xi} \int_0^1 \sigma^{\alpha+2}(1-\sigma)^3 d\sigma \\
&\quad = |\Psi'''(\varrho_1)|^{\xi} B(\alpha+4,3) + |\Psi'''(\varrho_2)|^{\xi} B(\alpha+3,4)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \sigma^{\alpha+3}(1-\sigma)|\Psi'''(\sigma\varrho_1+(1-\sigma)\varrho_2)|^{\xi}d\sigma \\
&\quad \leq |\Psi'''(\varrho_1)|^{\xi} \int_0^1 \sigma^{\alpha+4}(1-\sigma)d\sigma + |\Psi'''(\varrho_2)|^{\xi} \int_0^1 \sigma^{\alpha+3}(1-\sigma)^2 d\sigma \\
&\quad = |\Psi'''(\varrho_1)|^{\xi} B(\alpha+5,2) + |\Psi'''(\varrho_2)|^{\xi} B(\alpha+4,3).
\end{aligned}
$$

Thus, we have

$$|I_1| \leq B(\alpha+3,3)^{1-\frac{1}{\xi}} \left[|\Psi'''(\varrho_1)|^{\xi} B(\alpha+4,3) + |\Psi'''(\varrho_2)|^{\xi} B(\alpha+3,4)\right]^{\frac{1}{\xi}}$$
$$+ B(\alpha+4,2)^{1-\frac{1}{\xi}} \left[|\Psi'''(\varrho_1)|^{\xi} B(\alpha+5,2) + |\Psi'''(\varrho_2)|^{\xi} B(\alpha+4,3)\right]^{\frac{1}{\xi}}.$$

Similarly for $|I_2|$ from (8), we obtain

$$|I_2| \leq B(\alpha+3,3)^{1-\frac{1}{\xi}} \left[|\Psi'''(\varrho_2)|^{\xi} B(\alpha+4,3) + |\Psi'''(\varrho_1)|^{\xi} B(\alpha+3,4)\right]^{\frac{1}{\xi}}$$
$$+ B(\alpha+4,2)^{1-\frac{1}{\xi}} \left[|\Psi'''(\varrho_2)|^{\xi} B(\alpha+5,2) + |\Psi'''(\varrho_1)|^{\xi} B(\alpha+4,3)\right]^{\frac{1}{\xi}}.$$

Upon summing the integrals and considering the accepted notations, we get

$$|I_1| + |I_2| \leq B(\alpha+3,3)^{1-\frac{1}{\xi}} (\mathbf{P}_1 + \mathbf{P}_3) + B(\alpha+4,2)^{1-\frac{1}{\xi}} (\mathbf{P}_2 + \mathbf{P}_4).$$

By taking into account the last inequality and (8), we obtain (14). ☑

Corollary 2.12. *If we choose* $\alpha = 1$*, then from* (14)*, we obtain*

$$\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2} - \frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2} \Psi(\sigma)d\sigma\right| \leq \frac{\varrho_2-\varrho_1}{12} |\Psi'(\varrho_2) - \Psi'(\varrho_1)|$$
$$+ \frac{(\varrho_2-\varrho_1)^3}{12} \left[B(4,3)^{1-\frac{1}{\xi}} \left(\tilde{\mathbf{P_1}} + \tilde{\mathbf{P_3}}\right) + B(5,2)^{1-\frac{1}{\xi}} \left(\tilde{\mathbf{P_2}} + \tilde{\mathbf{P_4}}\right)\right], \quad (15)$$

where

$$\tilde{\mathbf{P_1}} = \left[|\Psi'''(\varrho_1)|^{\xi} B(5,3) + |\Psi'''(\varrho_2)|^{\xi} B(4,4)\right]^{\frac{1}{\xi}},$$
$$\tilde{\mathbf{P_2}} = \left[|\Psi'''(\varrho_1)|^{\xi} B(6,2) + |\Psi'''(\varrho_2)|^{\xi} B(5,3)\right]^{\frac{1}{\xi}},$$
$$\tilde{\mathbf{P_3}} = \left[|\Psi'''(\varrho_2)|^{\xi} B(5,3) + |\Psi'''(\varrho_1)|^{\xi} B(4,4)\right]^{\frac{1}{\xi}},$$
$$\tilde{\mathbf{P_4}} = \left[|\Psi'''(\varrho_2)|^{\xi} B(6,2) + |\Psi'''(\varrho_1)|^{\xi} B(5,3)\right]^{\frac{1}{\xi}}.$$

2.1. Examples

To verify the accuracy of the main results, the following examples are provided in order to analyze them from various viewpoints.

Example 2.13. Case 1: Considering $\Psi(\sigma) = e^{\sigma}$, where $\sigma > 0$, taking $[\varrho_1, \varrho_2] = [1, 2]$ and $\xi \in [1.1, 10]$, we find that the mapping $\Psi'''(\sigma) = e^{\sigma}$ is convex for $\sigma > 0$, inequality (9) becomes

$$-\frac{1}{12}(e^2-e)-\frac{1}{6}\left(\frac{1}{252}\right)^{1-\frac{1}{\xi}}\left[\frac{e^{\xi}+e^{2\xi}}{2}\right]^{\frac{1}{\xi}}$$
$$\leq \frac{e\,(3-e)}{2}\leq \frac{1}{12}(e^2-e)+\frac{1}{6}\left(\frac{1}{252}\right)^{1-\frac{1}{\xi}}\left[\frac{e^{\xi}+e^{2\xi}}{2}\right]^{\frac{1}{\xi}}. \tag{16}$$

Case 2: Let $\Psi(\sigma)=e^{\sigma},\sigma>0$, if we take $\xi=2$ and $\varrho_1\in[1,2],\varrho_2\in[3,4]$ then inequality (9) changes to

$$-\frac{\varrho_2-\varrho_1}{12}(e^{\varrho_2}-e^{\varrho_1})-\frac{(\varrho_2-\varrho_1)^3}{6}\left(\frac{1}{252}\right)^{\frac{1}{2}}\left[\frac{e^{2\varrho_1}+e^{2\varrho_2}}{2}\right]^{\frac{1}{2}}$$
$$\leq \left(\frac{e^{\varrho_2}+e^{\varrho_1}}{2}-\frac{e^{\varrho_2}-e^{\varrho_1}}{\varrho_2-\varrho_1}\right)$$
$$\leq \frac{\varrho_2-\varrho_1}{12}(e^{\varrho_2}-e^{\varrho_1})+\frac{(\varrho_2-\varrho_1)^3}{6}\left(\frac{1}{252}\right)^{\frac{1}{2}}\left[\frac{e^{2\varrho_1}+e^{2\varrho_2}}{2}\right]^{\frac{1}{2}}. \tag{17}$$

Figure 1 illustrates the three mappings corresponding to the Right, Middle, and Left sides of inequality (16) plotted as a function of $\xi\in[1.1,10]$. Additionally, Figure 2 displays the three mappings realized on the Right, Middle, and Left sides of inequality (17) graphed for $\varrho_1\in[1,2]$ and $\varrho_2\in[3,4]$.

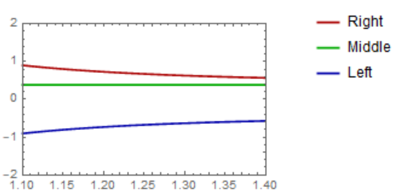

FIGURE 1. The graphical representation of Example 2.13 for $\varrho_1=1$, $\varrho_2=2$ and $\xi\in[1.1,10]$.

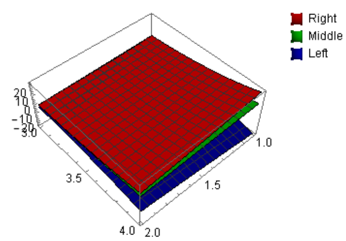

FIGURE 2. The graphical representation of Example 2.13 for $\varrho_1\in[1,2]$, $\varrho_2\in[3,4]$.

Example 2.14. Case 1 : Considering $\Psi(\sigma)=\ln\sigma$, where $\sigma>0$, taking $[\varrho_1,\varrho_2]=[2,3]$ and $\xi\in[1.1,100]$, we find that the mapping $\Psi'''(\sigma)=\frac{2}{\sigma^3}$ is convex for $\sigma>0$, inequality (9) becomes

$$-\frac{1}{72}-\frac{1}{6}\left(\frac{1}{252}\right)^{1-\frac{1}{\xi}}\left[\frac{\left|\frac{1}{4}\right|^{\xi}+\left|\frac{2}{27}\right|^{\xi}}{2}\right]^{\frac{1}{\xi}}$$
$$\leq \frac{\ln 2}{2}-2\ln 2+1\leq \frac{1}{72}+\frac{1}{6}\left(\frac{1}{252}\right)^{1-\frac{1}{\xi}}\left[\frac{\left|\frac{1}{4}\right|^{\xi}+\left|\frac{2}{27}\right|^{\xi}}{2}\right]^{\frac{1}{\xi}}. \tag{18}$$

Case 2: Let $\Psi(\sigma) = \ln\sigma, \sigma > 0$, if we take $\xi = 2$ and $\varrho_1 \in [2, 2.5], \varrho_2 \in [3, 4]$, then inequality (9) becomes

$$
\begin{aligned}
&-\frac{1}{12}\left|\frac{1}{\varrho_2}-\frac{1}{\varrho_1}\right| - \frac{(\varrho_2-\varrho_1)^3}{6}\left(\frac{1}{252}\right)^{\frac{1}{2}}\left[\frac{\left|\frac{2}{\varrho_1^3}\right|^2+\left|\frac{2}{\varrho_2^3}\right|^2}{2}\right]^{\frac{1}{2}} \\
&\quad \le \frac{\ln\varrho_1+\ln\varrho_2}{2} - \frac{1}{\varrho_2-\varrho_1}\left(\ln\left(\frac{\varrho_2^{\varrho_2}}{\varrho_1^{\varrho_1}}\right)+\varrho_1-\varrho_2\right) \\
&\quad \le \frac{1}{12}\left|\frac{1}{\varrho_2}-\frac{1}{\varrho_1}\right| + \frac{(\varrho_2-\varrho_1)^3}{6}\left(\frac{1}{252}\right)^{\frac{1}{2}}\left[\frac{\left|\frac{2}{\varrho_1^3}\right|^2+\left|\frac{2}{\varrho_2^3}\right|^2}{2}\right]^{\frac{1}{2}}. \qquad (19)
\end{aligned}
$$

Figure 1 illustrates the three mappings corresponding to the Right, Middle, and Left sides of inequality (18) plotted as a function of $\xi \in [1.1, 100]$. Additionally, Figure 2 displays the three mappings realized on the Right, Middle, and Left sides of inequality (19) graphed for $\varrho_1 \in [2, 2.5]$ and $\varrho_2 \in [3, 4]$.

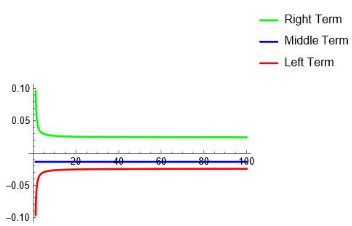

FIGURE 3. The graphical representation of Example 20 for $\varrho_1 = 2$, $\varrho_2 = 3$ and $\xi \in [1.1, 100]$.

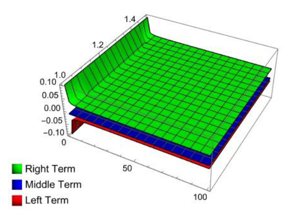

FIGURE 4. The graphical representation of Example 20 for $\varrho_1 \in [2, 2.5]$, $\varrho_2 \in [3, 4]$.

Example 2.15. Case 1: Let $\Psi(\sigma) = \frac{1}{6}\sigma^6, \sigma > 0$. If we take $\varrho_1 = 1, \varrho_2 = 2$ and $\xi \in [1.1, 10]$, then the mapping $\Psi'''(\sigma) = 20\sigma^3$ is convex for $\sigma > 0$ and inequality (13) becomes

$$
\begin{aligned}
&-\frac{31}{12} - \frac{1}{240}\left\{\left[\frac{2\cdot 20^{\xi}+160^{\xi}}{3}\right]^{\frac{1}{\xi}} + \left[\frac{20^{\xi}+2\cdot 160^{\xi}}{3}\right]^{\frac{1}{\xi}}\right\} \\
&\quad \le \frac{67}{28} \le \frac{31}{12} + \frac{1}{240}\left\{\left[\frac{2\cdot 20^{\xi}+160^{\xi}}{3}\right]^{\frac{1}{\xi}} + \left[\frac{20^{\xi}+2\cdot 160^{\xi}}{3}\right]^{\frac{1}{\xi}}\right\}. \qquad (20)
\end{aligned}
$$

Case 2: Let $\Psi(\sigma)=\frac{1}{6}\sigma^6, \sigma>0$. If we take $\xi=2$ and $\varrho_1\in[1,2], \varrho_2\in[3,4]$, then inequality (13) takes the following form:

$$\begin{aligned}
&-\frac{\varrho_2-\varrho_1}{12}(\varrho_2^5-\varrho_1^5)-\frac{(\varrho_2-\varrho_1)^3}{240}\left\{\left[\frac{800\varrho_1^6+400\varrho_2^6}{3}\right]^{\frac{1}{2}}+\left[\frac{400\varrho_1^6+800\varrho_2^6}{3}\right]^{\frac{1}{2}}\right\}\\
&\quad\leq \frac{\varrho_2^6+\varrho_1^6}{12}-\frac{\varrho_2^7-\varrho_1^7}{42(\varrho_2-\varrho_1)}\\
&\quad\quad\leq \frac{\varrho_2-\varrho_1}{12}(\varrho_2^5-\varrho_1^5)+\frac{(\varrho_2-\varrho_1)^3}{240}\left\{\left[\frac{800\varrho_1^6+400\varrho_2^6}{3}\right]^{\frac{1}{2}}+\left[\frac{400\varrho_1^6+800\varrho_2^6}{3}\right]^{\frac{1}{2}}\right\}.
\end{aligned} \tag{21}$$

The three mappings realized on the Right, Middle and Left sides of inequality (20) are plotted in Figure 3 as a function of $\xi\in[1.1,10]$. While the three mappings corresponding to the Right, Middle and Left sides of inequality (21) are graphed in Figure 4 for $\varrho_1\in[1,2]$ and $\varrho_2\in[3,4]$.

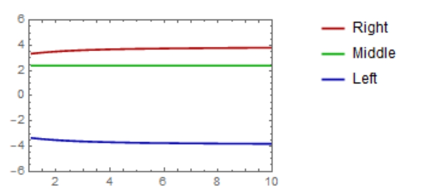

FIGURE 5. The graphical representation of Example 2.15 for $\varrho_1=1$, $\varrho_2=2$ and $\xi\in[1.1,10]$.

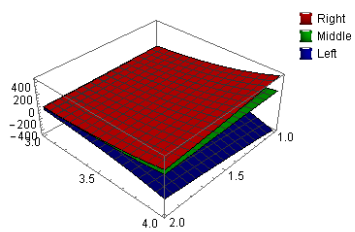

FIGURE 6. The graphical representation of Example 2.15 for $\varrho_1\in[1,2]$, $\varrho_2\in[3,4]$.

2.2. Comparison analysis of classical and improved bounds

Example 2.16. If one chooses $\Psi(\sigma)=\sigma^4, \sigma>0$, then for $\xi>1$, function $|\Psi'''(\sigma)|=24\sigma$ is convex. In case of $\alpha=1$, $[\varrho_1,\varrho_2]=[1,2]$ and $\xi=2$, let us find the right hand side of inequalities (9) and (11).

a) For (9), we have

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right| \\
&\leq \frac{1}{12}\cdot 28+\frac{1}{6}\cdot\left(B\left(7,3\right)\right)^{\frac{1}{2}}\cdot\left[\frac{24^2+48^2}{2}\right]^{\frac{1}{2}} \\
&\approx 2.333334+\frac{1}{6}\cdot\left(0.0039683\right)^{\frac{1}{2}}\cdot\left[\frac{24^2+48^2}{2}\right]^{\frac{1}{2}} \\
&\approx 2.333334+0.398412=2.731746.
\end{aligned}$$

b) For (11), we get

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int\limits_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right| \\
&\leq 2.333334+\frac{1}{12}\left[\sqrt{B\left(7,4\right)}+\sqrt{B\left(8,3\right)}\right]\left(\sqrt{\frac{24^2+2\cdot 48^2}{6}}+\sqrt{\frac{2\cdot 24^2+48^2}{6}}\right) \\
&\approx 2.333334+\frac{1}{12}\left(0.034504+0.052705\right)\left(29.393877+24\right) \\
&\approx 2.333334+\frac{0.087209\cdot 53.393877}{12}=2.333334+0.388036\approx 2.72137.
\end{aligned}$$

As 2.731746 is greater than 2.72137, we observe that the improved Hölder inequality provides a superior estimate compared to the classical Hölder inequality.

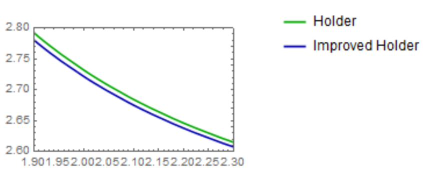

FIGURE 7. The graphical representation of Example 2.16 for $\varrho_1 = 1$, $\varrho_2 = 2$ and $\xi \in [1.1, 10]$.

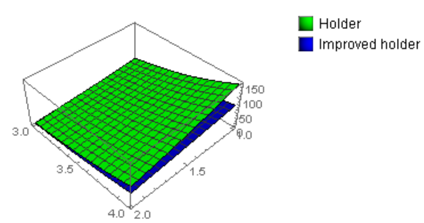

FIGURE 8. The graphical representation of Example 2.16 for $\varrho_1 \in [1, 2]$, $\varrho_2 \in [3, 4]$.

Example 2.17. If one chooses $\Psi(\sigma) = e^{2\sigma}, \sigma > 0$, then for $\xi > 1$ and $\sigma > 0$ function $|\Psi'''(\sigma)| = 8e^{2\sigma}$ is convex. In the case $\alpha = 1, [\varrho_1, \varrho_2] = [1, 2]$ and $\xi = 2$, let us find the right hand side of inequalities (13) and (15).

a) For (13), we have

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right| \\
&\quad\le \frac{e^4-e^2}{6}+\frac{8e^2}{240}\left[\sqrt{\frac{2+e^4}{3}}+\sqrt{\frac{1+2e^4}{3}}\right] \\
&\approx \frac{54.598150-7.389056}{6}+\frac{7.389056}{30}(4.343507+6.060701) \\
&\approx 7.868182+2.562576\approx 10.430758.
\end{aligned}$$

b) For (15), we get

$$\begin{aligned}
&\left|\frac{\Psi(\varrho_2)+\Psi(\varrho_1)}{2}-\frac{1}{\varrho_2-\varrho_1}\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma\right| \\
&\le \frac{\varrho_2-\varrho_1}{12}\left|\Psi'(\varrho_2)-\Psi'(\varrho_1)\right| \\
&\quad+\frac{(\varrho_2-\varrho_1)^3}{12}\left[\sqrt{B(4,3)}\left(\tilde{\mathbf{P_1}}+\tilde{\mathbf{P_3}}\right)+\sqrt{B(5,2)}\left(\tilde{\mathbf{P_2}}+\tilde{\mathbf{P_4}}\right)\right] \\
&\approx 7.868182+\frac{8e^2}{12}\cdot\sqrt{\frac{1}{60}}\left(\sqrt{B(5,3)+e^4B(4,4)}+\sqrt{B(5,3)\,e^4+B(4,4)}\right) \\
&\quad+\frac{8e^2}{12}\cdot\sqrt{\frac{1}{30}}\left(\sqrt{B(6,2)+e^4B(5,3)}+\sqrt{B(6,2)\,e^4+B(5,3)}\right) \\
&\approx 7.868182+\frac{59.112449}{12}(0.175330+0.343558)\approx 10.424244.
\end{aligned}$$

As 10.430758 is greater than 10.424244, we can conclude that the improved power mean inequality offers a more accurate estimate compared to the classical power mean inequality.

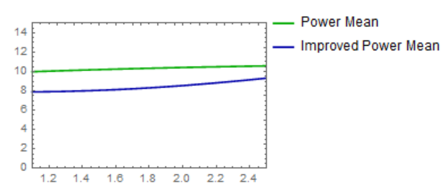

FIGURE 9. The graphical representation of Example 2.17 for $\varrho_1 = 1, \varrho_2 = 2$ and $\xi \in [1.1, 10]$.

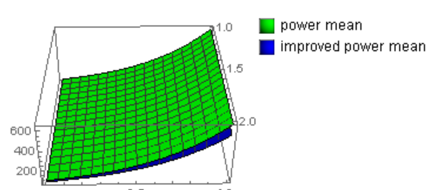

FIGURE 10. The graphical representation of Example 2.17 for $\varrho_1 \in [1,2], \varrho_2 \in [3,4]$.

3. Applications

3.1. Special Means

Next, we will explore the means for arbitrary real numbers ϱ_1 and ϱ_2 ($\varrho_1 \neq \varrho_2$). Let us consider the following ones.

(1) Arithmetic mean:

$$A(\varrho_1, \varrho_2) = \frac{\varrho_1 + \varrho_2}{2}, \quad \varrho_1, \varrho_2 \in \mathbb{R}.$$

(2) Logarithmic mean:

$$L(\varrho_1, \varrho_2) = \frac{\varrho_1 - \varrho_2}{\ln|\varrho_1| - \ln|\varrho_2|}, \quad |\varrho_1| \neq |\varrho_2|, \ \varrho_1, \varrho_2 \neq 0, \ \varrho_1, \varrho_2 \in \mathbb{R}.$$

(3) Generalized log-mean:

$$L_n(\varrho_1, \varrho_2) = \left[\frac{\varrho_2^{n+1} - \varrho_1^{n+1}}{(n+1)(\varrho_2 - \varrho_1)}\right]^{\frac{1}{n}}, \ n \in \mathbb{Z}\backslash\{-1, 0\}, \ \varrho_1, \varrho_2 \in \mathbb{R}^+, \ \varrho_1 < \varrho_2.$$

(4) Harmonic mean

$$H = H(\varrho_1, \varrho_2) = \frac{2\varrho_1\varrho_2}{\varrho_1 + \varrho_2}, \quad \varrho_1, \varrho_2 > 0.$$

We present some applications to special means of real numbers using our main results.

Proposition 3.1. *Let $\varrho_1, \varrho_2 \in [0, \infty)$, $\varrho_1 < \varrho_2$. We have*

$$\left|A\left(e^{\varrho_1}, e^{\varrho_2}\right) - L\left(e^{\varrho_1}, e^{\varrho_2}\right)\right| \leq \frac{\varrho_2 - \varrho_1}{12}\left(e^{\varrho_2} - e^{\varrho_1}\right) + \frac{(\varrho_2 - \varrho_1)^3}{120} A\left(e^{\varrho_1}, e^{\varrho_2}\right).$$

***Proof*.** This is a consequence of applying Corollary 2.4 to the convex function

$$\Psi : [0, \infty) \to \mathbb{R}, \ \Psi(\sigma) = e^{\sigma}.$$

☑

Proposition 3.2. *Let $\varrho_1, \varrho_2 \in [0, \infty)$, $\varrho_1 < \varrho_2$, $n \in \mathbb{Z}^+$, $n \geq 3$, $\xi > 1$. We have*

$$\begin{aligned} \left|A\left(\varrho_1^n, \varrho_2^n\right) - L_n^n\left(\varrho_1, \varrho_2\right)\right| &\leq \frac{n\left(\varrho_2 - \varrho_1\right)\left(\varrho_2^{n-1} - \varrho_1^{n-1}\right)}{12} \\ &+ \frac{n(n-1)(n-2)(\varrho_2 - \varrho_1)^3}{6}\left(B\left(3\zeta + 1, \zeta + 1\right)\right)^{\frac{1}{\zeta}} \cdot \left(A\left(\varrho_1^{(n-3)\xi}, \varrho_2^{(n-3)\xi}\right)\right)^{\frac{1}{\xi}}, \end{aligned}$$

where $B(.,.)$ is the Euler Beta function and $\frac{1}{\zeta} + \frac{1}{\xi} = 1$.

Proof. This is a consequence of applying Corollary 2.6 to the convex function

$$\Psi : [0,\infty) \to \mathbb{R},\ \Psi(\sigma) = \sigma^n.$$

☑

Proposition 3.3. *Let $\varrho_1, \varrho_2 \in [0,\infty)$, $\varrho_1 < \varrho_2$, $\xi > 1$, $\frac{1}{\zeta} + \frac{1}{\xi} = 1$. We have*

$$\begin{aligned} &\left|A\left(e^{\varrho_1}, e^{\varrho_2}\right) - L\left(e^{\varrho_1}, e^{\varrho_2}\right)\right| \\ &\le \frac{\varrho_2 - \varrho_1}{12}\left(e^{\varrho_2} - e^{\varrho_1}\right) + \frac{(\varrho_2 - \varrho_1)^3}{6}\left(B\left(3\zeta + 1, \zeta + 1\right)\right)^{\frac{1}{\zeta}} \cdot \left(A\left(e^{\varrho_1 \xi}, e^{\varrho_2 \xi}\right)\right)^{\frac{1}{\xi}}. \end{aligned}$$

Proof. This is a consequence of applying Corollary 2.6 to the convex function

$$\Psi : [0,\infty) \to \mathbb{R},\ \Psi(\sigma) = e^{\sigma}.$$

☑

Proposition 3.4. *Let $\varrho_1, \varrho_2 \in [0,\infty)$, $\varrho_1 < \varrho_2$, $n \in \mathbb{Z}^+$, $n \ge 3$, $\xi \ge 1$. We have*

$$\begin{aligned} &\left|A\left(\varrho_1^n, \varrho_2^n\right) - L_n^n\left(\varrho_1, \varrho_2\right)\right| \\ &\le \frac{n\left(\varrho_2 - \varrho_1\right)\left(\varrho_2^{n-1} - \varrho_1^{n-1}\right)}{12} + \frac{n(n-1)(n-2)(\varrho_2 - \varrho_1)^3}{120}\left(\frac{2}{3}\right)^{\frac{1}{\xi}} \\ &\times A\left(\left(A\left(2\varrho_1^{(n-3)\xi}, \varrho_2^{(n-3)\xi}\right)\right)^{\frac{1}{\xi}}, \left(A\left(\varrho_1^{(n-3)\xi}, 2\varrho_2^{(n-3)\xi}\right)\right)^{\frac{1}{\xi}}\right). \end{aligned}$$

Proof. This is a consequence of applying Corollary 2.10 to the convex function

$$\Psi : [0,\infty) \to \mathbb{R},\ \Psi(\sigma) = \sigma^n.$$

☑

3.2. Quadrature formula

Now, we present an application related to a quadrature formula. Assume d is a partition where $\varrho_1 = \sigma_0 < \sigma_1 ... < \sigma_{m-1} < \sigma_m = \varrho_2$, which represents the interval $[\varrho_1, \varrho_2]$. Let us take a look at the associated quadrature formula:

$$\int_{\varrho_1}^{\varrho_2} \Psi(\sigma) d\sigma = T(\Psi, d) + E(\Psi, d).$$

Here

$$T(\Psi, d) = \sum_{i=0}^{m-1}\left[\frac{\Psi(\sigma_{i+1}) + \Psi(\sigma_i)}{2}\left(\sigma_{i+1} - \sigma_i\right)\right],$$

and it computes the approximation error $E(\Psi, d)$. In this context, we provide certain error evaluations applicable to the quadrature formula.

Proposition 3.5. *According to the stipulations in Corollary 2.4, the following inequality holds:*

$$
\begin{aligned}
&\left|\int_{\varrho_1}^{\varrho_2}\Psi(\sigma)d\sigma - T(\Psi,d)\right| \\
&\quad\le \sum_{i=0}^{m-1}\left[\frac{(\sigma_{i+1}-\sigma_i)^2}{12}\left|\Psi'(\sigma_{i+1})-\Psi'(\sigma_i)\right| + \frac{(\sigma_{i+1}-\sigma_i)^4}{240}\left(\left|\Psi'''(\sigma_{i+1})\right|+\left|\Psi'''(\sigma_i)\right|\right)\right].
\end{aligned}
$$

Proof. By applying Corollary 2.4, the desired result is achieved. ☑

3.3. $\tilde{q}$-digamma function

The $\tilde{q}$-digamma mapping is determined by the next expression:

$$
\delta_{\tilde{q}}(\sigma) = -\ln(\tilde{q}-1) + \ln(\tilde{q})\left(\sigma - \frac{1}{2} - \sum_{j=1}^{\infty}\frac{\tilde{q}^{-j\sigma}}{1-\tilde{q}^{-j\sigma}}\right),
$$

with $\tilde{q} > 1$ and $\sigma > 0$ (see [27]).

Proposition 3.6. *For* $0 < \varrho_1 < \varrho_2$, *we have*

$$
\begin{aligned}
&\left|\frac{\delta_{\tilde{q}}'(\varrho_2)+\delta_{\tilde{q}}'(\varrho_1)}{2} - \frac{\delta_{\tilde{q}}(\varrho_2)-\delta_{\tilde{q}}(\varrho_1)}{\varrho_2-\varrho_1}\right| \\
&\quad\le \frac{\varrho_2-\varrho_1}{12}\left[\delta_{\tilde{q}}''(\varrho_2)-\delta_{\tilde{q}}''(\varrho_1)\right] + \frac{(\varrho_2-\varrho_1)^3}{240}\left[\left|\delta_{\tilde{q}}''''(\varrho_1)\right| + \left|\delta_{\tilde{q}}''''(\varrho_2)\right|\right].
\end{aligned}
$$

Proof. Applying Corollary 2.4 to the function $\Psi(\sigma) = \delta_{\tilde{q}}'(\sigma)$ for $\sigma > 0$ yields the desired result. ☑

Proposition 3.7. *For* $0 < \varrho_1 < \varrho_2$, $\xi > 1$, *we have*

$$
\begin{aligned}
\left|\frac{\delta_{\tilde{q}}'(\varrho_2)+\delta_{\tilde{q}}'(\varrho_1)}{2} - \frac{\delta_{\tilde{q}}(\varrho_2)-\delta_{\tilde{q}}(\varrho_1)}{\varrho_2-\varrho_1}\right| &\le \frac{\varrho_2-\varrho_1}{12}\left[\delta_{\tilde{q}}''(\varrho_2)-\delta_{\tilde{q}}''(\varrho_1)\right] \\
&+ \frac{(\varrho_2-\varrho_1)^3}{6}\left(B\left(3\zeta+1,\zeta+1\right)\right)^{\frac{1}{\zeta}}\left[\frac{\left|\delta_{\tilde{q}}''''(\varrho_1)\right|^{\varrho_1} + \left|\delta_{\tilde{q}}''''(\varrho_2)\right|^{\xi}}{2}\right]^{\frac{1}{\xi}}.
\end{aligned}
$$

Proof. Applying $\Psi(\sigma) = \delta_{\tilde{q}}'(\sigma)$ for $\sigma > 0$ to Corollary 2.6 and obtains the desired result. ☑

3.4. Modified Bessel function

Let the function $\mathcal{I}_p : \mathbb{R} \to [1, 0)$ be defined by

$$\mathcal{I}_p(\sigma) = 2^p \Gamma(1+p) \sigma^{-p} I_p(\sigma),$$

where we used the modified Bessel function of the first kind I_p, which is defined as

$$I_p(\sigma) = \sum_{n \geq 0} \frac{(\frac{\sigma}{2})^{p+2n}}{n! \Gamma(p+n+1)},$$

in which $p > -1$ and $\Gamma(.)$ is the Euler Gamma function (see [25]).

The first-, second-, and nth-order derivative formulas of $\mathcal{I}_p(\sigma)$ are respectively given by (see [15])

$$\mathcal{I}'_p(\sigma) = \frac{\sigma}{2(1+p)} \mathcal{I}_{1+p}(\sigma),$$

$$\mathcal{I}''_p(\sigma) = \frac{1}{2p+2} \mathcal{I}_{p+1}(\sigma) + \frac{\sigma^2}{4p^2+12p+8} \mathcal{I}_{p+2}(\sigma).$$

$$\frac{\partial^n \mathcal{I}_p(\sigma)}{\partial^n \sigma} = 2^{n-2p} \sqrt{\pi}\, \sigma^{p-n} \Gamma(1+p) {}_2F_3\left(\frac{1+p}{2}, \frac{2+p}{2}; \frac{1+p-n}{2}, \frac{2+p-n}{2}, 1+p; \frac{\sigma^2}{4}\right),$$

where ${}_2F_3(.;.;.)$ is the hyper-geometric function defined by (see [15])

$${}_2F_3\left(a_1, a_2; b_1, b_2, b_3; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b_1)_k (b_2)_k (b_3)_k} \frac{z^k}{k!}.$$

Proposition 3.8. *Let $\varrho_1, \varrho_2 \in \mathbb{R}$ with $0 < \varrho_1 < \varrho_2$. For each $p > -1$, we have*

$$\begin{aligned} &\left| \frac{\varrho_2 \mathcal{I}_{1+p}(\varrho_2) + \varrho_1 \mathcal{I}_{1+p}(\varrho_1)}{4(1+p)} - \frac{\mathcal{I}_p(\varrho_2) - \mathcal{I}_p(\varrho_1)}{\varrho_2 - \varrho_1} \right| \leq \frac{\varrho_2 - \varrho_1}{12} \left| \mathcal{I}''_p(\varrho_2) - \mathcal{I}''_p(\varrho_1) \right| \\ &+ \frac{(\varrho_2 - \varrho_1)^3}{240} 2^{4-2p} \sqrt{\pi}\, \Gamma(1+p) \left[|\varrho_1|^{p-4} \left| {}_2F_3\left(\frac{1+p}{2}, \frac{2+p}{2}; \frac{p-3}{2}, \frac{p-2}{2}, 1+p; \frac{\varrho_1^2}{4}\right) \right| \right. \\ &\left. + |\varrho_2|^{p-4} \left| {}_2F_3\left(\frac{1+p}{2}, \frac{2+p}{2}; \frac{p-3}{2}, \frac{p-2}{2}, 1+p; \frac{\varrho_2^2}{4}\right) \right| \right]. \end{aligned}$$

Proof. Applying Corollary 2.4 to $\Psi(\sigma) = \mathcal{I}'_p(\sigma)$ yields the desired result. ☑

4. Conclusion

To the best of our knowledge, the current investigation provides new perspectives of fractional Hermite–Hadamard integral inequalities involving three times differentiable functions. We develop a new fractional integral identity for three times differentiable functions via Riemann–Liouville fractional integral operators. Taking advantage of the established identity, combining with classical convexity, we achieve a series of new Hadamard-type integral inequalities.

It is also worth noting that the main results obtained here are transformed into results of classical calculus, which are also new. It should be emphasized that the extended versions of Hölder's inequality perform slightly better than the classical version. With the ideas developed in this paper, we hope to motivate interested researchers to further explore other types of results by considering fractional integral operators other than Riemann–Liouville, or, by considering general convexity instead of convexity, or, by considering higher-order derivatives (greater than third). Moreover, one can explore such extensions in Fractal, Fractal-Fractional and Quantum calculus.

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