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Existence of solutions for some degenerate elliptic equation under Fourier boundary conditions

Existencia de soluciones para alguna ecuación elíptica degenerada en condiciones de contorno de Fourier

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Abstract. This paper aims to study the existence of renormalized solutions for the anisotropic elliptic problem with a Hardy potential and Fourier boundary conditions

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \alpha |u|^{r(x)-1}u = \nu \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} + f(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u)\cdot n_i + \lambda u = g(x) & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $I\!R^N$ $(N \geq 2)$, the data f belongs to $L^1(\Omega)$, $g \in L^1(\partial\Omega)$ and $\alpha, \lambda, \nu > 0$. with $a_i(x,s,\xi)$ are Carathéodory functions that verifying some nonstandard conditions.

Key words and phrases. Anisotropic variable exponent Sobolev space, quasilinear elliptic problem, Hardy potential, Fourier boundary conditions, renormalized solution, L^1-data.

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Resumen. Este artículo tiene como objetivo estudiar la existencia de soluciones renormalizadas para el problema elíptico anisotrópico con un potencial de

Hardy y condiciones de borde de Fourier

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \alpha |u|^{r(x)-1} u = \nu \dfrac{|u|^{p_0(x)-2} u}{|x|^{p_0(x)}} + f(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u) \cdot n_i + \lambda u = g(x) & \text{on } \partial\Omega, \end{cases}$$

donde Ω es un subconjunto abierto y acotado de $I\!R^N$ $(N \geq 2)$, los datos f pertenecen a $L^1(\Omega)$, $g \in L^1(\partial\Omega)$ y $\alpha, \lambda, \nu > 0$. con $a_i(x,s,\xi)$ son funciones de Carathéodoria que verifican algunas condiciones no estándar.

Palabras y frases clave. Espacio de Sobolev de exponente variable anisotrópico, problema elíptico cuasilineal, potencial de Hardy, condiciones de contorno de Fourier, solución renormalizada, L^1-datos.

1. Introduction

In recent years, increasing attention has been paid to anisotropic elliptic problems in the study of nonlinear elliptic equations involving lower-order terms. The particular interest in these equations arises from their usefulness in the mathematical modeling of physical and mechanical processes in anisotropic continuums. It is recognized that lower-order terms can influence the existence, uniqueness, regularity, and asymptotic behavior of solutions to partial differential equations, as evidenced by various studies (see, e.g., [11, 15]).

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ containing the origin, with Lipschitz boundary $\partial\Omega$. In [24], the authors studied a nonlinear anisotropic elliptic problem with a Fourier-type boundary condition, formulated as follows:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, D^i u) + \beta(u) \ni \mu & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, D^i u) \cdot n_i + \lambda u = g(x) & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where β is a maximal monotone graph on $\mathbb{R}$ with $0 \in \beta(0)$, and μ is a bounded Radon measure. They proved the existence and uniqueness of renormalized or entropy solutions to the general elliptic problem (1). For related results, see also [6, 8]. In [23], I. Ibrango and S. Ouaro investigated the following nonlinear elliptic problem with a Fourier-type boundary condition:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, D^i u) + b(u) = f & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, D^i u) \cdot n_i + \lambda u = g(x) & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where f and g belong to $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively. Using monotone operator techniques in Banach spaces, they established the existence of a weak solution. Furthermore, via approximation methods, they proved the existence and uniqueness of entropy solutions. Additional details can be found in [7, 9, 11].

In [2], the authors have studied the nonlinear elliptic problem

$$\begin{cases} -\triangle u \pm |\nabla u|^2 = \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with $\lambda > 0$, they have demonstrated the existence of positive solutions for the problem (3) in the absorption case $(+|\nabla u|^2)$ with $f \in L^1(\Omega)$. In the diffusion case $(-|\nabla u|^2)$, the non-existence of solution is proved even in a very weak sense, we refer to [1], [2] and [3]). Porzio has considered in [29] the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(M(x,u)\nabla u) + \nu |u|^{s-1}u = \lambda \dfrac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

with $s > \dfrac{N}{N-2}$. the author has established the existence of solution $u \in W_0^{1,q}(\Omega)$ for every $1 < q < 2s/(s+1)$, we refer the reader also to [30]. The authors have investigated in [5] the existence of entropy solutions for the quasilinear elliptic problem

$$\begin{cases} Au + |u|^{s(x)-1}u = f + \lambda \dfrac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

in the anisotropic variable exponent Soblev spaces, where $Au = -\displaystyle\sum_{i=1}^{N} a_i(x, u, \nabla u)$ is a Leray-Lions operator, such that the Carathéodory functions $a_i(x, u, \nabla u)$ verifies sone non-standard assumptions, with $\lambda \geq 0, f \in L^1(\Omega)$ and

$$s(x) > \max\Big(\frac{N(p_0(x)-1)}{N-p_0(x)}, \frac{1}{p_0(x)-1}\Big) \qquad a.e \quad \text{in} \quad \Omega.$$

For more results involving Hardy potential, we refer the reader to [15], [16], and [17].

Since f belongs to $L^1(\Omega)$, we cannot expect a solution in the sense of distributions, as there is no guarantee that the field $a_i(x, u, \nabla u) \in L^1_{Loc}(\Omega)$. To address this difficulty, we adopt the framework of renormalized solutions in this work. The notion of a renormalized solution was introduced for the first time

P.-L. Lions and Di Perna in [20] by studying the Boltzmann equation. This notion was adapted to elliptic problem with L^1 data in the reference [14] (see also [27, 28]). Also, this adaptation was applied to elliptic problems with bounded measure data in [18]. For parabolic equations with L^1 data, this notion was adapted by D. Blanchard and F. Murat in [12] and by D. Blanchard et al. in [13]. At the same time, the equivalent notion of entropy solutions has been developed independently by Bénilan et al. in [10] for the study of nonlinear elliptic problems.

In this paper, we study the existence of renormalized solutions for the following quasi-linear anisotropic problem with degenerate coercivity, and the Fourier boundary condition :

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \alpha|u|^{r(x)-1}u = \nu\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} + f(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,u,\nabla u)\cdot n_i + \lambda u = g(x) & \text{on } \partial\Omega, \end{cases} \tag{6}$$

for all $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$ and for all $\alpha, \nu > 0$ and $\lambda > 0$. We prove the existence of solutions in the sense of distributions for the case of $L^\infty-$data. Furthermore, we conclude the existence and regularity of renormalized solutions in the anisotropic variable exponent Soblev spaces for L^1-data.

The paper is organized as follows: In Section 2 we recall some definitions and lemmas concerning the anisotropic Sobolev spaces with variable exponents. Section 3 is devoted to presenting the assumptions on the Carathéodory functions $a_i(x,u,\nabla u)$ under which our problem has at least one solution. Section 4 is dedicated to study the existence of solution in the sense of distributions for our elliptic equation with right-hand side $F(x) \in L^\infty(\partial\Omega)$ and $G(x) \in L^\infty(\partial\Omega)$. In the last section, we establish the existence of renormalized solutions for the non-coercive elliptic equation (6) with the right-hand side $f(x) \in L^1(\Omega)$ and $g(x) \in L^1(\partial\Omega)$.

2. Main results

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$, with Lipschitz boundary $\partial\Omega$, we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function } p(\cdot) : \Omega \longmapsto I\!R \text{ such that } 1 < p^- \leq p^+ < N\},$$

where

$$p^- = ess\inf\{p(x) \,/\, x \in \Omega\} \qquad \text{and} \qquad p^+ = ess\sup\{p(x) \,/\, x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \mapsto I\!R$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)}dx$$

is finite. If the exponent is bounded, i.e. if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm.

The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv\ dx\right| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \tag{7}$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

An important role in manipulating the generalized Lebesgue spaces is played by the modular $\rho_{p(\cdot)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result :

Proposition 2.1. (see [21], [31]) *If $u_n, u \in L^{p(\cdot)}(\Omega)$, then the following properties hold true:*

(i) $\|u\|_{p(\cdot)} < 1 \quad (resp, = 1, > 1) \quad \Longleftrightarrow \quad \rho_{p(\cdot)}(u) < 1 \quad (resp, = 1, > 1)$,

(ii) $\|u\|_{p(\cdot)} > 1 \implies \|u\|^{p^-}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^+}_{p(\cdot)}$ *and* $\|u\|_{p(\cdot)} < 1 \implies \|u\|^{p^+}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq \|u\|^{p^-}_{p(\cdot)}$,

(iii) $\|u_n\|_{p(\cdot)} \to 0 \iff \rho_{p(\cdot)}(u_n) \to 0$, *and* $\|u_n\|_{p(\cdot)} \to \infty \iff \rho_{p(\cdot)}(u_n) \to \infty$,

which implies that the norm convergence and the modular convergence are equivalent.

We refer the reader to [19] for more details concerning Lebesgue space with variable exponent.

Now, we define the anisotropic variable exponent Sobolev space used in the study of our quasilinear elliptic problem (6).

Let $p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot)$ be N variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = (1, p_1(\cdot), \ldots, p_N(\cdot)),\ D^0 u = u \ \text{ and } \ D^i u = \frac{\partial u}{\partial x_i} \ \text{ for } \ i = 1, \ldots, N,$$

and we define

$$\underline{p} = \min\{p_1^-, \ldots, p_N^-\} \qquad \text{then} \qquad \underline{p} > 1. \tag{8}$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in W^{1,1}(\Omega) \quad \text{and} \quad D^i u \in L^{p_i(\cdot)}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}. \tag{9}$$

The space $\big(W^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)}\big)$ is a reflexive Banach space (cf. [26]).

Lemma 2.2. *We have the following continuous and compact embedding*

- *if* $\underline{p} < N$ *then* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ *for* $q \in [\underline{p}, \underline{p}^*[$, *where* $\underline{p}^* = \dfrac{N\underline{p}}{N-\underline{p}}$,
- *if* $\underline{p} = N$ *then* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \qquad \forall q \in [\underline{p}, +\infty[$,
- *if* $\underline{p} > N$ *then* $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.

The proof of the lemma follows from the fact that the embedding $W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W^{1,\underline{p}}(\Omega)$ is continuous, and in view of the compact embedding theorems of Sobolev spaces.

Definition 2.3. Let $k > 0$, we consider the truncation function $T_k(\cdot) : I\!R \longmapsto I\!R$, given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega) := \{u : \Omega \mapsto I\!R$ measurable, such that $T_k(u) \in W^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0\}$.

Proposition 2.4. *For any* $u \in \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$, *there exists a unique measurable function* $v_i : \Omega \mapsto I\!R$ *for any* $i \in \{1, \dots, N\}$ *such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \qquad a.e. \quad x \in \Omega,$$

where χ_E *represents the characteristic function of a measurable set* E. *The functions* v_i *are called the weak partial derivatives of* u *and are still denoted* $D^i u$. *Moreover, if* u *belongs to* $W^{1,1}(\Omega)$, *then* v_i *coincides with the standard distributional derivative of* u, *that is,* $v_i = D^i u$.

Definition 2.5. We introduce the set $\mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$ as a subset of $\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$ for which a generalized notion of trace may be defined (see also [4] for the case of constant exponent). More precisely, $\mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$ is the set of function u in $\mathcal{T}^{1,\vec{p}(\cdot)}(\Omega)$, such that : there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ and a measurable function v defined on $\partial\Omega$ and verifying

(a) $u_n \longrightarrow u$ a.e. in Ω.

(b) $D^i T_k(u_n) \longrightarrow D^i T_k(u)$ in $L^1(\Omega)$ for every $k > 0$.

(c) $u_n \longrightarrow v$ a.e. on $\partial\Omega$.

The function v is called the trace of u in the generalized sense introduced in [4].

Proposition 2.6. *Let $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $\tau(u)$.*

For any $u \in \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\omega)$, the trace of u on $\partial\Omega$will be denote by $tr(u)$ or u, the operator $tr(\cdot)$ satisfied the following properties:

(i) *if $u \in \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$, then $\tau(T_k(u)) = T_k(tr(u))$ for any $k > 0$.*

(ii) *if $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega)$, then, for any $u \in \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$, we have $u - \varphi \in \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$ and $tr(u - \varphi) = tr(u) - \tau(\varphi)$.*

In the case where $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$. Obviously, we have

$$W^{1,\vec{p}(\cdot)}(\Omega) \subset \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega) \subset \mathcal{T}^{1,\vec{p}(\cdot)}(\Omega).$$

Lemma 2.7. (see [22], Theorem 13.47) *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that*

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $u_n \geq 0$ *and* $u \geq 0$ *a.e. in* Ω,

(iii) $\displaystyle\int_\Omega u_n\,dx \to \int_\Omega u\,dx,$

then $u_n \to u$ strongly in $L^1(\Omega)$.

3. Essential assumptions

In this paper, we establish the existence of renormalized solutions for the non-coercive quasilinear anisotropic elliptic equation given by

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + \alpha |u|^{r(x)-1}u = \nu \dfrac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} + f(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u)\cdot n_i + \lambda u = g(x) & \text{on } \partial\Omega, \end{cases} \tag{10}$$

Let $p_i(\cdot) : \overline{\Omega} \to (1,+\infty)$ measurable for $i = 1,\dots,N$. We consider the Leray-Lions operator A, acted from $W^{1,\vec{p}(\cdot)}(\Omega)$ into its dual, defined by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u),$$

where $a_i : \Omega \times I\!R \times I\!R^N \longmapsto I\!R$ are Carathéodory functions for $i = 1,\dots,N$ (measurable with respect to x in Ω for every (s,ξ) in $I\!R \times I\!R^N$, and continuous with respect to (s,ξ) in $I\!R \times I\!R^N$ for almost every x in Ω), which satisfy the following conditions :

$$\sum_{i=1}^{N} (a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0 \qquad \text{for} \quad \xi_i \neq \xi_i', \tag{11}$$

$$|a_i(x,s,\xi)| \leq \beta(K_i(x) + |s|^{p_i(x)-1} + |\xi_i|^{p_i(x)-1}) \qquad \text{for} \quad i = 1,\dots,N, \tag{12}$$

for a.e. $x \in \Omega$ and all $(s,\xi) \in I\!R \times I\!R^N$, where the nonnegative functions $K_i(\cdot)$ are assumed to be in $L^{p_i'(\cdot)}(\Omega)$ for $i = 1,\dots,N$, with $\dfrac{1}{p_i(x)} + \dfrac{1}{p_i'(x)} = 1$ and β is a positive constants.

$$a_i(x,s,\xi)\xi_i \geq b(|s|)|\xi_i|^{p_i(x)} \quad \text{for} \quad i = 1,\dots,N, \tag{13}$$

where $b(\cdot) : I\!R^+ \mapsto I\!R^+$ is a decreasing function. Furthermore there exists a positive constant b_0 and $\delta(\cdot)$ is a measurable function such that

$$b(|s|) \geq \frac{b_0}{(1+|s|)^{\delta(x)}},$$

and $0 < \delta(x) < p_i(x) - 1$ a.e. in Ω for $i = 1,\cdots,N$.

The data $f(x) \in L^1(\Omega)$ and $g(x) \in L^1(\partial\Omega)$, the constants $\alpha, \nu > 0$ and $\lambda > 0$, the exponents $r(x)$ and δ verifying

$$r(x) > \max\Big\{\frac{1+\delta(x)}{p_i(x)-1}, \frac{N(p_0(x)-1)}{N-p_0(x)}\Big\}.$$

We are going now to recall the following technical Lemma, useful to prove our main results.

Lemma 3.1. (see [9]) *Let $k > 0$, assuming that* (12) − (11) *hold true, and let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $W^{1,\vec{p}(\cdot)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *weakly in* $W^{1,\vec{p}(\cdot)}(\Omega)$ *and*

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}&(a_i(x,T_k(u_n),\nabla u_n)-a_i(x,T_k(u_n),\nabla u))(D^iu_n-D^iu)\,dx\\ &+\int_{\Omega}(|u_n|^{\underline{p}-2}u_n-|u|^{\underline{p}-2}u)(u_n-u)\,dx\to 0\quad\text{as}\quad n\to\infty,\end{aligned}\tag{14}$$

then $u_n \to u$ strongly in $W^{1,\vec{p}(\cdot)}(\Omega)$ for a subsequence.

4. Existence of solutions in the sense of distributions for $L^\infty-$ data

We consider the quasilinear elliptic problem

$$\begin{cases}-\mathrm{div}\ a(x,T_n(u),\nabla u)+\alpha|u|^{r(x)-1}u=\nu\dfrac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+1/n}+F(x) & \text{in }\Omega,\\ a(x,T_n(u),\nabla u)\cdot n_i+\lambda T_n(u)=G(x) & \text{on }\partial\Omega,\end{cases}\tag{15}$$

with

$$G(x)\in L^\infty(\partial\Omega)\qquad\text{and}\qquad |F(x)|\le C_0\quad\text{for any}\quad x\in\Omega,\tag{16}$$

where C_0 is a positive constant.

Definition 4.1. A measurable function u is called a solution in the sense of distributions for the quasilinear anisotropic elliptic equation (15), if $u \in W^{1,\vec{p}(\cdot)}(\Omega)$ and $|u|^{r(x)+1} \in L^1(\Omega)$, such that u verifies the following equality

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u),\nabla u)D^iv\,dx&+\alpha\int_{\Omega}|u|^{r(x)-1}uv\,\mathrm{d}x+\lambda\int_{\partial\Omega}T_n(u)\ v\,\mathrm{d}\sigma\\ =\int_{\Omega}F\ v\,\mathrm{d}x&+\nu\int_{\Omega}\frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+1/n}v\,\mathrm{d}x+\int_{\partial\Omega}G\ v\,\mathrm{d}\sigma,\end{aligned}\tag{17}$$

for any $v \in W^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$, where $r(x) > \dfrac{N(p_0(x)-1)}{N-p_0(x)}$ a.e. in Ω.

Theorem 4.2. *Assuming that* (12) − (11) *and* (16) *hold true. Then there exists at least one solution in the sense of distributions for the quasilinear elliptic equation* (15).

Proof of Theorem 4.2

Step 1 : Approximate problem

We consider the following approximate problem for the quasilinear elliptic equation (15), giving by

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_m), \nabla u_m) + \alpha |T_m(u_m)|^{r(x)-1} T_m(u_m) & \\ \qquad\qquad + \frac{1}{m}|u_m|^{\underline{p}-2}u_m = \nu \frac{|T_n(u_m)|^{p_0(x)-2}T_n(u_m)}{|x|^{p_0(x)}+1/n} + F(x) & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x, T_n(u_m), \nabla u_m).n_i + \lambda T_n(u_m) = G(x) & \text{on } \partial\Omega. \end{cases} \tag{18}$$

We define the operator H_m acted from $W^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $(W^{1,\vec{p}(\cdot)}(\Omega))'$ by

$$\begin{aligned} \langle H_m u, v\rangle &= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v \mathrm{d}x + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2} uv \, \mathrm{d}x \\ &\quad + \lambda \int_{\partial\Omega} T_n(u) v \, \mathrm{d}\sigma - \int_{\partial\Omega} G(x) v \, \mathrm{d}\sigma. \end{aligned} \tag{19}$$

We consider the operator $K_m : W^{1,\vec{p}(\cdot)}(\Omega) \longmapsto (W^{1,\vec{p}(\cdot)}(\Omega))'$ given by

$$\langle K_m u, v\rangle = \alpha \int_\Omega |T_m(u)|^{r(x)-1} T_m(u) v \, \mathrm{d}x - \nu \int_\Omega \frac{|T_n(u)|^{p_0(x)-2}T_n(u)}{|x|^{p_0(x)}+1/n} v \, \mathrm{d}x, \tag{20}$$

for any $u, v \in W^{1,\vec{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned} |\langle K_m u, v\rangle| &= \alpha \int_\Omega |T_m(u)|^{r(x)} |v| \, \mathrm{d}x + \nu \int_\Omega \frac{|T_n(u)|^{p_0(x)-1}}{|x|^{p_0(x)}+1/n} |v| \, \mathrm{d}x \\ &\leq \alpha m^{r_+} \int_\Omega |v| \, \mathrm{d}x + \nu n^{p_0^+} \int_\Omega |v| \, \mathrm{d}x \\ &\leq C_1 \|v\|_{1,\vec{p}(\cdot)}. \end{aligned} \tag{21}$$

Lemma 4.3. *The bounded operator $B_m = H_m + K_m$ acting from $W^{1,\vec{p}(\cdot)}(\Omega)$ into it dual $(W^{1,\vec{p}(\cdot)}(\Omega))'$ is a pseudo-monotone operator. Moreover, B_m is coercive in the following sense :*

$$\frac{\langle B_m v, v\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}(\cdot)} \longrightarrow \infty, \tag{22}$$

for any $v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

The proof of Lemma 4.3 is similar to the arguments of the proof in [8] (see also [6]) with very few modifications.

In view of Lemma 4.3 (cf. [25], Theorem 8.2) there exists at least one weak solution $u_m \in W^{1,\vec{p}(\cdot)}(\Omega)$ for the problem (18), i.e.

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x, T_n(u_m), \nabla u_m) D^i v\, dx + \alpha \int_{\Omega} |T_m(u_m)|^{r(x)-1} T_m(u_m) v\, dx \\
&\qquad + \frac{1}{m}\int_{\Omega} |u_m|^{\underline{p}-2} u_m v\, dx + \lambda \int_{\partial\Omega} T_n(u_m) v\, \mathrm{d}\sigma \\
&= \int_{\Omega} F(x) v\, dx + \nu \int_{\Omega} \frac{|T_n(u_m)|^{p_0(x)-2} T_n(u_m)}{|x|^{p_0(x)} + 1/n} v\, dx + \int_{\partial\Omega} G(x) v\, d\sigma,
\end{aligned} \tag{23}$$

for any $v \in W^{1,\vec{p}(\cdot)}(\Omega)$.

Step 2: Weak convergence of the sequence $(u_m)_m$

Let $m \geq n \geq 1$, by taking $v = u_m$ as a test function for the approximate problem (18), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x, T_n(u_m), \nabla u_m) D^i u_m\, dx + \alpha \int_{\Omega} |T_m(u_m)|^{r(x)}\ |u_m|\, dx \\
&\qquad + \frac{1}{m}\int_{\Omega} |u_m|^{\underline{p}}\, dx + \lambda \int_{\partial\Omega} |T_n(u_m)|\ |u_m|\, d\sigma \\
&= \int_{\Omega} F(x) u_m\, dx + \nu \int_{\Omega} \frac{|T_n(u_m)|^{p_0(x)-1} |u_m|}{|x|^{p_0(x)} + 1/n}\, dx + \int_{\partial\Omega} G(x) u_m\, d\sigma.
\end{aligned} \tag{24}$$

We set $\delta^+ = \max\{\delta_i^+, \quad \text{for } i = 1, \cdots, N\}$, in view of (13) and (16) we obtain

$$\begin{aligned}
&\frac{b_0}{(1+n)^{\delta^+}}\sum_{i=1}^{N}\int_{\Omega} |D^i u_m|^{p_i(x)}\, dx + \alpha \int_{\Omega} |T_m(u_m)|^{r(x)}\ |u_m|\, dx \\
&\qquad + \frac{1}{m}\int_{\Omega} |u_m|^{\underline{p}}\, dx + \lambda \int_{\partial\Omega} |T_n(u_m)|\ |u_m|\, d\sigma \\
&\leq C_0 \int_{\Omega} |u_m|\, dx + \nu \int_{\Omega} \frac{|T_n(u_m)|^{p_0(x)-1} |u_m|}{|x|^{p_0(x)} + \frac{1}{n}}\, dx + \|G\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u_m|\, d\sigma.
\end{aligned} \tag{25}$$

For the first and the second terms on the right-hand side of (25), by applying Young's inequality we have

$$
\begin{aligned}
C_0\int_\Omega |u_m| dx &\leq C_0\int_{\{|u_m|\leq m\}} |u_m|\,dx + C_0\int_{\{|u_m|>m\}} |u_m|\,dx\\
&\leq C_1 + \varepsilon\int_{\{|u_m|<m\}} |T_m(u_m)|^{r(x)}\,|u_m|\,dx + C_0\int_{\{|u_m|>m\}} |u_m|\,dx\\
&\leq C_1 + \varepsilon\int_\Omega |T_m(u_m)|^{r(x)}\;|u_m|\,dx + C_0\int_{\{|u_m|>m\}} |u_m|\,dx.
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\nu\int_\Omega \frac{|T_n(u_m)|^{p_0(x)-1}|u_m|}{|x|^{p_0(x)}+\frac{1}{n}}dx &\leq \nu n\int_\Omega |T_m(u_m)|^{p_0(x)-1}|u_m|\,dx\\
&\leq \frac{\alpha}{4}\int_\Omega |T_m(u_m)|^{r(x)}|u_m|\,dx + C_3(n)\int_\Omega |u_m|\,dx\\
&\leq \Big(\frac{\alpha}{4}+\frac{C_3(n)}{C_0}\varepsilon\Big)\int_\Omega |T_m(u_m)|^{r(x)}|u_m|\,dx\\
&\qquad + C_3(n)\int_{\{|u_m|>m\}} |u_m|\,dx + C_4(n).
\end{aligned}
\tag{27}
$$

On the other hand, For the last term on the right-hand side of (25), in view of the trace theorem, and thanks to (26) and Young's inequality, we obtain

$$
\begin{aligned}
&\|G\|_{L^\infty(\partial\Omega)}\int_{\partial\Omega} |u_m|\,dx \leq C_5\Big(\int_\Omega |u_m|\,dx + \sum_{i=1}^N\int_\Omega\,|D^iu_m|\,dx\Big)\\
&\leq \frac{C_5}{C_0}\varepsilon\int_\Omega |T_m(u_m)|^{r(x)}|u_m|\,dx + C_5\int_{\{|u_m|>m\}} |u_m|\,dx\\
&\quad + \frac{b_0}{2(1+n)^{\delta+}}\sum_{i=1}^N\int_\Omega\,|D^iu_m|^{p_i(x)}\,dx + C_7(n).
\end{aligned}
\tag{28}
$$

By taking $\varepsilon>0$ small enough such that $(1+\dfrac{C_3(n)}{C_0}+\dfrac{C_5}{C_0})\varepsilon\leq\dfrac{\alpha}{4}$ and putting (25) and $(26)-(28)$ together, we conclude that

$$
\begin{aligned}
&\frac{b_0}{2(1+n)^{\delta+}}\sum_{i=1}^N\int_\Omega |D^iu_m|^{p_i(x)}\,dx + \frac{\alpha}{2}\int_\Omega |T_m(u_m)|^{r(x)}\;|u_m|\,dx\\
&\quad + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}}\,dx + \lambda\int_{\partial\Omega} |T_n(u_m)|\;|u_m|\,d\sigma\\
&\leq C_8(n) + (C_0+C_3(n)+C_5)\int_{\{|u_m|>m\}} |u_m|\,dx.
\end{aligned}
\tag{29}
$$

Therefore, by taking $m \geq 1$ large enough, for example $\frac{\alpha}{4} m^{r^+} - (C_0 + C_3(n) + C_5)) > 0$, a.e. in Ω, we get

$$\begin{aligned} &\frac{b_0}{2(1+n)^{\delta^+}} \sum_{i=1}^{N} \int_\Omega |D^i u_m|^{p_i(x)}\, dx + \frac{\alpha}{4} \int_\Omega |T_m(u_m)|^{r(x)}\, |u_m|\, dx \\ &\quad + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}}\, dx + \lambda \int_{\partial\Omega} |T_n(u_m)|\, |u_m|\, d\sigma \\ &\leq C_8(n). \end{aligned} \tag{30}$$

Moreover, Thanks to (30), there exists a positive constant $C_9(n)$ that not depending on m such that

$$\|u_m\|_{1,\vec{p}(\cdot)} \leq C_9(n). \tag{31}$$

Thus, the sequence $(u_m)_m$ is uniformly bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$, and there exists a subsequence still denoted by $(u_m)_m$ such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}(\cdot)}(\Omega), \\ u_m \longrightarrow u & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega, \\ u_m \rightharpoonup u & \text{weakly in } L^1(\partial\Omega) \text{ and a.e. on } \partial\Omega. \end{cases} \tag{32}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-2} u_m \longrightarrow 0 \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega) \tag{33}$$

and in view of Lebesgue's dominated convergence theorem we deduce that

$$\frac{|T_n(u_m)|^{p_0(x)-2} T_n(u_m)}{|x|^{p_0(x)} + \frac{1}{n}} \longrightarrow \frac{|T_n(u)|^{p_0(x)-2} T_n(u)}{|x|^{p_0(x)} + \frac{1}{n}} \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{34}$$

Moreover, in view of (29) we conclude that $(T_m(u_m))_m$ is bounded in $L^{r(\cdot)+1}(\Omega)$, and since $T_m(u_m) \to u$ almost everywhere in Ω, we get

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in} \quad L^{r(\cdot)+1}(\Omega). \tag{35}$$

Step 3 : The convergence almost everywhere of the gradient

By taking $v = u_m - u$ as a test function for the approximated problem (15) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i u)dx \\
&\qquad + \alpha \int_\Omega |T_m(u_m)|^{r(x)-1} T_m(u_m)(u_m - u)dx \\
&+ \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m(u_m - u)\, dx + \lambda \int_{\partial\Omega} T_n(u_m)\ (u_m - u)\, d\sigma \\
&= \int_\Omega F(x)\ (u_m - u)\, dx + \nu \int_\Omega \frac{|T_n(u_m)|^{p_0(x)-2} T_n(u_m)}{|x|^{p_0(x)} + \frac{1}{n}} (u_m - u)\, dx \\
&\qquad + \int_{\partial\Omega} G(x)\ (u_m - u)\, d\sigma, \qquad (36)
\end{aligned}$$

it follows that

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega \left(a_i(x, T_n(u_m), \nabla u_m) - a_i(x, T_n(u_m), \nabla u)\right)(D^i u_m - D^i u)\, dx \\
&\qquad + \alpha \int_\Omega \Big(|T_m(u_m)|^{r(x)-1} T_m(u_m) - |T_m(u)|^{r(x)-1} T_m(u)\Big)(u_m - u)\, dx \\
&\qquad + \lambda \int_{\partial\Omega} (T_n(u_m) - T_n(u))\ (u_m - u)\, d\sigma \\
&= -\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u)(D^i u_m - D^i u)dx - \alpha \\
&\qquad \int_\Omega |T_m(u)|^{r(x)-1} T_m(u)(u_m - u)\, dx \\
&\quad - \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m(u_m - u)\, dx - \lambda \int_{\partial\Omega} T_n(u)\ (u_m - u)\, d\sigma \\
&\quad + \int_\Omega F(x)(u_m - u)\, dx + \nu \int_\Omega \frac{|T_n(u_m)|^{p_0(x)-2} T_n(u_m)}{|x|^{p_0(x)} + \frac{1}{n}} (u_m - u)\, dx \\
&\quad + \int_{\partial\Omega} G(x)(u_m - u)\, d\sigma \\
&\le \sum_{i=1}^{N} \int_\Omega |a_i(x, T_n(u_m), \nabla u)|\ |D^i u_m - D^i u|\, dx + \alpha \int_\Omega |T_m(u)|^{r(x)}\ |u_m - u| dx \\
&\quad + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-1}\ |u_m - u|\, dx + \lambda \int_{\partial\Omega} |T_n(u)|\ |u_m - u|\, d\sigma \\
&\quad + \int_\Omega |F(x)|\ |u_m - u|\, dx + \int_\Omega \frac{|T_n(u_m)|^{p_0(x)-1} |u_m - u|}{|x|^{p_0(x)} + \frac{1}{n}} dx + \\
&\qquad \int_{\partial\Omega} |G(x)|\ |u_m - u|\, d\sigma. \qquad (37)
\end{aligned}$$

For the first term on the right-hand side of (37), we have $T_n(u_m) \to T_n(u)$ strongly in $L^{p_i(\cdot)}(\Omega)$ then $|a_i(x, T_n(u_m), \nabla u)| \longrightarrow |a_i(x, T_n(u), \nabla u)|$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and since $D^i u_m \to D^i u$ weakly in $L^{p_i(\cdot)}(\Omega)$, it follows that

$$\sum_{i=1}^{N} \int_\Omega |a_i(x, T_n(u_m), \nabla u)| \, |D^i u_m - D^i u| \, dx \longrightarrow 0 \qquad \text{as} \quad m \to \infty. \tag{38}$$

Concerning the second term on the right-hand side of (37), we have $|T_m(u)|^{r(x)} \to |u|^{r(x)}$ strongly in $L^{\frac{r(\cdot)+1}{r(\cdot)}}(\Omega)$ and since $u_m \rightharpoonup u$ weakly in $L^{r(\cdot)+1}(\Omega)$, it follows that

$$\alpha \int_\Omega |T_m(u)|^{r(x)} |u_m - u| \, dx \longrightarrow 0 \qquad \text{as} \quad m \to \infty. \tag{39}$$

Moreover, in view of (33) and (34), we deduce that

$$\frac{1}{m} \int_\Omega |u_m|^{\underline{p}-1} \, |u_m - u| \, dx \longrightarrow 0 \qquad \text{as} \quad m \to \infty, \tag{40}$$

$$\int_\Omega \frac{|T_n(u_m)|^{p_0(x)-1}}{|x|^{p_0(x)} + \frac{1}{n}} \, |u_m - u| \, dx \longrightarrow 0 \qquad \text{as} \quad m \to \infty \tag{41}$$

and

$$\int_\Omega |F(x)| \, |u_m - u| \, dx \longrightarrow 0 \qquad \text{as} \quad m \to \infty. \tag{42}$$

Furthermore, we have $T_n(u)$ and $G(x)$ belongs to $L^\infty(\partial\Omega)$, and since $u_m \to u$ strongly in $L^1(\partial\Omega)$, it follows that

$$\lambda \int_{\partial\Omega} |T_n(u)| \, |u_m - u| \, d\sigma \longrightarrow 0 \qquad \text{as} \quad m \to \infty \tag{43}$$

and

$$\int_{\partial\Omega} |G(x)| \, |u_m - u| \, d\sigma \longrightarrow 0 \qquad \text{as} \quad m \to \infty. \tag{44}$$

By combining (37) and (38) – (44) we conclude that

$$\begin{aligned} \lim_{m\to\infty} \Big(\sum_{i=1}^{N} \int_\Omega \left(a_i(x, T_n(u_m), \nabla u_m) - a_i(x, T_n(u_m), \nabla u)\right) (D^i u_m - D^i u) \, dx \\ + \int_\Omega \left(|u_m|^{\underline{p}-2} u_m - |u|^{\underline{p}-2} u\right) (u_m - u) \, dx \Big) = 0. \end{aligned} \tag{45}$$

In view of Lemma 3.1, we conclude that

$$\begin{cases} u_m \to u \quad \text{strongly in} \quad W^{1,\vec{p}(\cdot)}(\Omega), \\ D^i u_m \to D^i u \qquad \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, ..., N. \end{cases} \tag{46}$$

Thus, we have $a_i(x, T_n(u_m), \nabla u_m) \to a_i(x, T_n(u), \nabla u)$ almost everywhere in Ω, and since $(a_i(x, T_n(u_m), \nabla u_m))_m$ is uniformly bounded in $L^{p_i'(\cdot)}(\Omega)$, it follows that

$$a_i(x, T_n(u_m), \nabla u_m) \rightharpoonup a_i(x, T_n(u), \nabla u) \qquad \text{weakly in} \quad L^{p_i'(\cdot)}(\Omega), \tag{47}$$

for $i = 1, \ldots, N$.

Step 4 : Passage to the limit

By taking $v \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ as a test function for the approximate problem (15) we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i v \, dx + \alpha \int_\Omega |T_m(u_m)|^{r(x)-1} T_m(u_m) v \, dx \\ &\quad + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-1} u_m v \, dx + \lambda \int_{\partial\Omega} T_n(u_m) \; v \, d\sigma \\ &= \int_\Omega F(x) \; v \, dx + \nu \int_\Omega \frac{|T_n(u_m)|^{p_0(x)-1} T_n(u_m)}{|x|^{p_0(x)} + \frac{1}{n}} \; v \, dx + \int_{\partial\Omega} G(x) \; v \, d\sigma. \end{aligned} \tag{48}$$

In view of $(33) - (35)$ and (47), then letting m tends to infinity we conclude that

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v \, dx + \alpha \int_\Omega |u|^{r(x)-1} u v \, dx + \lambda \int_{\partial\Omega} T_n(u) v \, d\sigma \\ &= \int_\Omega F(x) \; v \, dx + \nu \int_\Omega \frac{|T_n(u)|^{p_0(x)-1} T_n(u)}{|x|^{p_0(x)} + \frac{1}{n}} v \, dx + \int_{\partial\Omega} G(x) \; v \, d\sigma. \end{aligned} \tag{49}$$

Thus, the proof of the theorem 4.2 is concluded.

5. Main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ containing the origin, with Lipschitz boundary $\partial\Omega$,

Let $p_i(\cdot) \in \mathcal{C}_+(\overline{\Omega})$ for $i = 0, 1, \ldots, N$ where $p_0(x) = \max\{p_i(x), \; i = 0, 1, \ldots, N\}$ a.e. in Ω.

Definition 5.1. A measurable function u is called a renormalized solution of the quasilinear elliptic problem (6) if $u \in \mathcal{T}_{tr}^{1,\vec{p}(\cdot)}(\Omega)$, such that

$$\lim_{h \to \infty} \sum_{i=1}^{N} \int_{\{h < |u| \leq h+1\}} a_i(x, u, \nabla u) D^i u \, dx = 0,$$

and verifying the following equality

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i(x,u,\nabla u)(D^i uS'(u)\varphi+S(u)D^i\varphi)\mathrm{d}x+\alpha\\&\quad\int_{\Omega}|u|^{r(x)-1}uS(u)\varphi\mathrm{d}x+\lambda\int_{\partial\Omega}uS(u)\varphi\mathrm{d}\sigma\\&=\int_{\Omega}f(x)S(u)\varphi\mathrm{d}x+\nu\int_{\Omega}\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}}S(u)\varphi\mathrm{d}x+\int_{\partial\Omega}g(x)S(u)\varphi\mathrm{d}\sigma,\end{aligned}\tag{50}$$

for every $\varphi\in W^{1,\vec{p}(\cdot)}(\Omega)\cap L^{\infty}(\Omega)$. and for any smooth function $S(\cdot)\in W^{1,\infty}(\Omega)$ with a compact support.

We will establish the following result

Theorem 5.2. *Let $\alpha,\lambda,\nu>0$ and $f\in L^1(\Omega)$, $g\in L^1(\partial\Omega)$, assuming that (12)-(11) hold true. Then there exists at least one renormalized solution u for quasilinear elliptic problem* (6)*. Moreover, we have $u\in W^{1,\vec{q}(\cdot)}(\Omega)$ with*

$$\vec{q}(\cdot)=(r(\cdot),q_1(\cdot),...,q_N(\cdot))\ \text{ and } 1<q_i(x)<\frac{p_i(x)r(x)}{1+r(x)+\delta(x)}\ \text{for } i=1,2,\dots,N.\tag{51}$$

Proof of Theorem 5.2

Step 1: Approximate problem

Let $(f_n)_{n\in N^*}$ be a sequence of smooth functions in $L^1(\Omega)\cap L^{\infty}(\Omega)$ such that $f_n\to f$ strongly in $L^1(\Omega)$ (for example $f_n(x)=T_n(f(x))$), and let $(g_n(x))_n$ be a sequence in $L^1(\partial\Omega)\cap L^{\infty}(\partial\Omega)$ such that $g_n\to g$ strongly in $L^1(\partial\Omega)$. We consider the approximate problem of the equation (6), giving by

$$\begin{cases}-\sum_{i=1}^{N}D^i a_i(x,T_n(u_n),\nabla u_n)+\alpha|u_n|^{r(x)-1}u_n=f_n(x)+\nu\dfrac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}} & \text{in }\Omega,\\ \sum_{i=1}^{N}a_i(x,T_n(u_n),\nabla u_n)n_i+\lambda T_n(u_n)=g_n(x) & \text{on }\partial\Omega,\end{cases}\tag{52}$$

In view of theorem 4.2, there exists at least one solution in the sense of distributions $u_n\in W^{1,\vec{p}(\cdot)}(\Omega)\cap L^{r(\cdot)+1}(\Omega)$ for the approximate problem (52), i. e.

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_n),\nabla u_n)D^i v\,\mathrm{d}x+\alpha\int_{\Omega}|u_n|^{r(x)-1}u_n v\,\mathrm{d}x+\lambda\int_{\partial\Omega}T_n(u_n)v\,\mathrm{d}\sigma\\&=\int_{\Omega}f_n(x)v\,\mathrm{d}x+\nu\int_{\Omega}\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}v\,\mathrm{d}x+\int_{\partial\Omega}g_n(x)v\,\mathrm{d}\sigma,\end{aligned}\tag{53}$$

for every $v\in W^{1,\vec{p}(\cdot)}(\Omega)\cap L^{\infty}(\Omega)$.

Step 2 : A priori estimates

Lemma 5.3. *Let u_n be a solution in the sense of distributions for the approximate problem* (53), *then the following regularity results hold true:*

$$u_n \in W^{1,\vec{q}(\cdot)}(\Omega) \quad \text{with} \quad \vec{q}(\cdot) = (r(\cdot), q_1(\cdot), \ldots, q_N(\cdot)), \tag{54}$$

where the exponents $r(\cdot)$ and $q_i(\cdot)$ verifies $r(x) > \max\left\{\frac{1+\delta(x)}{p_0(x)-1}, \frac{N(p_0(x)-1)}{N-p_0(x)}\right\}$ and we have $1 < q_i(x) < \dfrac{r(x)p_i(x)}{r(x)+\delta(x)+1}$ almost everywhere in Ω. Then

$$\sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta+\delta(x)}}\,\mathrm{d}x \le C \qquad \text{for all} \quad \theta > 1, \tag{55}$$

$$\sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\,\mathrm{d}x \le C(1+k)^{\theta+\delta^+} \quad \text{for all} \quad k > 0, \tag{56}$$

with C is a positive constant that does not depend on k and n, where $\delta^+ = \max\{\delta_i^+,\ for\ i = 1, \cdots, N\}$.

Proof of Lemma 5.3.

Let $\theta > 1$, we consider the function $\varphi(t) : \mathbb{R} \mapsto \mathbb{R}$ defined by

$$\varphi(u_n) = (1 - \frac{1}{(1+|u_n|)^{\theta-1}})\operatorname{sign}(u_n).$$

By taking $\varphi(u_n)$ as a test function for the approximate problem (53), and since $\varphi(u_n)$ has the same sign as u_n, we get

$$\begin{aligned}
&(\theta-1)\sum_{i=1}^{N}\int_\Omega \frac{a_i(x,T_n(u_n),\nabla u_n)D^i u_n}{(1+|u_n|)^\theta}\mathrm{d}x + \alpha\int_\Omega |u_n|^{r(x)}|\varphi(u_n)|\mathrm{d}x \\
&\qquad +\lambda\int_{\partial\Omega} |T_n(u_n)|\ |\varphi(u_n)|\mathrm{d}\sigma \\
&\le \int_\Omega |f_n(x)|\ |\varphi(u_n)|\,\mathrm{d}x + \nu\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}|\varphi(u_n)|\,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega} |g_n(x)|\ |\varphi(u_n)|\,\mathrm{d}\sigma.
\end{aligned} \tag{57}$$

We have $0 \le |\varphi(u_n)| \le 1$, and in view of (13) we obtain

$$\begin{aligned}
&b_0(\theta-1)\sum_{i=1}^{N}\int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta+\delta(x)}}\,\mathrm{d}x + \alpha\int_\Omega |u_n|^{r(x)}|\varphi(u_n)|\,\mathrm{d}x + \lambda \\
&\qquad \int_{\partial\Omega} |T_n(u_n)||\varphi(u_n)|\,\mathrm{d}\sigma \\
&\le \int_\Omega |f_n(x)|\,\mathrm{d}x + \nu\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}|\varphi(u_n)|}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x + \int_{\partial\Omega} |g_n(x)|\,\mathrm{d}\sigma.
\end{aligned} \tag{58}$$

For the second term on the right hand side of (58) and by using Young's inequality, we have

$$2\nu\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}|\varphi(u_n)|}{|x|^{p_0(x)}+\frac{1}{n}}\mathrm{d}x \le \frac{\alpha}{2}\int_\Omega |u_n|^{r(x)}|\varphi(u_n)|\mathrm{d}x + C_{10}\int_\Omega \frac{\mathrm{d}x}{|x|^{\frac{r(x)p_0(x)}{r(x)-p_0(x)+1}}}. \tag{59}$$

Thus, we have

$$\begin{aligned} & b_0(\theta-1)\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta+\delta(x)}}\mathrm{d}x + \frac{\alpha}{2}\int_\Omega |u_n|^{r(x)}|\varphi(u_n)|\mathrm{d}x + \lambda \\ & \qquad \int_{\partial\Omega} |T_n(u_n)||\varphi(u_n)|\,\mathrm{d}\sigma \\ & \le \int_\Omega |f(x)|\,\mathrm{d}x + C_{10}\int_\Omega \frac{\mathrm{d}x}{|x|^{\frac{r(x)p_0(x)}{r(x)-p_0(x)+1}}} + \int_{\partial\Omega}|g(x)|\,\mathrm{d}\sigma. \end{aligned} \tag{60}$$

Having in mind $r(x) > \dfrac{N(p_0(x)-1)}{N-p_0(x)}$, then $\dfrac{1}{|x|^{\frac{r(x)p_0(x)}{r(x)-p_0(x)+1}}} \in L^1(\Omega)$. It follows that

$$\sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta+\delta(x)}}\mathrm{d}x + \int_\Omega |u_n|^{r(x)}\,|\varphi(u_n)|\mathrm{d}x + \int_{\partial\Omega}|T_n(u_n)|\,|\varphi(u_n)|\mathrm{d}\sigma \le C_{11}. \tag{61}$$

Then we conclude that estimate (55). For the estimate (56), in view of (55) we have

$$\begin{aligned} \sum_{i=1}^N \int_\Omega |D^i T_k(u_n)|^{p_i(x)}\,\mathrm{d}x & \le (1+k)^{\theta+\delta^+}\sum_{i=1}^N \int_\Omega \frac{|D^i T_k(u_n)|^{p_i(x)}}{(1+|u_n|)^{\theta+\delta(x)}}\,\mathrm{d}x \\ & \le C_{11}(1+k)^{\theta+\delta^+} \qquad \text{for all} \quad \theta > 1. \end{aligned} \tag{62}$$

We have $|\varphi(s)| \ge \frac{1}{2}$ for $|s| \ge R = 2^{\frac{1}{\delta^+}}$ large enough, we obtain

$$\begin{aligned} C_{11} & \ge \int_\Omega |u_n|^{r(x)}\,|\varphi(u_n)|\,\mathrm{d}x + \int_{\partial\Omega}|T_n(u_n)|\,|\varphi(u_n)|\,\mathrm{d}\sigma \\ & \ge \frac{1}{2}\int_{\{|u_n|\ge R\}} |u_n|^{r(x)}\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega\cap\{|u_n|\ge R\}} |T_n(u_n)|\,\mathrm{d}\sigma \\ & \ge \frac{1}{2}\int_\Omega |u_n|^{r(x)}\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}|T_n(u_n)|\,\mathrm{d}\sigma - R^{r^+}\mathrm{mes}(\Omega) - R\,\mathrm{mes}(\partial\Omega). \end{aligned} \tag{63}$$

It follows that

$$\int_\Omega |u_n|^{r(x)}\,\mathrm{d}x + \int_{\partial\Omega}|T_n(u_n)|\,\mathrm{d}\sigma \le C_{12}. \tag{64}$$

Now, taking $q_i(\cdot) \in \mathcal{C}_+(\Omega)$ such that $1 < q_i(x) < \dfrac{r(x)p_i(x)}{1+r(x)+\delta(x)}$ for $i = 1,\dots,N$, and by choosing $1 < \theta < \dfrac{r(x)(p_i(x)-q_i(x))}{q_i(x)} - \delta(x)$ for $i = 1,\dots,N$, we have

$$
\begin{aligned}
\int_\Omega |D^i u_n|^{q_i(x)}\,\mathrm{d}x &\leq \int_\Omega \frac{|D^i u_n|^{q_i(x)}}{(1+|u_n|)^{\frac{q_i(x)(\theta+\delta(x))}{p_i(x)}}}(1+|u_n|)^{\frac{q_i(x)(\theta+\delta(x))}{p_i(x)}}\,\mathrm{d}x \\
&\leq \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta+\delta(x)}}\,\mathrm{d}x + \int_\Omega (1+|u_n|)^{\frac{q_i(x)(\theta+\delta(x))}{p_i(x)-q_i(x)}}\,\mathrm{d}x \\
&\leq C_{11} + \int_\Omega (1+|u_n|)^{\frac{q_i(x)(\theta+\delta(x))}{p_i(x)-q_i(x)}}\,\mathrm{d}x \\
&\leq C_{13} + C_{14}\int_\Omega |u_n|^{r(x)}\,dx \\
&\leq C_{15}.
\end{aligned}
\tag{65}
$$

According to (63) and (65), we deduce that $u_n \in W_0^{1,\vec{q}(\cdot)}(\Omega)$, where $\vec{q}(\cdot) = (r(\cdot), q_1(\cdot), \cdots, q_N(\cdot))$, such that

$$
1 < q_i(x) < \frac{r(x)p_i(x)}{1+r(x)+\delta(x)} \qquad \text{and} \quad r(x) > \max\left\{\frac{1+\delta(x)}{p_0(x)-1}, \frac{N(p_0(x)-1)}{N-p_0(x)}\right\}
$$

a.e. in Ω.

Step 3 : The weak convergence of $(T_k(u_n))_n$ ***in*** $W^{1,\vec{p}(\cdot)}(\Omega)$

To show the weak convergence of $(T_k(u_n))_n$ in $W^{1,\vec{p}(\cdot)}(\Omega)$, we shall show that $(u_n)_n$ is a Cauchy sequence. Indeed, in view of (62) we have

$$
\|T_k(u_n)\|_{1,\vec{p}(\cdot)} \quad \leq C(k), \tag{66}
$$

where $C(k)$ is a positive constant not depending on n. Then, the sequence $(T_k(u_n))_n$ is uniformly bounded in $W^{1,\vec{p}(\cdot)}(\Omega)$, it follows that : there exists a subsequence still denoted $(T_k(u_n))_n$ and a measurable function $v_k \in W^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$
\begin{cases}
T_k(u_n) \rightharpoonup v_k & \text{weakly in} \quad W^{1,\vec{p}(\cdot)}(\Omega) \\
T_k(u_n) \to v_k & \text{strongly in} \quad L^{\underline{p}}(\Omega) \quad \text{and a.e. in} \quad \Omega \\
T_k(u_n) \rightharpoonup v_k & \text{weakly in} \quad L^1(\partial\Omega).
\end{cases}
\tag{67}
$$

Let $k \geq 1$, in view of (63) we have

$$
k^{r_-}\,\mathrm{meas}\{|u_n| > k\} \leq \int_{\{|u_n|>k\}} |u_n|^{r(x)}\,\mathrm{d}x \leq C_{11}, \tag{68}
$$

it follows that

$$
\mathrm{meas}\{|u_n| > k\} \leq \frac{C_{11}}{k^{r_-}} \to 0 \qquad \text{as} \quad k \to \infty. \tag{69}
$$

For all $\eta > 0$, we have

$$\text{meas}\{|u_n - u_m| > \eta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\}. \tag{70}$$

Let $\varepsilon > 0$, using (69) we can choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{71}$$

On the other hand, thanks to (67) we can assume that $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure. Thus, for any $k > 0$ and $\eta, \varepsilon > 0$, there exists $n_0 = n_0(k, \eta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \eta\} \leq \frac{\varepsilon}{3} \quad \forall n, m \geq n_0(k, \eta, \varepsilon). \tag{72}$$

By combining (70) and $(71) - (72)$, we conclude that : for any $\eta > 0$ and $\varepsilon > 0$, there exists $n_0(\eta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \eta\} \leq \varepsilon \quad \forall n, m \geq n_0(k, \eta, \varepsilon). \tag{73}$$

Thus, the sequence $(u_n)_n$ is a Cauchy sequence in measure, and there exists a measurable function u and a subsequence, still denoted by $(u_n)_n$, such that $u_n \to u$ almost everywhere in Ω. Consequently, we deduce that

$$\begin{cases} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad W^{1,\vec{p}(\cdot)}(\Omega) \\ T_k(u_n) \to T_k(u) & \text{strongly in} \quad L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega \\ T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in} \quad L^1(\partial\Omega). \end{cases} \tag{74}$$

Step 4: The equi-integrability of $(|u_n|^{r(x)-1}u_n)_n$ ***and*** $\Big(\dfrac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)} + \frac{1}{n}}\Big)$.

We shall prove that

$$|u_n|^{r(x)-1}u_n \to |u|^{r(x)-1}u \quad \text{strongly in} \quad L^1(\Omega), \tag{75}$$

and

$$\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)} + \frac{1}{n}} \to \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \quad \text{strongly in} \quad L^1(\Omega). \tag{76}$$

By using $T_1(u_n - T_h(u_n))$ as a test function in (53), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \,\mathrm{d}x \\ &\qquad + \alpha \int_{\{|u_n|\geq h\}} |u_n|^{r(x)} |T_1(u_n - T_h(u_n))| \,\mathrm{d}x + \lambda \int_{\{|u_n|\geq h+1\}\cap\partial\Omega} |T_n(u_n)| \,\mathrm{d}\sigma \\ &\leq \int_{\{|u_n|\geq h\}} |f_n(x)| \,\mathrm{d}x + \nu \int_{\{|u_n|\geq h\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + \frac{1}{n}} |T_1(u_n - T_h(u_n))| \,\mathrm{d}x \\ &\qquad + \int_{\{|u_n|\geq h\}\cap\partial\Omega} |g(x)| \,\mathrm{d}\sigma. \end{aligned} \tag{77}$$

Thanks to Young's inequality, we have

$$\begin{aligned}&\nu\int_{\{|u_n|\geq h\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}|T_1(u_n-T_h(u_n))|\,\mathrm{d}x\\&\leq\frac{\alpha}{3}\int_{\{|u_n|\geq h\}}|u_n|^{r(x)}|T_1(u_n-T_h(u_n))|\,\mathrm{d}x+C_{15}\int_{\{|u_n|\geq h\}}\frac{\mathrm{d}x}{|x|^{\frac{p_0(x)r(x)}{r(x)-p_0(x)+1}}}.\end{aligned}\tag{78}$$

Then

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,\mathrm{d}x+\frac{\alpha}{3}\int_{\{|u_n|\geq h+1\}}|u_n|^{r(x)}\,\mathrm{d}x\\&+\nu\int_{\{|u_n|\geq h+1\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x+\lambda\int_{\{|u_n|\geq h+1\}\cap\partial\Omega}|T_n(u_n)|\,\mathrm{d}\sigma\\&\leq\int_{\{|u_n|\geq h\}}|f(x)|\,\mathrm{d}x+2C_{15}\int_{\{|u_n|\geq h\}}\frac{\mathrm{d}x}{|x|^{\frac{p_0(x)r(x)}{r(x)-p_0(x)+1}}}+\int_{\{|u_n|\geq h\}\cap\partial\Omega}|g(x)|\,\mathrm{d}\sigma.\end{aligned}\tag{79}$$

We have $f(x)\in L^1(\Omega)$ and $\frac{1}{|x|^{\frac{p_0(x)r(x)}{r(x)-p_0(x)+1}}}\in L^1(\Omega)$, having in mind that $\mathrm{meas}(\{|u_n|\geq h\})\to 0$ as h tends to infinity, we obtain

$$\int_{\{|u_n|\geq h\}}|f_n(x)|\mathrm{d}x+2C_{15}\int_{\{|u_n|\geq h\}}\frac{\mathrm{d}x}{|x|^{\frac{p_0(x)r(x)}{r(x)-p_0(x)+1}}}\longrightarrow 0\qquad\text{as}\quad h\to\infty.\tag{80}$$

Similarly, we have $g(x)\in L^1(\partial\Omega)$ and since $\mathrm{meas}(\{|u_n|\geq h\}\cap\partial\Omega)\to 0$ as h tends to infinity, it follows that

$$\int_{\{|u_n|\geq h\}\cap\partial\Omega}|g(x)|\,\mathrm{d}\sigma\longrightarrow 0\qquad\text{as}\quad h\to\infty.\tag{81}$$

By combining (79) and (80) – (81) we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,\mathrm{d}x=0.\tag{82}$$

and

$$\lim_{h\to\infty}\Big(\int_{\{|u_n|\geq h\}}|T_n(u_n)|^{r(x)}\,\mathrm{d}x+\int_{\{|u_n|\geq h\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x\Big)=0.\tag{83}$$

Thus, for any $\eta>0$, there exists $h(\eta)>0$ such that

$$\int_{\{|u_n|\geq h(\eta)\}}|T_n(u_n)|^{r(x)}\,\mathrm{d}x+\int_{\{|u_n|\geq h(\eta)\}}\frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x\leq\frac{\eta}{2}.\tag{84}$$

On the other hand, for any measurable subset $E \subseteq \Omega$, we have

$$\begin{aligned}
&\int_E |T_n(u_n)|^{r(x)}\,\mathrm{d}x + \int_E \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x \\
&\leq \int_E |T_{h(\eta)}(u_n)|^{r(x)}\,\mathrm{d}x + \int_E \frac{|T_{h(\eta)}(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x \\
&\quad + \int_{\{|u_n|\geq h(\eta)\}} |T_n(u_n)|^{r(x)}\,\mathrm{d}x + \int_{\{|u_n|\geq h(\eta)\}} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x.
\end{aligned} \tag{85}$$

Then, in view of (74) there exists $\beta(\eta,h)>0$ such that : for any $E\subseteq\Omega$ we have

$$\int_E |T_{h(\eta)}(u_n)|^{r(x)}\,dx + \int_E \frac{|T_{h(\eta)}(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}} \leq \frac{\eta}{2} \quad \text{for } \operatorname{meas}(E)\leq\beta(\eta,h). \tag{86}$$

Finally, by combining (84), (85) and (86), it yields

$$\int_E |T_n(u_n)|^{r(x)}\mathrm{d}x + \int_E \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)}+\frac{1}{n}}\,\mathrm{d}x \leq \eta \text{ for any } E\subseteq\Omega \text{ with } \operatorname{meas}(E)\leq\beta(\eta). \tag{87}$$

We deduce that $(|T_n(u_n)|^{r(x)-1}T_n(u_n))_n$, and $\left(\dfrac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}\right)$ are equi-integrable. Having in mind that

$$|T_n(u_n)|^{r(x)-1}T_n(u_n) \to |u|^{r(x)-1}u \qquad \text{a.e. in} \quad \Omega,$$

and

$$\frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}} \to \frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} \qquad \text{a.e. in} \quad \Omega.$$

Thus, in view of Vitali's theorem, the convergence (75) and (76) are concluded.

Step 5 : Strong convergence of truncations

In the sequel, we denote by $\varepsilon_i(n)$ for $i=1,2,\dots$ some various functions of real variables that converges to 0 as n tends to infinity, respectively, we define $\varepsilon_i(h)$ and $\varepsilon_i(n,h)$.

Let $h>k>0$, we set $z_n = u_n - T_h(u_n) + T_k(u_n) - T_k(u)$ and $w_n := T_{2k}(z_n)$. By using w_n as a test function for the approximate problem (53), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i w_n\,\mathrm{d}x + \alpha\int_\Omega |u_n|^{r(x)-1}u_n w_n\,\mathrm{d}x + \lambda\int_{\partial\Omega} T_n(u_n)w_n\,\mathrm{d}\sigma \\
&= \int_\Omega f_n(x)w_n\,\mathrm{d}x + \nu\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}w_n\,\mathrm{d}x + \int_{\partial\Omega} g_n(x)w_n\,\mathrm{d}\sigma.
\end{aligned} \tag{88}$$

For $M = 4k + h$, it is easy to check that $D^i w_n = 0$ on the set $\{|u_n| \geq M\}$. we get

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\{|u_n|\leq M\}} a_i(x, T_M(u_n), \nabla T_M(u_n)) D^i w_n \, \mathrm{d}x + \lambda \int_{\partial\Omega} T_n(u_n) w_n \, \mathrm{d}\sigma \\ &\leq \int_{\Omega} |f(x)||w_n| \, \mathrm{d}x + \nu \int_{\Omega} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + \frac{1}{n}} |w_n| \, \mathrm{d}x + \int_{\partial\Omega} |g_n(x)||w_n| \, \mathrm{d}\sigma \\ &\qquad + \alpha \int_{\Omega} |u_n|^{r(x)} |w_n| \, \mathrm{d}x. \end{aligned} \tag{89}$$

For the first term on the right hand side of (89), we have $w_n \rightharpoonup T_{2k}(u - T_h(u))$ weak$-*$ in $L^\infty(\Omega)$, and since $f(x) \in L^1(\Omega)$ we deduce that

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega} |f(x)||w_n| \, \mathrm{d}x &= \int_{\Omega} |f(x)||T_{2k}(u - T_h(u))| \, \mathrm{d}x \\ &\leq 2k \int_{\{|u_n|>h\}} |f(x)| \, dx \longrightarrow 0 \qquad \text{as} \quad h \to \infty. \end{aligned} \tag{90}$$

Similarly, in view of (75) and (76) we have

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega} \frac{|T_n(u_n)|^{p_0(x)-1}}{|x|^{p_0(x)} + \frac{1}{n}} |w_n| \, \mathrm{d}x &= \int_{\Omega} \frac{|u|^{p_0(x)-1}}{|x|^{p_0(x)}} |T_{2k}(u - T_h(u))| \, \mathrm{d}x \qquad (91) \\ &\leq 2k \int_{\{|u|>h\}} \frac{|u|^{p_0(x)-1}}{|x|^{p_0(x)}} \, dx \longrightarrow 0 \text{ as } h \to \infty. \end{aligned}$$

and

$$\lim_{n\to\infty} \int_{\Omega} |u_n|^{r(x)} |w_n| \, \mathrm{d}x \quad \leq 2k \int_{\{|u|>h\}} |u|^{r(x)} \, dx \longrightarrow 0 \qquad \text{as} \quad h \to \infty. \tag{92}$$

Moreover, since $w_n \rightharpoonup T_{2k}(u - T_h(u))$ weak$-*$ in $L^\infty(\partial\Omega)$, and since $g(x) \in L^1(\partial\Omega)$ we deduce that

$$\lim_{n\to\infty} \int_{\partial\Omega} |g(x)||w_n| \, \mathrm{d}\sigma \quad \leq 2k \int_{\partial\Omega \cap \{|u|>h\}} |g(x)| \, d\sigma \longrightarrow 0 \qquad \text{as} \quad h \to \infty. \tag{93}$$

For the second term on the left-hand side of (89), since w_n has the same sign as u_n on the set $\{|u_n| > k\}$, and since $w_n = T_k(u_n) - T_k(u)$ on the set $\{|u_n| \leq k\}$ we have

$$\begin{aligned} \int_{\partial\Omega} T_n(u_n) w_n \mathrm{d}\sigma &= \int_{\{|u_n|\leq k\}\cap\partial\Omega} u_n w_n \, \mathrm{d}\sigma + \int_{\{|u_n|>k\}\cap\partial\Omega} |u_n||w_n| \, \mathrm{d}\sigma \\ &\geq \int_{\{|u_n|\leq k\}\cap\partial\Omega} T_k(u_n)(T_k(u_n) - T_k(u)) \mathrm{d}\sigma \to 0 \text{ as } n \to \infty. \end{aligned} \tag{94}$$

By combining (89) and (90) − (94), we deduce that

$$\sum_{i=1}^{N}\int_{\{|u_n|\leq M\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^i w_n \,\mathrm{d}x \leq \varepsilon_1(n,h). \tag{95}$$

On the other hand, we have $a(x,s,0)=0$ then

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_n|\leq M\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^i w_n \,\mathrm{d}x\\
&=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))D^i(T_k(u_n)-T_k(u))\,\mathrm{d}x\\
&\ +\sum_{i=1}^{N}\int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} a_i(x,T_n(u_n),\nabla u_n)(D^iu_n-D^iT_h(u_n)-D^iT_k(u))\mathrm{d}x\\
&=\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\mathrm{d}x\\
&\quad+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,\mathrm{d}x\\
&\quad+\sum_{i=1}^{N}\int_{\{h<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^iu_n\,\mathrm{d}x\\
&\quad-\sum_{i=1}^{N}\int_{\{k<|u_n|\leq M\}\cap\{|z_n|\leq 2k\}} a_i(x,T_M(u_n),\nabla T_M(u_n))D^iT_k(u)\,\mathrm{d}x\\
&\leq \varepsilon_1(n,h).
\end{aligned} \tag{96}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\mathrm{d}x\\
&\leq \sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|\;|D^iT_k(u_n)-D^iT_k(u)|\,\mathrm{d}x\\
&\quad+\sum_{i=1}^{N}\int_{\{k<|u_n|\leq M\}}|a_i(x,T_M(u_n),\nabla T_M(u_n))|\;|D^iT_k(u)|\,\mathrm{d}x+\varepsilon_1(n,h).
\end{aligned} \tag{97}$$

For the first term on the right-hand side of (97), we have $T_k(u_n)$ tends strongly to $T_k(u)$ in $L^{p_i}(\Omega)$ then $|a_i(x,T_k(u_n),\nabla T_k(u))| \to |a_i(x,T_k(u_n),\nabla T_k(u))|$ strongly in $L^{p_i'}(\Omega)$, and since $D^iT_k(u_n) \rightharpoonup D^iT_k(u)$ weakly in $L^{p_i}(\Omega)$, it follows that

$$\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_k(u_n),\nabla T_k(u))|\;|D^iT_k(u_n)-D^iT_k(u)|\,\mathrm{d}x \longrightarrow 0 \quad \text{as} \quad n\to\infty, \tag{98}$$

Concerning the second term on the right-hand side of (97), Thanks to (12) we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is uniformly bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a measurable function $\varphi_i \in L^{p_i'(\cdot)}(\Omega)$ such that $|a_i(x, T_M(u_n), \nabla T_M(u_n))| \rightharpoonup \varphi_i$ weakly in $L^{p_i'(\cdot)}(\Omega)$, it follows that

$$\begin{aligned}&\lim_{n\to\infty} \sum_{i=1}^{N} \int_{\{k<|u_n|\leq M\}} |a_i(x, T_M(u_n), \nabla T_M(u_n))||D^i T_k(u)| \, \mathrm{d}x \\ &= \sum_{i=1}^{N} \int_{\{k<|u_n|\leq M\}} \varphi_i |D^i T_k(u)| \, \mathrm{d}x = 0.\end{aligned} \tag{99}$$

By combining (97) and (98) – (99), we conclude that

$$\begin{aligned}\sum_{i=1}^{N} \int_{\Omega} &(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))) \\ &(D^i T_k(u_n) - D^i T_k(u)) \mathrm{d}x \leq \varepsilon_2(n, h).\end{aligned} \tag{100}$$

According to (74), and letting n and h tend to infinity, we obtain

$$\begin{aligned}\lim_{n\to\infty} \Big(&\sum_{i=1}^{N} \int_{\Omega} (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u)) \mathrm{d}x \\ &+ \int_{\Omega} (|T_k(u_n)|^{\underline{p}-2} T_k(u_n) - |T_k(u)|^{\underline{p}-2} T_k(u))(T_k(u_n) - T_k(u)) \, \mathrm{d}x \Big) = 0.\end{aligned} \tag{101}$$

In view of Lemma 3.1, we conclude that

$$\begin{cases} T_k(u_n) \to T_k(u) & \text{strongly in} \quad W^{1,\vec{p}(\cdot)}(\Omega), \\ D^i u_n \to D^i u & \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1 \ldots, N. \end{cases} \tag{102}$$

Therefore, for $i = 1, \cdots, N$ we have $a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n$ tends to $a_i(x, u, \nabla u) \cdot D^i u$ almost everywhere in Ω. Thanks to Fatou's Lemma and (82), we conclude that

$$\begin{aligned}&\lim_{h\to\infty} \int_{\{h<|u|\leq h+1\}} a_i(x, u, \nabla u) \cdot D^i u \, dx \\ &\leq \lim_{h\to\infty} \liminf_{n\to\infty} \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n \, dx \\ &\leq \lim_{h\to\infty} \limsup_{n\to\infty} \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n \, dx = 0.\end{aligned} \tag{103}$$

Step 6: Passage to the limit.

Let $\varphi \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and let $S(\cdot)$ be a smooth function in $W^{1,\infty}(\mathbb{R})$ such that supp $(S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$.

By choosing $S(u_n)\varphi$ as a test function for the approximate problem (53), we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n)D^i\varphi)\,\mathrm{d}x\\&\quad+\alpha\int_\Omega |u_n|^{r(x)-1}u_n S(u_n)\varphi\,\mathrm{d}x+\lambda\int_\Omega T_n(u_n)S(u_n)\varphi\,\mathrm{d}\sigma\\&=\int_\Omega f_n(x)S(u_n)\varphi\mathrm{d}x+\nu\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-2}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}S(u_n)\varphi\mathrm{d}x+\int_{\partial\Omega} g_n(x)S(u_n)\varphi\,\mathrm{d}\sigma.\end{aligned}\tag{104}$$

For the first term on the left-hand side of (104) we have

$$\begin{aligned}&\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n)D^i\varphi)\,dx\\&=\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\left(S'(u_n)\varphi D^i T_M(u_n)+S(T_M(u_n))D^i\varphi\right)\,dx.\end{aligned}$$

In view of (12), we have $(a_i(x,T_M(u_n),\nabla T_M(u_n)))_n$ is uniformly bounded in $L^{p_i'(\cdot)}(\Omega)$, and since $a_i(x,T_M(u_n),\nabla T_M(u_n))$ tends to $a_i(x,T_M(u),\nabla T_M(u))$ almost everywhere in Ω, it follows that

$$a_i(x,T_M(u_n),\nabla T_M(u_n))\rightharpoonup a_i(x,T_M(u),\nabla T_M(u))\quad\text{weakly in}\quad L^{p_i'(\cdot)}(\Omega),$$

and since $S'(u_n)\varphi D^iT_M(u_n)+S(T_M(u_n))D^i\varphi$ tends to $S'(u)\varphi D^iT_M(u)+S(T_M(u))D^i\varphi$ strongly in $L^{p_i(\cdot)}(\Omega)$, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n)D^i\varphi)\,dx\\&=\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\left(S'(u_n)\varphi D^iT_M(u_n)+S(T_M(u_n))D^i\varphi\right)dx\\&=\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u),\nabla T_M(u))\left(S'(u)\varphi D^iT_M(u)+S(T_M(u))D^i\varphi\right)\,dx\\&=\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)\left(S'(u)\varphi D^iu+S(u)D^i\varphi\right)\,dx.\end{aligned}\tag{105}$$

On the other hand, we have $f_n\to f$ strongly in $L^1(\Omega)$, and since $S(T_M(u_n))\varphi\to S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\Omega)$, we deduce that

$$\lim_{n\to+\infty}\int_\Omega f_n(x)S(T_M(u_n))\varphi\,dx=\int_\Omega f(x)S(T_M(u))\varphi\,dx=\int_\Omega f(x)S(u)\varphi\,dx.\tag{106}$$

Similarly, in view of (75) and (76) we obtain

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{r(x)-1}u_nS(T_M(u_n))\varphi\,dx=\int_\Omega |u|^{r(x)-1}uS(u)\varphi\,dx,\tag{107}$$

and

$$\lim_{n\to+\infty}\int_\Omega \frac{|T_n(u_n)|^{p_0(x)-1}T_n(u_n)}{|x|^{p_0(x)}+\frac{1}{n}}S(T_M(u_n))\varphi\,\mathrm{d}x=\int_\Omega \frac{|u|^{p_0(x)-1}u}{|x|^{p_0(x)}}S(u)\varphi\,\mathrm{d}x. \tag{108}$$

Moreover, we have $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\partial\Omega)$ then

$$\lim_{n\to\infty}\int_{\partial\Omega} g_n(x)S(T_M(u_n))\varphi\,d\sigma=\int_{\partial\Omega} g(x)S(T_M(u))\varphi\,d\sigma=\int_{\partial\Omega} g(x)S(u)\varphi\,d\sigma. \tag{109}$$

and

$$\lim_{n\to\infty}\int_{\partial\Omega} T_n(u_n)S(T_M(u_n))\varphi\mathrm{d}\sigma=\lim_{n\to\infty}\int_{\partial\Omega} T_M(u_n)S(T_M(u_n))\varphi\mathrm{d}\sigma=\int_{\partial\Omega} uS(u)\varphi\mathrm{d}\sigma. \tag{110}$$

By combining (104) and (105) $-$ (110) together, we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)S(u)\varphi\,\mathrm{d}x+\alpha\int_\Omega |u|^{r(x)-1}uS(u)\varphi\,\mathrm{d}x+\lambda\int_\Omega uS(u)\varphi\,\mathrm{d}\sigma\\&=\int_\Omega f(x)S(u)\varphi\,\mathrm{d}x+\nu\int_\Omega\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}}S(u)\varphi\,\mathrm{d}x+\int_{\partial\Omega} g(x)S(u)\varphi\,\mathrm{d}\sigma,\end{aligned} \tag{111}$$

which conclude the proof of the theorem 5.2.

References

[1] B. Abdellaoui, A. Attar, and S. E. Miri, *Nonlinear singular elliptic problem with gradient term and general datum*, Math. Anal. Appl. **409** (2014), no. 1, 362–377.

[2] B. Abdellaoui, I. Peral, and A. Primo, *Elliptic problems with a Hardy potential and critical growth in the gradient: non-resonance and blow-up results*, J. Differ. Equations **239** (2007), no. 2, 386–416.

[3] ______, *Breaking of resonance and regularizing effect of a first order quasi-linear term in some elliptic equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **25** (2008), no. 5, 969–985.

[4] F. Andereu, J.M. Mazón, S. Segura De león, and J. Teledo, *Quasi-linear elliptic and parabolic equations in L1 with non-linear boundary conditions*, Adv. Math. Sci. Appl. **7** (1997), 183–213.

[5] E. Azroul, M. Bouziani, H. Hjiaj, and A. Youssfi, *Existence of solutions for some quasilinear elliptic problem with Hardy potential*, Math. Bohem. **144** (2019), no. 3, 299–324.

[6] M. Benboubker, H. Benkhalou, H. Hjiaj, and I. Nyanquini, *Entropy solutions for elliptic Schrödinger type equations under Fourier boundary conditions*, Rend. Circ. Mat. Palermo **(2) 72** (2023), no. 4, 2831–2855.

[7] M. B. Benboubker, H. Benkhalou, and H. Hjiaj, *Weak and renormalized solutions for anisotropic Neumann problems with degenerate coercivity*, Bol. Soc. Parana. Mat. **46** (2023), no. 7, 181–212.

[8] M. B. Benboubker, R. Bentahar, M. El Lekhlifi, and H. Hjiaj, *Existence of renormalized solutions for some non-coercive anisotropic elliptic problems with Neumann boundary condition*, Rend. Mat. Appl. **46** (2025), no. 7, 181–212.

[9] M. B. Benboubker, H. Hjiaj, and S. Ouaro, *Entropy solutions to nonlinear elliptic anisotropic problem with variable exponent*, Appl. Anal. Comput. **4** (2014), no. 3, 245–270.

[10] Ph. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez, *An L1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **22** (1995), no. 4, 241–273.

[11] M. F. Betta, A. Mercaldo, F. Murat, and M. M. Porzio, *Existence of renormalized solutions to nonlinear elliptic equations with lower-order terms and right-hand side measure*, J. Math. Pures Appl. **81** (2002), no. 6, 533–566.

[12] D. Blanchard and F. Murat, *Renormalized solution for nonlinear parabolic problems with L^1 data, existence and uniqueness*, Proc. Roy. Soc. Edinburgh Sect. A **127** (1997), 1137–1152.

[13] D. Blanchard, F. Murat, and H. Redwane, *Existence and uniqueness of a renormalized solution for a fairly general class of nonlinear parabolic problems*, J. Differential Equations **177** (2001), 331–374.

[14] L. Boccardo, D. Giachetti, J. I. Diaz, and F. Murat, *Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms*, J. Differential Equations **106** (1993), no. 2, 215–237.

[15] L. Boccardo, L. Orsina, and I. Peral, *A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential*, Discrete Contin. Dyn. Syst. **16** (2006), no. 3, 513–523.

[16] H. Brezis and X. Cabré, *Some simple nonlinear PDE without solutions*, Boll. Unione Mat. Ital. **1-B** (1998), no. 2, 223–262.

[17] H. Brezis, L. Dupaigne, and A. Tesei, *On a semilinear equation with inverse-square potential*, Selecta Math. **11** (2005), 1–19.

[18] G. DalMaso, F.Murat, L. Orsina, and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **28(4)** (1999), no. 4, 741–808.

[19] L. Diening, P. Harjulehto, P. Hästö, and M. Råžička, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, Springer, Heidelberg, Germany, **2017** (2011).

[20] R.J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. of Math. **130(2)** (1989), no. 2, 321–366.

[21] X.L. Fan and D. Zhao, *On the generalised Orlicz-Sobolev Space* $W^{k,p(x)}(\Omega)$, J. Gansu Educ. College. **12** (1998), no. 1, 1–6.

[22] E. Hewitt and K. Stromberg, *Real and abstract analysis*, Springer-verlng, Berlin Heidelberg New York, 1965.

[23] I. Ibrango and S. Ouaro, *Entropy solution for doubly nonlinear elliptic anisotropic problems with Fourier boundary conditions*, Discuss. Math. Differ. Incl. Control Optim **35** (2015), no. 1, 123–150.

[24] I. Konaté and S. Ouaro, *Multivalued anisotropic problem with Fourier boundary condition involving diffuse Radon measure data and variable exponents*, Glas. Mat. Ser. III **54(74)** (2019), no. 1, 211–232.

[25] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod et Gauthiers-Villars, Paris, 1969.

[26] M. Mihailescu, P. Pucci, and V. Radulescu, *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Anal. Appl. **340** (2008).

[27] F. Murat, *Soluciones renormalizadas de EDP elipticas no lineales, Laboratoire d'Analyse Numerique*, Paris VI, R93023, 1993.

[28] ______, *Equations elliptiques non linéaires avec second membre l1 ou mesure*, Comptes Rendus du 26ème Congrés National d'Analyse Numèrique, Les Karellis (1994).

[29] M. M. Porzio, *On some quasilinear elliptic equations involving Hardy potential*, Rend. Mat. Appl. seven **27** (2007), no. 3-4, 277–297.

[30] A. Youssfi, E. Azroul, and H. Hjiaj, *On nonlinear elliptic equations with Hardy potential and L1-data*, Monatsh. Math. **173** (2014), no. 1, 107–129.

[31] D. Zhao, W. J. Qiang, and X.L. Fan, *On generalized Orlicz spaces* $L^{p(x)}(\Omega)$, J. Gansu Sci. **9** (1997), no. 2, 1–7.

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