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Pillai's Problem with Padovan Numbers and Prime Powers

El problema de Pillai con números de Padovan y potencias de primos

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Abstract. We consider the Padovan sequence $\{P_n\}_{n\geq 0}$, defined by $P_0 = 0$, $P_1 = P_2 = 1$, with subsequent terms given by the recurrence relation $P_{n+3} = P_{n+1} + P_n$ for all $n \geq 0$. In this paper, we use the methods of Baker-Davenport. We demonstrate that the Diophantine equation $P_n - p^m = P_{n_1} - p^{m_1}$ admits only finitely many non-negative integer solutions n, m, n_1, m_1, where p is a fixed prime number ≥ 5. Additionally, once the value of p is specified, these solutions can be obtained explicitly. We address the case where $p = 5$.

Key words and phrases. Padovan numbers, Linear form in logarithms, Reduction method.

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Resumen. Consideramos la succession de Padovan $\{P_n\}_{n\geq 0}$, definida inductivamente por $P_0 = 0$, $P_1 = P_2 = 1$ y la relación de recurrencia $P_{n+3} = P_{n+1} + P_n$ para $n \geq 0$. Este artículo utiliza el método de Baker-Davenport. Probamos que la ecuación Diofantina $P_n - p^m = P_{n_1} - p^{m_1}$ admite sólo finitas soluciones enteras positivas n, m, n_1, m_1, donde p es un número primo fijo ≥ 5. Más aún, una vez fijo el valor de p, las soluciones pueden ser listadas de manera explícita. Mostramos este proceso para el caso $p = 5$.

Palabras y frases clave. Números de Padovan, Formas lineales en logaritmos, método de reducción.

1. Introduction

Let $U = \{U_n\}_{n\geq 0}$ and $V = \{V_n\}_{n\geq 0}$ be two sequences of integers. We consider the problem of determining the existence of non-negative integer solutions (n, m, n_1, m_1) for the Diophantine equation

$$U_n - V_m = U_{n_1} - V_{m_1}, \quad (n, m) \neq (n_1, m_1). \tag{1}$$

A classical problem consists of finding integers c that can be expressed as the difference between elements of U and V; that is, solutions to the equation $U_n - V_m = c$. This problem was studied by Pillai for the case when U and V are sequences of powers of a and b, respectively, where a and b are integers greater than 2. This is known as the "Pillai problem" [14].

Herschfeld proved in [8] that this equation for $(a, b) = (2, 3)$ has at most one solution when $|c|$ is sufficiently large. This result was extended by Pillai in the case where a and b are co-prime integers [15]. Pillai conjectured that the equation $2^x - 3^y = 2^{x_1} - 3^{y_1}$ has the unique solutions $(x, y, x_1, y_1) = (3, 2, 1, 1)$, $(5, 3, 3, 1)$ and $(8, 5, 4, 1)$ [16]. This conjecture was proved later by Stroeker and Tijdeman in [19], by using Baker's theory on linear forms in the logarithm of algebraic numbers. The study of equation (1) in the case where U and V are linear recurrence sequences was initiated by Ddamulira, Luca, and Rakotomalala, who studied the case when U and V are the sequences of Fibonacci numbers and powers of 2, respectively [11]. Other cases of this type of equation have been studied (see [4, 5, 10, 12, 3, 9, 7]).

In this context, we consider the Diophantine equation

$$P_n - p^m = P_{n_1} - p^{m_1} \tag{2}$$

where $\{P_n\}_{n\geq 0}$ is the Padovan recurrence sequence given by the relation

$$P_{n+3} = P_{n+1} + P_n,$$

with $P_0 = 0$, $P_1 = P_2 = 1$ and $p \geq 5$ being a prime number. The first few terms of this sequence are as follows:

$$(P_n)_{n\geq 0} = \{0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, \ldots\}.$$

In this paper, we first demonstrate that the Diophantine equation (2) admits only finitely many non-negative integer solutions. This result is obtained by applying Baker's method to derive lower bounds for linear forms in logarithms of algebraic numbers. Secondly, for $p = 5$, this bound is reduced using a variant of a result by Baker and Davenport [2], and then, using Maple, we explicitly compute the different solutions of (2) in the case $p = 5$.

Since $P_1 = P_2 = P_3 = 1$, and $P_4 = P_5$, we identify P_1 and P_2 with P_3, and P_4 with P_5.

Theorem 1.1. *For each prime number $p \geq 5$, the number of non-negative integer solutions (n, m, n_1, m_1) of the equation (2) is finite.*

For $p = 2, 3$, similar results were obtained by Ana Cecilia García Lomelí, Santos Hernández Hernández, and Mahadi Ddamulira (see [6], [9]). For $p = 5$, we have the following theorem:

Theorem 1.2. *The solutions (n, m, n_1, m_1) in non-negative integers for the equation $P_n - 5^m = P_{n_1} - 5^{m_1}$, with $n > n_1$ and $m > m_1$ are*

$$(7,1,0,0), (8,1,3,0), (9,1,6,0), (10,1,8,0), (12,1,11,0),$$

$$(13,2,3,1), (14,2,7,0), (25,4,12,2), (36,6,27,5).$$

As a consequence, we arrive at the following corollary:

Corollary 1.3. *The distinct integers c that have two or more representations in the form $P_n - 5^m$ are limited to $\{0, -1, 2, 4, 11, -4, 3, -9, -2044\}$.*

Additionally, the corresponding representations of these numbers with integers n and m as $P_n - 5^m$ are enumerated as follows,

$$\begin{array}{rcccl}
-1 & = & P_7 - 5^1 & = & P_0 - 5^0 \\
0 & = & P_8 - 5^1 & = & P_3 - 5^0 \\
2 & = & P_9 - 5^1 & = & P_6 - 5^0 \\
4 & = & P_{10} - 5^1 & = & P_8 - 5^0 \\
11 & = & P_{12} - 5^1 & = & P_{11} - 5^0 \\
-4 & = & P_{13} - 5^2 & = & P_3 - 5^1 \\
3 & = & P_{14} - 5^2 & = & P_7 - 5^0 \\
-9 & = & P_{25} - 5^4 & = & P_{12} - 5^2 \\
-2044 & = & P_{36} - 5^6 & = & P_{27} - 5^5.
\end{array} \tag{3}$$

2. Auxiliary result

We begin by reviewing the fundamental characteristics of the Padovan sequence $\{P_n\}_{n\geq 0}$, as detailed in [17]. Specifically, the characteristic polynomial for this sequence is

$$\psi(x) := x^3 - x - 1.$$

The roots of ψ are α, β, and $\gamma = \overline{\beta}$, (the complex conjugate of β), where

$$\alpha = \frac{r_1 + r_2}{6}, \ \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$c_\alpha = \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1},$$
$$c_\beta = \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2+3\beta+1},$$
$$c_\gamma = \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2+3\gamma+1} = \bar{c_\beta}.$$

The following is Binet's formula for P_n:

$$P_n = c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n, \ for\ all\ n \geq 0. \tag{4}$$

Numerically, we can observe that

$$\begin{gathered} 1.32 < \alpha < 1.33, \\ 0.86 < |\beta| = |\gamma| < 0.87, \\ 0.72 < c_\alpha < 0.73, \\ 0.24 < |c_\beta| = |c_\gamma| < 0.25. \end{gathered} \tag{5}$$

It is easily verifiable that

$$|\beta| = |\gamma| = \alpha^{-\frac{1}{2}}.$$

We can demonstrate via induction that

$$\alpha^{n-3} \leq P_n \leq \alpha^{n-1} \quad \text{for all } n \geq 4. \tag{6}$$

Let $K := \mathbb{Q}(\alpha, \beta)$ denote the splitting field of the polynomial ψ over $\mathbb{Q}$; this extension has degree $[K : \mathbb{Q}] = 6$. Moreover, the Galois group of K over $\mathbb{Q}$ is represented by $\{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\}$ which is isomorphic to S_3. Thus, we identify the elements of G with the permutations of the zeros of the polynomial ψ. Standard references for transcendental methods in Diophantine equations include ([1], [2], [13], [14]). For any non-zero algebraic number γ of degree d over $\mathbb{Q}$, with minimal primitive polynomial over $\mathbb{Z}$ given by $a_0 \prod_{j=1}^{d}(X - \gamma^{(j)})$, where the $\gamma^{(i)}$'s denote the conjugates of γ, and the leading coefficient a_0 is positive, the logarithmic height of γ is defined by

$$h(\gamma) := \frac{1}{d}(\log a_0 + \sum_{i=1}^{d} \log(\max\{|\gamma^{(i)}|, 1\})).$$

In particular, $h(\gamma) = \log(\max|p|, q)$ if $\gamma = \dfrac{p}{q}$ is a rational integer with $\gcd(p, q) = 1$ and $q > 0$. In the following sections, we will use the following properties of the logarithmic height function $h(\cdot)$.

$$\begin{aligned} h(\eta \pm \gamma) &\leq h(\eta) + h(\gamma) + \log 2, \\ h(\eta\gamma^{\pm 1}) &\leq h(\eta) + h(\gamma), \\ h(\eta^s) &= |s| h(\eta) \quad (s \in \mathbb{Z}). \end{aligned} \tag{7}$$

A key result in our approach to the Diophantine equation is the following theorem, established by Matveev in [13] (see also Theorem 2 in [10]).

Theorem 2.1. *In a real algebraic number field* $\mathbb{K}$ *of degree* D*, let us consider positive real algebraic numbers* $\gamma_1, \ldots, \gamma_t$*. Let* $b_1, \ldots, b_t$ *be non-zero rational integers. Suppose that*

$$\Lambda := \gamma_1^{b_1} \ldots \gamma_t^{b_t} - 1 \tag{8}$$

is non-zero. Then we have

$$\log|\Lambda| > -1.4 \times 30^{t+3} \times t^{4.5} \times D^2(1+\log D)(1+\log B)A_1 \ldots A_t,$$

where

$$B \geq \max\{|b_1|, \ldots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log\gamma_i|, 0.16\} \quad \text{for all} \quad i = 1, \ldots, t.$$

2.1. Baker-Davenport reduction lemma

During our work, we find large upper bounds that should be reduced. Specifically, for a non-homogeneous linear form in two integer variables, we utilize a slightly modified version of a result by Dujella and Pethö (see [2]), which is a generalization of the result of Baker and Davenport [1]. For a real number X, we denote $\|X\|$:=min$\{|X-n|, n \in Z\}$ as the distance from X to the nearest integer. Recall that the distance between a real number X and the nearest integer is expressed as follows:

$$\|X\| := \min\{|X-n|, n \in Z\}.$$

Lemma 2.2 ([2])**.** *Let* τ *be a real number and* M *a positive integer. Let* $\frac{p}{q}$ *be convergent in the continued fraction of* τ *such that* $q > 6M$*, and let* μ, A, B *be real numbers such that* $A > 0$ *and* $B > 1$*. Define* $\epsilon = \|\mu q\| - M\|\tau q\|$*. If* $\epsilon > 0$*, the following inequality cannot be solved for positive integers* $u, v,$ *and* k*,*

$$0 < |u\tau - v + \mu| < AB^{-k},$$

where $u \leq M$ *and* $k \geq \dfrac{\log(Aq/\epsilon)}{\log B}$.

Additionally, we require the following lemma from [18]:

Lemma 2.3 (Sánchez, Luca)**.** *If* $m \geq 1$ *,* $Z \geq (4m^2)^m$ *and* $Z > \dfrac{t}{(\log t)^m}$*, then*

$$t \leq 2^m Z(\log Z)^m.$$

3. Proof of Theorem 1.1

We notice that when $m = m_1$ then $n = n_1$; therefore, we will assume that $m > m_1$ in the following.

By applying equations (2) and (6), we obtain

$$\alpha^{n-8} \leq P_n - P_{n_1} = p^m - p^{m_1} < p^m,$$

The inequality on the left-hand side is evident for $n_1 = 0$ and $n = 3$. When $n_1 \geq 3$ and $n \geq 5$, we observe that

$$P_n - P_{n_1} \geq P_n - P_{n-1} = P_{n-5} \geq \alpha^{n-8}.$$

These inequalities are straightforward. Consequently, we see that $\alpha^{n-8} < p^m$.

Likewise,

$$\alpha^{n-1} \geq P_n \geq P_n - P_{n_1} = p^m - p^{m_1} > p^{m-1}.$$

Thus,

$$(n-8)\frac{\log \alpha}{\log p} < m < (n-1)\frac{\log \alpha}{\log p} + 1. \tag{9}$$

Finding a bound for n allows us to find a bound for m, which shows that the number of solutions is finite. Thus, establishing an upper bound for n is essential before addressing the Diophantine equation (2).

3.1. Bounding n:

Using (2) and (4) in conjunction with the estimates from (5), we arrive at

$$\begin{aligned} c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n - p^m &= c_\alpha \alpha^{n_1} + c_\beta \beta^{n_1} + c_\gamma \gamma^{n_1} - p^{m_1} \\ |c_\alpha \alpha^n - p^m| &= |c_\alpha \alpha^{n_1} + c_\beta(\beta^{n_1} - \beta^n) + c_\gamma(\gamma^{n_1} - \gamma^n) - p^{m_1}| \\ &< c_\alpha \alpha^{n_1} + |c_\beta|(|\beta|^{n_1} + |\beta|^n) + |c_\gamma|(|\gamma|^{n_1} + |\gamma|^n) + p^{m_1} \\ &< c_\alpha \alpha^{n_1} + 4|c_\beta||\beta|^{n_1} + p^{m_1} \\ &< \alpha^{n_1-1} + 1 + p^{m_1} \\ &< 2\alpha^{n_1} + p^{m_1} \\ &< 2\max\{\alpha^{n_1+3}, p^{m_1}\}. \end{aligned}$$

Dividing both sides by p^m and using the fact that $2 < \alpha^3$, we obtain

$$\begin{aligned} |c_\alpha \alpha^n p^{-m} - 1| &< \max\left\{\frac{\alpha^{n_1+6}}{p^m}, \frac{p^{m_1+1}}{p^m}\right\} \\ &< \max\left\{\alpha^{n_1-n+14}, p^{m_1-m+1}\right\}, \end{aligned} \tag{10}$$

whenever we have made use of $\alpha^{n-8} < p^m$.

We put
$$\Lambda = c_\alpha \alpha^n p^{-m} - 1.$$
Assuming that $\Lambda = 0$, we have $c_\alpha \alpha^n = p^m \in \mathbb{Z}$. By applying the Galois automorphism corresponding to $(\alpha\beta)$, we obtain $c_\beta \beta^n = p^m$. However, since $|c_\beta \beta^n| < 1$, this creates a contradiction. Consequently, we conclude that $\Lambda \neq 0$.

By applying Matveev's inequality, we calculate Λ by taking
$$t = 3, \quad \gamma_1 = c_\alpha, \quad \gamma_2 = \alpha, \quad \gamma_3 = p,$$
$$b_1 = 1, \quad b_2 = n, \quad b_3 = -m.$$
We often use $D = 3$ and the field $\mathbb{K} = \mathbb{Q}(\alpha)$ in our study. We choose $B := n$ since $\max\{1, n, m\} \leq n$. Moreover, $c_\alpha = \dfrac{\alpha(\alpha+1)}{3\alpha^2 - 1}$. The polynomial $23x^3 - 23x^2 + 6x - 1$ is the minimal polynomial of c_α, with roots c_α, c_β and c_γ. According to (5), we also have $\max\{|c_\alpha|, |c_\beta|, |c_\gamma|\} < 1$. Thus, $h(\gamma_1) = \dfrac{1}{3}\log 23$. Subsequently, we choose $A_1 = 3h(\gamma_1) = \log 23$. In the same way, $A_2 = 3h(\gamma_2) = \log(\alpha)$ and $A_3 = 3h(\gamma_3) = 3\log p$.

We prove that the left-hand side of (10) is bounded by using the theorem of Matveev,
$$\log|\Lambda| > -1.4 \times 30^6 \times 3^{4.5} \times 9 \times (1 + \log 3) \times (1 + \log n) \times \log 23 \times \log\alpha \times 3\log p$$
and we compare this with (10), we get
$$\min\{(n - n_1 - 14)\log\alpha, (m - m_1 - 1)\log p\} < 7.154 \times 10^{12} \times (1 + \log n) \times \log p.$$
Thus, this leads to
$$\min\{(n - n_1)\log\alpha, (m - m_1)\log p\} < 7.16 \times .10^{12} \times (1 + \log n) \times \log p. \quad (11)$$
Now, let us examine each of these two cases individually.

Case 1 : $min\{(n - n_1)\log\alpha, (m - m_1)\log p\} = (n - n_1)\log\alpha$.

Under these conditions, we state (2) as
$$\begin{aligned} |c_\alpha \alpha^n - c_\alpha \alpha^{n_1} - p^m| &< |c_\beta|(|\beta|^{n_1} + |\beta|^n) + |c_\gamma|(|\gamma|^{n_1} + |\gamma|^n) + p^{m_1} \\ &< 4|c_\beta||\beta|^{n_1} + p^{m_1} \\ &< p^{m_1+1} \end{aligned}$$
leading to
$$|c_\alpha \alpha^{n_1}(\alpha^{n-n_1} - 1)p^{-m} - 1| < p^{m_1 - m + 1}. \quad (12)$$
We put
$$\Lambda_1 = c_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1} p^{-m} - 1.$$

To demonstrate that $\Lambda_1 \neq 0$, suppose for contradiction that $\Lambda_1 = 0$. Then, we have $c_\alpha \alpha^{n_1}(\alpha^{n-n_1} - 1) = p^m$.

By applying the Galois automorphism corresponding to $(\alpha\beta)$ to the previous relation, we obtain

$$|c_\beta \beta^{n_1}(\beta^{n-n_1} - 1)| \leq |c_\beta|(|\beta|^n + |\beta^{n_1}|) < 2|c_\beta||\beta|^{n_1} < 1,$$

which contradicts $p^m \geq 1$ for all $m \geq 0$. Hence $\Lambda_1 \neq 0$.

We can use Matveev's theorem on Λ_1, let us take

$$\gamma_1 = c_\alpha(\alpha^{n-n_1} - 1), \quad \gamma_2 = \alpha, \quad \gamma_3 = p,$$

$$b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m.$$

By (9) we have $\max\{1, n_1, m\} < n$, we can choose $B = n$.

Since

$$\begin{aligned} h(\gamma_1) &\leq h(c_\alpha) + h(\alpha^{n-n_1} - 1) \\ &< \frac{1}{3}\log 23 + (n - n_1)h(\alpha) + \log 2 \\ &= \frac{1}{3}(\log 23 + \log 8 + (n - n_1)\log\alpha) \\ &= \frac{1}{3}(\log 184 + (n - n_1)\log\alpha) \\ &< \frac{1}{3}(5.22 + (n - n_1)\log\alpha) \\ &< \frac{1}{3}(5.22 + 7.16 \times 10^{12} \times (1 + \log n) \times \log p, \quad \text{by (11)}, \end{aligned}$$

then

$$3h(\gamma_1) < 7.17 \times 10^{12} \times (1 + \log n) \times \log p,$$

therefore, we can select $A_1 = 7.17 \times 10^{12} \times (1 + \log n) \times \log p$. Additionally, as before, we choose $A_2 = 3h(\alpha) = \log\alpha$ and $A_3 = 3\log p$.

As a result of Matveev's theorem, we may conclude that

$$\begin{aligned} \log|\Lambda_1| > -1.4 \times 30^6 \times 3^{4.5} \times 9 \times (1 + \log 3) \times (1 + \log n) \times 7.17 \times 10^{12} \\ \times (1 + \log n) \times \log p \times \log\alpha \times 3\log p \end{aligned}$$

$$\log|\Lambda_1| > -1.64 \times 10^{25} \times (1 + \log n)^2 \times (\log p)^2.$$

By comparing the above relation with (12), we notice that

$$(m - m_1)\log p < 1.65 \times 10^{25} \times (1 + \log n)^2(\log p)^2. \tag{13}$$

Case 2 : $\min\{(n - n_1)\log\alpha, (m - m_1)\log p\} = (m - m_1)\log p$.

In this case, we rewrite our equation as,

$$\begin{aligned}
|c_\alpha\alpha^n - p^m + p^{m_1}| &= |c_\alpha\alpha^{n_1} + c_\beta\beta^{n_1} + c_\gamma\gamma^{n_1} - c_\beta\beta^n - c_\gamma\gamma^n| \\
|c_\alpha\alpha^n - (p^{m-m_1} - 1)p^{m_1}| &< c_\alpha\alpha^{n_1} + |c_\beta|(|\beta|^{n_1} + |\beta|^n) + |c_\gamma|(|\gamma|^{n_1} + |\gamma|^n) \\
&< \frac{3}{4}\alpha^{n_1} + 4|c_\beta||\beta|^{n_1} \\
&\leq \frac{3}{4}\alpha^{n_1} + 1 \\
&\leq \frac{3}{4}\alpha^{n_1} + \alpha^{n_1} \\
&= \frac{7}{4}\alpha^{n_1} \\
&< \alpha^{n_1+2}.
\end{aligned}$$

It suggests that

$$\begin{aligned}
\left|\frac{c_\alpha}{p^{m-m_1} - 1}.\alpha^n.p^{-m_1} - 1\right| &< \frac{\alpha^{n_1+2}}{p^m - p^{m_1}} \\
&< \frac{p.\alpha^{n_1+2}}{p^m} \\
&< \frac{p.\alpha^{n_1+2}}{\alpha^{n-8}} \\
&< p.\alpha^{n_1-n+10}
\end{aligned}$$

$$\left|\frac{c_\alpha}{p^{m-m_1} - 1}\alpha^n.p^{-m_1} - 1\right| < p.\alpha^{n_1-n+10}. \tag{14}$$

On the left-hand side of the absolute value, the expression (14) should be represented by Λ_2. To demonstrate that, $\Lambda_2 \neq 0$, assume otherwise. We derive $c_\beta\beta^n = p^m - p^{m_1}$ by applying the Galois automorphism $(\alpha\beta)$ to the previously described connection, we obtain $c_\beta\beta^n = p^m - p^{m_1}$. This leads to a contradiction by taking the absolute value, indeed, $|c_\beta\beta^n| < \frac{1}{4}$ and $|p^m - p^{m_1}| = |p^{m_1}(p^{m-m_1} - 1)| \geq 4$. We use Λ_2 and Matveev's theorem again. For this, we take into consideration

$$t = 3, \quad \gamma_1 = \frac{c_\alpha}{p^{m-m_1} - 1}, \quad \gamma_2 = \alpha, \quad \gamma_3 = p,$$

$$b_1 = 1, \quad b_2 = n, \quad b_3 = -m_1.$$

Therefore, $B = n$. The heights of γ_2 and γ_3 had previously been determined. Using the height for γ_1, we derive,

$$\begin{aligned}
h(\gamma_1) &\leq h(c_\alpha) + h(p^{m-m_1} - 1) \\
&\leq \frac{1}{3}\log 23 + (m - m_1)h(p) + \log 2 \\
&< \log(p^{m-m_1+1}) \\
&< 7.17 \times 10^{12} \times (1 + \log n) \times \log p \qquad \text{by } (11).
\end{aligned}$$

We can select, as a result $A_1 = 2.16 \times 10^{13} \times (1 + \log n) \times \log p$. Furthermore, we set $A_2 = \log\alpha$ and $A_3 = 3\log p$, similar to the preceding cases.

Since $\max\{1, n, m_1\} \leq n$, we can fix $B := n$.

After that, we obtain

$$\log|\Lambda_2| > -1.4 \times 30^6 \times 3^{4.5} \times 9 \times (1 + \log 3) \times (1 + \log n) \times 2.16 \times 10^{13} \times (1 + \log n) \times \log p \times \log\alpha \times 3\log p,$$

thus

$$\log|\Lambda_2| > -4.93\ldots \times 10^{25} \times (1 + \log n)^2 \times (\log p)^2.$$

By contrasting the aforementioned relation with (14), we conclude that

$$(n - n_1 - 10)\log(\alpha) < \log p + 4.93 \times 10^{25} \times (1 + \log n)^2 \times (\log p)^2,$$

subsequently

$$(n - n_1)\log\alpha < 4.94 \times 10^{25} \times (1 + \log n)^2 \times (\log p)^2. \tag{15}$$

Therefore, let us combine the two cases

$$\min\{(n - n_1)\log\alpha, (m - m_1)\log p\} < 7.16 \times 10^{12} \times (1 + \log n) \times \log p,$$

and

$$\max\{(n-n_1)\log\alpha, (m-m_1)\log p\} < 4.94 \times 10^{25} \times (1+\log n)^2 \times (\log p)^2. \tag{16}$$

Finally, we can represent equation (2) as follows

$$\begin{aligned} |c_\alpha\alpha^n - c_\alpha\alpha^{n_1} - p^m + p^{m_1}| &= |c_\beta\beta^{n_1} - c_\beta\beta^n + c_\gamma\gamma^{n_1} - c_\gamma\gamma^n| \\ &< |c_\beta\beta^{n_1}| + |c_\beta\beta^n| + |c_\gamma\gamma^{n_1}| + |c_\gamma\gamma^n| \\ &< 4|c_\beta||\beta|^{n_1} < 1. \end{aligned}$$

We obtain the following by dividing both sides by $p^m - p^{m_1}$,

$$\begin{aligned} \left|\frac{c_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1}p^{-m_1}}{p^{m-m_1} - 1} - 1\right| &< \frac{1}{p^m - p^{m_1}} \\ &< \frac{p}{p^m} \\ &< \frac{p}{\alpha^{n-8}}. \end{aligned}$$

Then

$$\left|\frac{c_\alpha(\alpha^{n-n_1} - 1)\alpha^{n_1}p^{-m_1}}{p^{m-m_1} - 1} - 1\right| < \frac{p}{\alpha^{n-8}}, \tag{17}$$

Here, we applied the inequality $\alpha^{n-8} < p^m$.

If we put

$$\Lambda_3 = \frac{c_\alpha(\alpha^{n-n_1}-1)}{p^{m-m_1}-1}\alpha^{n_1}p^{-m_1} - 1.$$

We must show that $\Lambda_3 \neq 0$. Assuming the opposite, $\Lambda_3 = 0$, implies

$$c_\alpha(\alpha^n - \alpha^{n_1}) = p^m - p^{m_1}.$$

On the above equation, we apply the Galois automorphism $(\alpha\beta)$ and discover that

$$c_\beta(\beta^n - \beta^{n_1}) = p^m - p^{m_1}.$$

Subsequently

$$|c_\beta(\beta^n - \beta^{n_1})| \leq |c_\beta|(|\beta|^n + |\beta|^{n_1}) < \frac{1}{2}$$

and

$$|p^m - p^{m_1}| = p^{m_1}(p^{m-m_1} - 1) \geq 4$$

which is a contradiction. It follows that $\Lambda_3 \neq 0$. Applying Matveev's theorem to it using the given parameters

$$t = 3, \quad \gamma_1 = \frac{c_\alpha(\alpha^{n-n_1}-1)}{p^{m-m_1}-1}, \quad \gamma_2 = \alpha, \quad \gamma_3 = p,$$

$$b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m_1.$$

Using the logarithmic height function, we can derive

$$\begin{aligned} h(\gamma_1) &\leq h(c_\alpha(\alpha^{n-n_1}-1)) + h(p^{m-m_1}-1) \\ &\leq h(c_\alpha) + (n-n_1)h(\alpha) + \log 2 + (m-m_1)h(p) + \log 2 \\ &\leq \frac{1}{3}\log 23 + \frac{1}{3}(n-n_1)\log\alpha + (m-m_1)\log p + 2\log 2. \end{aligned}$$

Then, using (16) we have

$$\begin{aligned} 3h(\gamma_1) &< \log 23 + (n-n_1)\log\alpha + 3(m-m_1)\log p + 6\log 2 \\ &< 2\times 10^{26}\times(1+\log n)^2 + (\log p)^2. \end{aligned}$$

We can choose $A_1 := 2\times 10^{26}\times(1+\log n)^2(\log p)^2$ and as before $A_2 := \log\alpha$ and $A_3 := 3\log(p)$.

we get,

$\log|\Lambda_3| > -1.4\times 30^6\times 3^{4.5}\times 9\times(1+\log 3)\times(1+\log n)\times 2\times 10^{26}\times(1+\log n)^2\times(\log p)^2\log\alpha\times 3\log p$.

Then,

$$\log|\Lambda_3| > -4.57\times 10^{38}\times(1+\log n)^3\times(\log p)^3.$$

Comparing those yields to those from (17),

$$\begin{aligned}(n-8)\log\alpha &< 4.58\times 10^{38}\times(1+\log n)^3\times(\log p)^3\\ n &< 1.63\times 10^{39}\times(1+\log n)^3\times(\log p)^3\\ n &< 10^{41}\times(\log n)^3\times(\log p)^3.\end{aligned}$$

From Lemma 2.3, we can now deduce

$$n < 8\times 10^{41}\times(\log p)^3\times\log(10^{40}\times(\log p)^3. \tag{18}$$

Therefore, for each prime number p, n is bounded. This proves, Theorem 1.1.

3.2. Proof of Theorem 1.2

Now, let us take $p=5$. By 18 we obtain the bound

$$n < 3.63224\cdot 10^{47}$$

Assuming $n\le 500$, then $m\le 88$, we use (9). Within the range $0\le n_1<n\le 500$ and $0\le m_1<m\le 88$, We executed a Maple program, and all solutions as in Theorem 1.2, were obtained. Assuming $500<n$ in what follows, we can deduce $m>88$ based on (9). Hence, it is crucial to establish an upper bound for n before tackling the Diophantine equation (2) when $p=5$.

To reduce the previously mentioned constraint for n, we use Lemma 2.2 iteratively.

Let's revisit (10) again

$$\Gamma := n\log\alpha - m\log 5 + \log c_\alpha.$$

We assume $\min\{n-n_1, m-m_1\}\ge 20$, (If not, we use (12) in the case $n-n_1\le 20$ and $m-m_1\ge 20$, we use (14) in the case $n-n_1\ge 20$ and $m-m_1\le 20$, and finally if $n-n_1>20$ and $m-m_1>20$ we use formula (17)).

Keep in mind that $e^\Gamma-1=\Lambda\neq 0$, therefore $\Gamma\neq 0$.

If $\Gamma>0$, we have

$$0<\Gamma<e^\Gamma-1=|\Lambda|<\max\{\alpha^{n_1-n+14}, 5^{m_1-m+1}\}.$$

If $\Gamma<0$, we have $1-e^\Gamma=|e^\Gamma-1|<\dfrac{1}{4}$, thus $e^{|\Gamma|}<\dfrac{4}{3}$, and we obtain

$$0<|\Gamma|<e^{|\Gamma|}-1=e^{|\Gamma|}|\Lambda|<\frac{4}{3}\max\{\alpha^{n_1-n+14}, 5^{m_1-m+1}\}.$$

Therefore, in each case, we arrive at

$$0<|\Gamma|<\frac{4}{3}\max\{\alpha^{n_1-n+14}, 5^{m_1-m+1}\}.$$

After dividing by $\log 5$, we obtain

$$0 < \left| n\frac{\log \alpha}{\log 5} - m + \frac{\log c_\alpha}{\log 5} \right| < \max\{43 \times \alpha^{-(n-n_1)}, 5 \times 5^{-(m-m_1)}\}.$$

Lemma 2.2 is utilized with the given data

$$\tau = \frac{\log \alpha}{\log 5}, \quad \mu = \frac{\log c_\alpha}{\log 5}, \quad (A, B) = (43, \alpha) \ \text{ or } \ (5, 5).$$

The $89 - th$ convergent of τ is examined, and $M := 3.63224 \times 10^{47}$ is taken as the upper bound on n to accomplish this

$$\frac{p}{q} = \frac{p_{89}}{q_{89}} = \frac{306021468323955514815822059178789731056956669902464}{1751505329711686341039694121547160756821189158240 11},$$

$q > 6M$ is satisfied by the chosen value of q. It also yields $\epsilon = \|q\mu\| - M\|q\tau\| = 0.04071840822609\ldots > 0$, Lemma 2.2 enables us to deduce that either

$$n - n_1 < \frac{\log(\frac{43 \times q}{\epsilon})}{\log \alpha} = 436.17422133\ldots, \ \text{ or } \ m - m_1 < \frac{\log(\frac{5 \times q}{\epsilon})}{\log 5} = 74.87101070\ldots$$

Then

$$n - n_1 \leq 436, \quad \text{or} \quad m - m_1 \leq 74.$$

We now examine each of these two instances separately.

The first assumption is that $n - n_1 \leq 436$. In this instance, we consider

$$\Gamma_1 = n_1 \log \alpha - m \log 5 + \log(c_\alpha(\alpha^{n-n_1} - 1)).$$

Here, when we proceed to (12), we observe that $e^{\Gamma_1} - 1 = \Lambda_1 \neq 0$, indicating that $\Gamma_1 \neq 0$. Since $|e^{\Gamma_1} - 1| = |\Lambda_1| < 5^{m_1 - m + 1} < \dfrac{1}{4}$, then $|\Gamma_1| < \dfrac{1}{4}$.

As $|x| < 2|e^x - 1|$ for all $x \in \left(\dfrac{-1}{2}; \dfrac{1}{2}\right)$, we derive

$$|\Gamma_1| < 2|e^{\Gamma_1} - 1| = 2|\Lambda_1|,$$

then

$$|\Gamma_1| < 2 \times 5^{m_1 - m + 1} = \frac{10}{5^{m - m_1}}.$$

When we divide both sides by $\log 5$, we get

$$0 < |n_1 \tau - m + \mu_k| < 7 \times 5^{-(m - m_1)},$$

where τ remains the same as before, and for every value of $k := n - n_1 \in \{1, 2, \ldots, 436\}$, we have $\mu_k = \dfrac{\log(c_\alpha(\alpha^k - 1))}{\log 5}$ instead of μ.

For all values of k, we get $\epsilon > 0.001728193$.

Therefore, we use the $87-th$ convergent $\frac{p}{q} = \frac{p_{87}}{q_{87}}$ of τ that satisfies $q_{87} > 6M$, as stated by Lemma 2.2. As a result, for all $k = 1, 2, \ldots, 421$, we compute $\dfrac{\log(\frac{7\times q_{87}}{\epsilon_k})}{\log 5}$, we discover that the greatest value of these is at most 75. Consequently, $m - m_1 \leq 75$. Let us now consider the case where $m - m_1 \leq 74$, let us take

$$\Gamma_2 = n \log \alpha - m_1 \log 5 + \log\left(\frac{c_\alpha}{5^{m-m_1} - 1}\right).$$

Next, we have

$$|e^{\Gamma_2} - 1| = |\Lambda_2| < \alpha^{n_1 - n + 16} < \frac{1}{4}.$$

Then $|\Gamma_2| < \dfrac{1}{2}$ as previously, we can conclude that

$$0 < |\Gamma_2| < \alpha^{n_1 - n + 19}.$$

The result of dividing by $\log 5$ is

$$0 < \left| n\frac{\log \alpha}{\log 5} - m_1 + \frac{\log\left(\frac{c_\alpha}{5^{m-m_1} - 1}\right)}{\log 5} \right| < 130 \times \alpha^{-(n-n_1)}.$$

Using the same τ, $q = q_{88}$, M, and taking $A = 130$, $B = \alpha$ and $\mu_l = \dfrac{\log\left(\frac{c_\alpha}{5^l - 1}\right)}{\log 5}$ for $l = 1, 2, \ldots, 74$, we reiterate the application of Lemma 2.2. We verify, using Maple, that for every $l = 1, 2, \ldots, 72$, the 88-th convergent of τ, represented as q_{88}, satisfies, $q_{88} > 6M$ and $\epsilon_l > 0.014094654$. For every $l = 1, 2, \ldots, 74$, we calculated $\dfrac{\log\left(\frac{130 \times q_{88}}{\epsilon_l}\right)}{\log \alpha} = 435.511774064\cdots$, and discovered that the maximum value is < 436.

Finally, our analysis gives the combined results $n - n_1 \leq 435$ and $m - m_1 \leq 75$.

In the last step, we consider the inequality involving Λ_3. Let

$$\Gamma_3 = n_1 \log \alpha - m_1 \log 5 + \log\left(\frac{c_\alpha(\alpha^{n-n_1} - 1)}{5^{m-m_1} - 1}\right).$$

Since $n \geq 500$, we utilize $\dfrac{\alpha^{14}}{\alpha^n} < \dfrac{1}{4}$, and apply (17), to obtain $0 < |\Gamma_3| < \dfrac{1}{2}$, consequently, we have $|\Gamma_3| < 2 \times \dfrac{\alpha^{14}}{\alpha^n}$.

After dividing by $\log 5$, we arrive at

$$|n_1\tau - m_1 + \mu_{k,l}| < 64 \times \alpha^{-n},$$

where τ is as defined earlier, $\mu_{k,l} := \dfrac{\log\left(\dfrac{c_\alpha(\alpha^k-1)}{5^l-1}\right)}{\log 5}$ for $k := n - n_1 \in \{1, 2, \ldots 435\}$, and $l := m - m_1 \in \{1, 2, \ldots, 75\}$.

As a result, we can use Lemma 2.2. The same 88-th convergent above of τ, satisfy $q_{88} > 6M$. And $\epsilon_{k,l} \geq 0.01409465477$ for $1 \leq k \leq 435$ and $1 \leq l \leq 75$, works well in this case based on our analysis. Using Maple, we set $A := 64$, $B := \alpha$, we computed $\dfrac{\log\left(\dfrac{64 \times q_{88}}{\epsilon_{k,l}}\right)}{\log \alpha}$ for all k in the set $\{1, 2, \ldots, 435\}$, and $l \in \{1, 2, \ldots, 75\}$, and found that the highest value is < 432.

Therefore, $n \leq 431$, which contradicts our assumption that $n \geq 500$. Thus, Theorem 1.2 is now fully proven.

In conclusion, the general problem remains widely open for arbitrary values of the prime p. In this regard, it would be relevant to extend the analysis to a broader set of prime numbers, for instance considering values $p \in [5, 1000]$. Such an extension would constitute a natural direction for future research.

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