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Entropy solutions for a class of doubly nonlinear parabolic systems involving measure data in non-reflexive Orlicz spaces

Soluciones de entropía para una clase no lineal de sistemas parabólicos que involucran conjuntos de datos medibles en espacios de Orlicz no reflexivos

Jabir Ouazzani Chandi[1,✉], Mohamed Bourahma[2], Hassane Hjiaj[1], Jaouad Bennouna[2], Abdelmoujib Benkirane[2]

[1]Abdelmalek Essaadi University, Tetouan, Morocco

[2]Sidi Mohamed Ben Abdellah University, Fez, Morocco

Abstract. This paper deals with an existence result of entropy solutions for a nonlinear parabolic systems of the form

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \operatorname{div}\Big(a(x,t,u_i,\nabla u_i) + \Phi_i(x,t,u_i)\Big) = f_i(x,u_1,u_2) - \operatorname{div}(F_i) & \text{in } Q_T \\ u_i = 0 & \text{on } \Gamma \\ u_i(t=0) = u_{i,0} & \text{in } \Omega, \end{cases}$$

where the lower order term Φ satisfies a growth condition prescribed by the N-function M defining the framework spaces (see section 2.1) and the right hand side is a measure datum. The main term which contains the space derivatives and a non-coercive lower order term are considered in divergence form satisfying only the original Orlicz growths. We don't assume any restriction neither on M nor on its complementary $\overline{M}$. Therefor, we work in a nonreflexive Orlicz spaces.

Key words and phrases. Parabolic systems; generalized growth; Orlicz-Sobolev spaces; Entropy solutions.

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Resumen. Este artículo se centra en probar la existencia de soluciones entrópicas para sistemas no lineales parabólicos de la forma

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \operatorname{div}\Big(a(x,t,u_i,\nabla u_i) + \Phi_i(x,t,u_i)\Big) = f_i(x,u_1,u_2) - \operatorname{div}(F_i) & \text{in } Q_T \\ u_i = 0 & \text{on } \Gamma \\ u_i(t=0) = u_{i,0} & \text{in } \Omega, \end{cases}$$

donde el término Φ satisface una condición de crecimiento dada por la N-función M definida en los espacios descritos en la sección 2.1 y el lado derecho de la ecuación corresponde a las condiciones medibles asociadas al problema. El término principal que contiene los términos con derivadas espaciales y los términos no coercitivos de menor orden, que aparecen forma divergente, satisfacen las tasas de crecimiento de Orlicz. No vamos a suponer ninguna restricción sobre la función M o la función complementaria $\overline{M}$. Por lo tanto, trabajaremos en espacios de Orlicz no reflexivos.

Palabras y frases clave. Sistemas parabólicos, crecimiento generalizado, espacios de Orlicz-Sobolev, soluciones de entropía.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, Q_T be the cylinder $\Omega \times (0,T)$ where T is a positive real number and M is an Orlicz function. Let $A : D(A) \subset W_0^{1,x} L_M(Q_T) \to W^{-1,x} L_{\overline{M}}(Q_T)$ be an operator of Leray-Lions type of the form:

$$\mathrm{A}(u) := -\operatorname{div} a(x,t,u,\nabla u).$$

In this paper we prove an existence theorem of entropy solutions in the setting of Orlicz spaces for the nonlinear parabolic problem

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \operatorname{div}\Big(a(x,t,u_i,\nabla u_i) + \Phi_i(x,t,u_i)\Big) = f_i(x,u_1,u_2) - \operatorname{div}(F_i) & \text{in } Q_T \\ u_i = 0 & \text{on } \Gamma \\ u_i(t=0) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{1}$$

where $u_0 \in L^1(\Omega)$, $f_i \in L^1(Q_T)$ and Φ_i satisfies the following natural growth condition

$$|\Phi_i(x,t,s)| \leq \gamma(x,t) + \overline{M}^{-1}(M(\delta|s|)), \text{ with } \gamma \in E_{\overline{M}}(Q_T). \tag{2}$$

The problem (1), with a single equation, has been studied in different particular directions. In the classical Sobolev spaces, in some elliptic cases, Guibé et al.(see [3]), have assumed on Φ, the condition

$$|\Phi(x,s)| \leq c(x)\Big(1+|s|\Big)^{p-1}, \tag{3}$$

and in some parabolic cases (see [17]), they have assumed the condition

$$
\begin{aligned}
&|\Phi(x,t,s)| \leq c(x,t)\Big(1+|s|^{\gamma}\Big), \text{ with} \\
&\gamma = \frac{N+2}{N+p}(p-1) \text{ and } c \in L^{r}(Q_T) \text{ for an } r>0.
\end{aligned} \tag{4}
$$

The parabolic equations in Orlicz spaces have been widely studied since the year 2005 starting from the works of Meskine et al. (see [6, 7]) and with later results by several authors, for instance the work of Moussa, Rhoudaf and Mabdaoui, (see [14]). Concerning the case of Orlicz spaces with a single equation, the existence of entropy solution for problem (1) has been established in the case $f \in L^1(Q_T)$ under the non natural growth condition

$$
|\Phi(x,t,s)| \leq \gamma(x,t)\cdot \overline{P}^{-1}(P(\delta|s|)) \text{ where } \gamma \in L^{\infty}(Q_T) \text{ and } P \prec\prec M. \tag{5}
$$

For the double equation, in the classical Sobolev spaces, the system (1) has been solved by Azroul et al. in [4] in the case where Φ_i are independent of x using the concept of renormalized solution. For the study of (1) in some particular cases one can consult [5, 11, 12, 15, 19].

The approach of this paper is how to deal with the existence of entropy solutions for system (1) in Orlicz spaces involving measure data where Φ_i satisfies the original Orlicz growth condition

$$
|\Phi_i(x,t,s)| \leq \gamma(x,t) + \overline{M}^{-1}(M(\delta|s|)), \text{ where } \gamma \in E_{\overline{M}}(Q_T), \tag{6}
$$

without assuming any restriction on the modular function M neither on its complementary $\overline{M}$, the described problem lives in non reflexive Orlicz spaces. The existence result in this context generalizes all cases mentioned above.

The space equipped with M and $\overline{M}$ satisfying the Δ_2 condition is reflexive and separable, which essentially simplifies methods of PDEs, then, also the modular topology introduced in section 2.1 coincides with the norm one. The challenges resulting from the lack of the reflexivity of the framework spaces are significant and require precise handling with general N-functions that do not satisfy the Δ_2 condition. Also, the imposed condition on the lower order term Φ_i is less restrictive and leads to serious difficulties in proving existence of solution to the approximate problem and it's convergence. Our approach to overcome such difficulties is the use of Young's inequality combined with a nice algebraic trick on a good decomposition of the constant of coercivity α.

Regarding applications, parabolic equations have many applications, among them image processing and electro-rheological fluids modeling. A model of applications of these operators is the Boussinesq's system:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u - 2\,\mathrm{div}(\mu(\theta)\varepsilon(u)) + \nabla p = F(\theta) & \text{in } Q_T \\ \dfrac{\partial b(\theta)}{\partial t} + u \cdot \nabla b(\theta) - \triangle\theta = 2\mu(\theta)|\varepsilon(u)|^2 & \text{in } Q_T \\ u = 0, \quad \theta = 0 & \text{on } \Gamma \\ u(t=0) = u_0 \quad b(\theta)(t=0) = b(\theta_0) & \text{in } \Omega, \end{cases}$$

where the first equation is the motion conservation equation, the unknowns are the fields of displacement $u : Q_T \to \mathbb{R}^N$ and temperature $\theta : Q_T \to \mathbb{R}$, the field $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the strain rate tensor.

Let us briefly summarize the contents of this article: in section 2 we collect some well-known preliminaries, results and properties of Orlicz-Sobolev spaces and inhomogeneous Orlicz-Sobolev spaces. Section 3 is devoted to basic assumptions, problem setting and the proof of the main result.

2. Preliminaries

2.1. Orlicz-Sobolev spaces

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous and convex function with:

$$M(t) > 0 \text{ for } t > 0, \lim_{t\to 0} \frac{M(t)}{t} = 0 \text{ and } \lim_{t\to +\infty} \frac{M(t)}{t} = +\infty.$$

The function M is said an N-function or an Orlicz function, the N-function complementary to M being defined as

$$\overline{M}(t) = \sup\Big\{st - M(s), s \geq 0\Big\}.$$

We recall that (see [1])

$$M(t) \leq t\overline{M}^{-1}(M(t)) \leq 2M(t) \quad \text{for all } t \geq 0 \tag{7}$$

and Young's inequality: for all $s, t \geq 0$,

$$st \leq \overline{M}(s) + M(t).$$

We said that M satisfies the Δ_2-condition if for some $k > 0$,

$$M(2t) \leq kM(t) \quad \text{for all } t \geq 0, \tag{8}$$

and if (8) holds only for $t \geq t_0$, then M is said to satisfy the Δ_2-condition near infinity.

Let M_1 and M_2 be two N-functions. The notation $M_1 \prec\prec M_2$ means that M_1 grows essentially less rapidly than M_2, i.e.

$$\forall \epsilon > 0, \quad \lim_{t \to \infty} \frac{M_1(t)}{M_2(\epsilon t)} = 0,$$

that is the case if and only if

$$\lim_{t \to \infty} \frac{(M_2)^{-1}(t)}{(M_1)^{-1}(t)} = 0.$$

Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence class of) real-valued measurable functions u on Ω such that:

$$\int_\Omega M(u(x))dx < \infty \quad (\text{resp.} \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)dx < \infty \text{ for some } \lambda > 0).$$

Endowed with the norm

$$\|u\|_M = \inf\Big\{\lambda > 0 : \int_\Omega M\Big(\frac{u(x)}{\lambda}\Big)dx \leq 1\Big\},$$

$L_M(\Omega)$ is a Banach space and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The Orlicz-Sobolev space $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$).

This is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M.$$

Thus, $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $(N+1)$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the norm closure of the Schwartz space $\mathfrak{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathfrak{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that a sequence $\{u_n\}$ converges to u for the modular convergence in $W^1L_M(\Omega)$ if, for some $\lambda > 0$,

$$\int_\Omega M\Big(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\Big)\, dx \to 0 \quad \text{for all } |\alpha| \leq 1;$$

this implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then the modular convergence coincides with norm convergence. Recall that the norm $\|Du\|_M$ defined on $W_0^1 L_M(\Omega)$ is equivalent to $\|u\|_{1,M}$ (see [8]).

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denotes the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open Ω has the segment property then the space $\mathfrak{D}(\Omega)$ is dense in $W_0^1 L_M(\Omega)$ for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (see [8]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1 L_M(\Omega)$ is well defined. For more details one can see for example [1] or [13].

2.2. Inhomogeneous Orlicz-Sobolev spaces

Let Ω be a bounded open subset of $\mathbb{R}^N$, $T > 0$ and set $Q_T = \Omega \times (0,T)$. For each $\alpha \in \mathbb{N}^N$, denote by D_x^α the distributional derivative on Q_T of order α with respect to the variable $x \in \Omega$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows,

$$W^{1,x}L_M(Q_T) = \Big\{u \in L_M(Q_T) : D_x^\alpha u \in L_M(Q_T) \quad \text{for all} \quad |\alpha| \leq 1\Big\}$$

and

$$W^{1,x}E_M(Q_T) = \Big\{u \in E_M(Q_T) : D_x^\alpha u \in E_M(Q_T) \quad \text{for all} \quad |\alpha| \leq 1\Big\}.$$

The last space is a subspace of the first one, and both are Banach spaces under the norm,

$$\|u\| = \sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{M,Q_T}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q_T)$ which have as many copies as there is α-order derivatives, $|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$). If $u \in W^{1,x}L_M(Q_T)$ then the function : $t \mapsto u(t) = u(t,\cdot)$ is defined on $(0,T)$ with values in $W^1 L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q_T)$ then the concerned function is a $W^1 E_M(\Omega)$-valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_M(Q_T) \subset L^1(0,T; W^1 E_M(\Omega))$. The space $W^{1,x}L_M(Q_T)$ is not in general separable, if $u \in W^{1,x}L_M(Q_T)$, we can not conclude that the function $u(t)$ is measurable on $(0,T)$. However, the scalar function $t \mapsto \| u(t) \|_{M,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_M(Q_T)$ is defined as the (norm) closure in $W^{1,x}E_M(Q_T)$ of $\mathfrak{D}(Q_T)$. It is proved that when Ω has the segment property, then each element u of the closure of $\mathfrak{D}(Q_T)$ with respect of the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W^{1,x}L_M(Q_T)$, of some subsequence $(u_n) \subset \mathfrak{D}(Q_T)$ for the modular convergence; i.e., if, for some $\lambda > 0$, such that for all $|\alpha| \leq 1$;

$$\int_{Q_T} M\Big(\frac{D_x^\alpha u_n - D_x^\alpha u}{\lambda}\Big) dx\, dt \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

This implies that (u_n) converges to u in $W^{1,x}L_M(Q_T)$ for the weak topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$. Consequently,

$$\overline{\mathfrak{D}(Q_T)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathfrak{D}(Q_T)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q_T)$. Furthermore,

$$W_0^{1,x}E_M(Q_T) = W_0^{1,x}L_M(Q_T) \cap \Pi E_M.$$

We have then the following complementary system

$$\Big(W_0^{1,x}L_M(Q_T), F, W_0^{1,x}E_M(Q_T), F_0\Big),$$

F being the dual space of $W_0^{1,x}E_M(Q_T)$. It is also, modulo isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q_T)^{\perp}$, and will be denoted by $F = W^{-1,x}L_{\overline{M}}(Q_T)$ and it is shown in [8] that

$$W^{-1,x}L_{\overline{M}}(Q_T) = \Big\{f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in L_{\overline{M}}(Q_T)\Big\},$$

this space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\leq 1} \|f_\alpha\|_{\overline{M}, Q_T},$$

where the infimum is taken on all possible decompositions

$$f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha, f_\alpha \in L_{\overline{M}}(Q_T).$$

The space F_0 is then given by,

$$W^{-1,x}L_{\overline{M}}(Q_T) = \Big\{f = \sum_{|\alpha|\leq 1} D_x^\alpha f_\alpha : f_\alpha \in E_{\overline{M}}(Q_T)\Big\},$$

and is denoted by $F_0 = W^{-1,x}E_{\overline{M}}(Q_T)$.

Definition 2.1. [8] Recall that an open domain $\Omega \subset \mathbb{R}^N$ has the segment property if there exist a locally finite open covering O_i of the boundary $\partial\Omega$ of Ω and a corresponding vectors y_i such that if $x \in \overline{\Omega} \cap O_i$ for some i, then $x + ty_i \in \Omega$ for $0 < t < 1$.

Definition 2.2. (Truncation function)

Define the following real function of a real variable, called the truncation at height $k > 0$,

$$T_k(s) = \max\Big(-k, \min(k,s)\Big) = \begin{cases} s & \text{if } |s| \le k \\ k\dfrac{s}{|s|} & \text{if } |s| > k, \end{cases}$$

and its primitive is defined by

$$\widetilde{T}_k(s) = \int_0^s T_k(t)\,dt.$$

Note that $\widetilde{T}_k$ has the properties: $\widetilde{T}_k(s) \ge 0$ and $\widetilde{T}_k(s) \le k|s|$.

Lemma 2.3. (cf. [16, Lemma 2.6]) *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure. let M, P and Q be N-functions such that $Q \prec\prec P$, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$,*

$$|f(x,s)| \le c(x) + k_1 P^{-1} M(k_2|s|),$$

where k_1, k_2 are real constants and $c(x) \in E_Q(\Omega)$. Then the Nemytskii operator N_f, defined by $N_f(u)(x) = f(x,u(x))$, is strongly continuous from $P(E_M, \frac{1}{k_2}) = \{u \in L_M(\Omega) : d(u, E_M(\Omega)) < \frac{1}{k_2}\}$ into $E_Q(\Omega)$.

Lemma 2.4. [9, Lemma 6] *Let $u_k, u \in L_M(\Omega)$. If $u_k \to u$ for the modular convergence, then $u_k \to u$ for $\sigma(L_M, L_{\overline{M}})$.*

Lemma 2.5. [2, Lemma 1] *If $u_n \to u$ for the modular convergence (with every $\lambda > 0$) in $L_M(Q_T)$, then $u_n \to u$ strongly in $L_M(Q_T)$.*

Lemma 2.6. [14, Lemma 2.2] *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be a Orlicz function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Then, $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{a.e. in} \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & \text{a.e. in} \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

Lemma 2.7. [14, Lemma 2.3] *Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be a Orlicz function. we assume that the set of discontinuity points D of F' is finite, then the mapping $F : W^1L_M(\Omega) \to W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Lemma 2.8. [6, Lemma 5] *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \ge 2$, satisfying the segment property, then*

$$\Big\{u \in W_0^{1,x}L_{\overline{M}}(Q_T) : \frac{\partial u}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)\Big\} \subset \mathcal{C}\Big([0,T], L^1(\Omega)\Big).$$

Lemma 2.9. [14, Lemma 2.7] *(Integral Poincaré's type inequality in inhomogeneous Orlicz spaces) Let Ω be a bounded open subset of $\mathbb{R}^N$ and M is an Orlicz function, then there exists two positive constants $\delta, \lambda > 0$ such that*

$$\int_{Q_T} M(\delta|u(x,t)|)\,dx\,dt \leq \int_{Q_T} \lambda M(|\nabla u(x,t)|)\,dx\,dt \quad \forall u \in W_0^1 L_M(Q_T).$$

Lemma 2.10. (cf. [16, Lemma 2.5]) *If $f_n \subset L^1(\Omega)$ with $f_n \to f \in L^1(\Omega)$ a.e. in Ω, $f_n, f \geq 0$ a. e. in Ω and $\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx$, then $f_n \to f$ in $L^1(\Omega)$.*

Lemma 2.11. (cf. [16, Lemma 2.8]) *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_M(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathfrak{D}(\Omega)$ such that $u_n \to u$ for the modular convergence in $W_0^1 L_M(\Omega)$. Furthermore, if $u \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega)$ then*

$$\|u_n\|_\infty \leq \ (N+1)\|u\|_\infty.$$

Lemma 2.12. (cf. [14, Lemma 2.8]) *Let M be an N-function. Let (u_n) be a sequence of $W^{1,x}L_M(Q_T)$ such that, $u_n \rightharpoonup u$ weakly in $W^{1,x}L_M(Q_T)$ for $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\frac{\partial u_n}{\partial t} = h_n + k_n$ in $\mathfrak{D}'(Q_T)$ with h_n is bounded in $W^{-1,x}L_{\overline{M}}(Q_T)$ and k_n is bounded in $L^1(Q_T)$. Then, $u_n \to u$ strongly in $L^1_{Loc}(Q_T)$. If further, $u_n \in W_0^{1,x}L_M(Q_T)$ then $u_n \to u$ strongly in $L^1(Q_T)$.*

3. Basic assumptions and main result

Through this paper Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, satisfying the segment property and M is an Orlicz function. Let $A : D(A) \subset W_0^{1,x}L_M(Q_T) \to W^{-1,x}L_{\overline{M}}(Q_T)$ be an operator of Leray-Lions type of the form:

$$\mathrm{A}(u) := -\operatorname{div}\ a(x,t,u,\nabla u).$$

This work aims to prove existence of entropy solutions in Orlicz spaces for the nonlinear differential system

$$\begin{cases} \dfrac{\partial u_i}{\partial t} - \operatorname{div}\Big(a(x,t,u_i,\nabla u_i) + \Phi_i(x,t,u_i)\Big) = f_i(x,u_1,u_2) - \operatorname{div}(F_i) & \text{in } Q_T \\ u_i = 0 & \text{on } \Gamma \\ u_i(t=0) = u_{i,0} & \text{in } \Omega, \end{cases} \tag{9}$$

where $a : Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for almost every $(x,t) \in Q_T$ and for all $s \in \mathbb{R}, \xi, \eta \in \mathbb{R}^N (\xi \neq \eta)$ the following conditions:

(H_1) there exist a function $c(x,t) \in E_{\overline{M}}(Q_T)$ and some positive constants k_1, k_2, k_3 and an Orlicz function $P \prec\prec M$ such that

$$|a(x,t,s,\xi)| \leq \beta\Big[c(x,t) + k_1 \overline{M}^{-1}(P(k_2|s|)) + \overline{M}^{-1}(M(k_3|\xi|))\Big].$$

(H_2)

$$\Big(a(x,t,s,\xi)-a(x,t,s,\eta)\Big)\cdot\Big(\xi-\eta\Big)>0.$$

(H_3)

$$a(x,t,s,\xi)\cdot\xi\geq\alpha M(|\xi|).$$

For the lower order term, we assume $\Phi_i: Q_T\times\mathbb{R}\to\mathbb{R}^N$ be a Carathéodory function satisfying:

(H_4) for all $s\in\mathbb{R}$, $\delta>0$ and for almost every $x\in\Omega$,

$$|\Phi_i(x,t,s)|\leq\gamma(x,t)+\overline{M}^{-1}(M(\delta|s|))\text{ where }\gamma\in E_{\overline{M}}(Q_T).$$

Moreover, we suppose that for $i=1,2$, $F_i\in(E_{\overline{M}}(Q_T))^N$ and $f_i:\Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function with

$$f_1(x,0,s)=f_2(x,s,0)=0\quad\text{a.e. }x\in\Omega,\forall s\in\mathbb{R}\tag{10}$$

and for almost every $x\in\Omega$, for every $s_1,s_2\in\mathbb{R}$,

$$sign(s_i)f_i(x,s_1,s_2)\geq 0.\tag{11}$$

Finally, we assume the following condition on the initial data $u_{i,0}$:

$$u_{i,0}\text{ is a measurable function for }i=1,2.\tag{12}$$

Lemma 3.1. (cf. [14, Lemma 3.1]) *Under assumptions (H_1)-(H_3), let (Z_n) be a sequence in $W_0^{1,x}L_M(Q_T)$ such that*

$$Z_n\rightharpoonup Z\quad\text{in }W_0^{1,x}L_M(Q_T)\text{ for }\sigma(\Pi L_M(Q_T),\Pi E_{\overline{M}}(Q_T)),\tag{13}$$

$$\Big(a(x,t,Z_n,\nabla Z_n)\Big)_n\quad\text{is bounded in }\Big(L_{\overline{M}}(Q_T)\Big)^N,\tag{14}$$

$$\lim_{n,s\to\infty}\int_{Q_T}\Big(a(x,t,Z_n,\nabla Z_n)-a(x,t,Z_n,\nabla Z\chi_s)\Big)\cdot\Big(\nabla Z_n-\nabla Z\chi_s\Big)dxdt=0,\tag{15}$$

where χ_s denote the characteristic function of the set $\Omega_s=\Big\{x\in\Omega:|\nabla Z|\leq s\Big\}$. Then,

$$\nabla Z_n\to\nabla Z\quad\text{a.e. in }Q_T,\tag{16}$$

$$\lim_{n\to\infty}\int_{Q_T}a(x,t,Z_n,\nabla Z_n)\nabla Z_n\,dx=\int_{Q_T}a(x,t,Z,\nabla Z)\nabla Z\,dxdt,\tag{17}$$

$$M(|\nabla Z_n|)\longrightarrow M(|\nabla Z|)\quad\text{in }L^1(Q_T).\tag{18}$$

Definition 3.2. A couple of measurable functions (u_1, u_2) defined on Q_T is said an entropy solution for system (9) if for $i = 1, 2$, $u_i \in L^\infty(0, T, L^1(\Omega))$, $T_k(u_i) \in D(A) \cap W_0^{1,x} L_M(Q_T)$, $\forall k > 0$ and $\widetilde{T}_k(u_i(\cdot, t)) \in L^1(Q_T)$ for every $\tau \in [0, T]$, we have

$$\begin{aligned} \int_\Omega \widetilde{T}_k(u_i - v)\, dx &+ \Big\langle \frac{\partial v}{\partial t}, T_k(u_i - v) \Big\rangle_{Q_\tau} \\ &+ \int_{Q_\tau} a(x, t, u_i, \nabla u_i) \nabla T_k(u_i - v))\, dx\, dt \\ &\quad + \int_{Q_\tau} \Phi_i(x, t, u_i) \nabla T_k(u_i - v))\, dx\, dt \\ &\quad\quad \le \int_{Q_\tau} f_i T_k(u_i - v))\, dx\, dt + \int_{Q_\tau} F_i \nabla T_k(u_i - v))\, dx\, dt \\ &\quad\quad\quad + \int_\Omega \widetilde{T}_k(u_{i,0} - v(0))\, dx, \end{aligned} \tag{19}$$

and

$$u_i(x, 0) = u_{i,0}(x) \text{ for a.e } x \in \Omega, \tag{20}$$

for every $\tau \in [0, T]$, $k > 0$ and for all $v \in W_0^{1,x} L_M(Q_T) \cap L^\infty(Q_T)$ such that $\dfrac{\partial v}{\partial t} \in W^{-1,x} L_{\overline{M}}(Q_T) + L^1(Q_T)$, where $\widetilde{T}_k$ is the primitive function of the truncation function T_k defined above.

***Remark* 3.3.** Equation (19) is formally obtained by multiplication of the problem (9) by $T_k(u_i - v)$. Notice that each term in (19) has a sense since $T_k(u_i - v) \in W_0^{1,x} L_M(Q_T) \cap L^\infty(Q_T)$. Moreover by Lemma 2.8, we have $v \in C\Big([0, T]; L^1(\Omega)\Big)$ and then the first and the last terms of (19) are well defined.

The following theorem is our main result.

Theorem 3.4. *Suppose that the assumptions* $(H_1) - (H_4)$ *and* (10), (11) *and* (12) *hold true, then there exists at least one solution* (u_1, u_2) *for the parabolic system* (9) *in sense of Definition* (19).

The proof of the above theorem is divided into four steps.

Step 1: Approximate problems. For each $n \in \mathbb{N}^*$, put

$$a_n(x, t, s, \xi) = a(x, t, T_n(s), \xi) \text{ a.e } (x, t) \in Q_T, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N$$

and

$$\Phi_{i,n}(x, t, s) = \Phi_i(x, t, T_n(s)) \text{ a.e } (x, t) \in \Omega_T, \forall s \in \mathbb{R},$$

$$f_{1,n}(x, s_1, s_2) = f_1(x, T_n(s_1), s_2) \text{ a.e in } \Omega, \forall s_1, s_2 \in \mathbb{R}, \tag{21}$$

$$f_{2,n}(x, s_1, s_2) = f_2(x, s_1, T_n(s_2)) \text{ a.e in } \Omega, \forall s_1, s_2 \in \mathbb{R}. \tag{22}$$

Let $u_{i,0n} \in \mathcal{C}_0^\infty(\Omega)$ be such that

$$\| u_{i,0n} \|_{L^1} \leq \| u_{i,0} \|_{L^1} \text{ and } u_{i,0n} \longrightarrow u_{i,0} \text{ in } L^1(\Omega).$$

Considering the following approximate problem

$$\begin{cases} \frac{\partial u_{i,n}}{\partial t} - \operatorname{div}\Big(a(x,t,u_{i,n},\nabla u_i) + \Phi_{i,n}(x,t,u_{i,n})\Big) = f_{i,n}(x,u_1,u_2) - \operatorname{div}(F_{i,n}) & \text{in } Q_T \\ u_{i,n} = 0 & \text{on } \Gamma \\ u_{i,n}(t=0) = u_{i,0n} & \text{in } \Omega. \end{cases} \tag{23}$$

Let $z_n(x,t,u_{i,n},\nabla u_n) = a_n(x,t,u_{i,n},\nabla u_{i,n}) + \Phi_{i,n}(x,t,u_{i,n}) + F_{i,n}$, which satisfies (A_1), (A_2), (A_3) and (A_4) of [10]. Indeed, it remains just to prove (A_4), to do this we use Young's inequality as follows:

$$\begin{aligned} |\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}| &\leq |\gamma(x,t)||\nabla u_{i,n}| + \overline{M}^{-1}(M(\delta|T_n(u_{i,n})|))|\nabla u_{i,n}| \\ &= \frac{\alpha^2}{\alpha+2}\frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla u_{i,n}| \\ &\quad + \frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(\delta|T_n(u_{i,n})|))\frac{\alpha}{\alpha+1}|\nabla u_{i,n}| \\ &\leq \frac{\alpha^2}{\alpha+2}\Big(\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + M\Big(|\nabla u_{i,n}|\Big)\Big) \\ &\quad + \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(\delta|T_n(u_{i,n})|))\Big) + M\Big(\frac{\alpha}{\alpha+1}|\nabla u_{i,n}|\Big). \end{aligned}$$

While $\dfrac{\alpha}{\alpha+1} < 1$, using the convexity of M and the fact that $\overline{M}$ and $\overline{M}^{-1} \circ M$ are increasing functions, one has

$$\begin{aligned} |\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}| \leq \frac{\alpha^2}{\alpha+2}\overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) + \frac{\alpha^2}{\alpha+2}M\Big(|\nabla u_{i,n}|\Big) \\ + \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(\delta n))\Big) + \frac{\alpha}{\alpha+1}M\Big(|\nabla u_{i,n}|\Big). \end{aligned}$$

Since $\gamma \in E_{\overline{M}}(Q_T)$, $\overline{M}\Big(\dfrac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big) \in L^1(\Omega)$, then we get

$$\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} \geq -\Big(\frac{\alpha^2}{\alpha+2} + \frac{\alpha}{\alpha+1}\Big)M\Big(|\nabla u_{i,n}|\Big) - C_n - \text{ fixed } L^1 function. \tag{24}$$

Moreover, by the same technic we have using Young's inequality

$$\begin{aligned} |F_{i,n}\nabla u_{i,n}| &= \frac{\alpha^2}{(\alpha+2)^2}\frac{(\alpha+2)^2}{\alpha^2}|F_{i,n}\nabla u_{i,n}| \\ &\leq \frac{\alpha^2}{(\alpha+2)^2}\overline{M}\Big(\frac{(\alpha+2)^2}{\alpha^2}|F_{i,n}|\Big) + \frac{\alpha^2}{(\alpha+2)^2}M\Big(|\nabla u_{i,n}|\Big)\, dx\, dt \\ &\leq C_F + \frac{\alpha^2}{(\alpha+2)^2}M\Big(|\nabla u_{i,n}|\Big) \text{ since } F_i \in (E_{\overline{M}}(Q_T))^N. \end{aligned}$$

Then,

$$F_{i,n}\nabla u_{i,n} \geq -\frac{\alpha^2}{(\alpha+2)^2} M\Big(|\nabla u_{i,n}|\Big) - C_F. \tag{25}$$

Using (24) and (25) and (H_3) we obtain

$$\begin{aligned} z_n(x,t,u_{i,n},\nabla u_{i,n})\nabla u_{i,n} &\geq \Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1} - \frac{\alpha^2}{(\alpha+2)^2}\Big) M\Big(|\nabla u_{i,n}|\Big) \\ &\qquad - C_n - \text{ fixed } L^1 function \\ &\geq \frac{\alpha^3+2\alpha^2}{(\alpha+1)(\alpha+2)^3} M\Big(|\nabla u_{i,n}|\Big) - \text{ fixed } L^1 function. \end{aligned}$$

Thus, from [6, Theorem 4], the approximate problem (23) has at least one weak solution $u_{i,n} \in W_0^{1,x} L_M(Q_T)$.

Step 2: A Priori Estimates.

Proposition 3.5. *Assume that the hypothesis $(H_1) - (H_4)$ hold true and let $u_{i,n}$ be a solution of the approximate problem (23). Then, for all $k > 0$, there exists a constant C_k (not depending on n), such that:*

$$\| T_k(u_{i,n}) \|_{W_0^{1,x} L_M(Q_T)} \leq C_k \tag{26}$$

and

$$\lim_{k\to\infty} \text{meas}\Big\{(x,t) \in Q_T : |u_{i,n}| > k\Big\} = 0. \tag{27}$$

Proof. Use $T_k(u_{i,n})\chi_{(0,\sigma)}$ as test function in the approximate problem (23), one has for every $\sigma \in (0,T)$

$$\begin{aligned} &\int_\Omega \widetilde{T}_k(u_{i,n})(\sigma)\,dx - \int_\Omega \widetilde{T}_k(u_{i,0n})\,dx + \int_{Q_\sigma} a_n(x,t,u_{i,n},\nabla u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt \\ &\quad + \int_{Q_\sigma} \Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt = \int_{Q_\sigma} f_{i,n} T_k(u_{i,n})\,dx\,dt \\ &\qquad + \int_{Q_\sigma} F_{i,n}\nabla T_k(u_{i,n})\,dx\,dt. \end{aligned} \tag{28}$$

Notice that $\Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})$ is different from zero only on the set $\{|u_{i,n}| \leq k\}$ where $T_k(u_{i,n}) = u_{i,n}$. From (H_4) and then Young's inequality for an arbitrary $\alpha > 0$ (the constant of coercivity), we have

$$\begin{aligned}
&\int_{Q_\sigma} \Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt \\
&\leq \int_{Q_\sigma} |\gamma(x,t)||\nabla T_k(u_n)|\,dx\,dt \\
&\quad + \int_{Q_\sigma} \overline{M}^{-1}(M(\delta|T_k(u_{i,n})|))|\nabla T_k(u_{i,n})|\,dx\,dt \\
&= \frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} \frac{\alpha+2}{\alpha^2}|\gamma(x,t)||\nabla T_k(u_{i,n})|\,dx\,dt \\
&\quad + \int_{Q_\sigma} \frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(\delta|T_k(u_{i,n})|))\frac{\alpha}{\alpha+1}|\nabla T_k(u_{i,n})|\,dx\,dt \\
&\leq \frac{\alpha^2}{\alpha+2}\Big(\int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx + \int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt\Big) \\
&\quad + \int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(\delta|T_k(u_{i,n})|))\Big)\,dx\,dt \\
&\quad\quad + \int_{Q_\sigma} M\Big(\frac{\alpha}{\alpha+1}|\nabla T_k(u_{i,n})|\Big)\,dx\,dt.
\end{aligned}$$

Since $\gamma \in E_{\overline{M}}(Q_\sigma)$, then $\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+2}{\alpha^2}|\gamma(x,t)|\Big)\,dx\,dt = \gamma_0 < +\infty$ and while $\frac{\alpha}{\alpha+1} < 1$, using the convexity of M and the fact that $\overline{M}$ and $\overline{M}^{-1} \circ M$ are increasing functions, then we get

$$\begin{aligned}
\int_{Q_\sigma} \Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})dx\,dt \leq \gamma_0 + &\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)dx\,dt \\
&+ \int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(\delta k))\Big)dx\,dt \\
&+ \frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)dx\,dt. \quad (29)
\end{aligned}$$

Using (7), there exists some constant C_k^α such that

$$\int_{Q_\sigma} \overline{M}\Big(\frac{\alpha+1}{\alpha}\overline{M}^{-1}(M(\delta k))\Big)\,dx\,dt \leq \int_{Q_\sigma} \overline{M}\Big(2\frac{\alpha+1}{\alpha\delta k}M(\delta k)\Big)\,dx\,dt = C_k^\alpha.$$

Which gives the estimate

$$\begin{aligned}
\int_{Q_\sigma} \Phi_{i,n}(x,t,u_{i,n})\nabla T_k(u_{i,n})\,dx\,dt \leq \gamma_0 + &\frac{\alpha^2}{\alpha+2}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt \\
&+ C_k^\alpha + \frac{\alpha}{\alpha+1}\int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\,dx\,dt. \quad (30)
\end{aligned}$$

On the other hand, due to (11), we have

$$\int_{Q_\sigma} f_{i,n}(x, u_{1,n}, u_{2,n}) T_k(u_{i,n})\, dx\, dt \geq 0. \tag{31}$$

Concerning the first integral in (28), since $\widetilde{T}_k \geq 0$, we have

$$\int_\Omega \widetilde{T}_k(u_{i,n})(\sigma)\, dx \geq 0 \tag{32}$$

and we have

$$\int_\Omega \widetilde{T}_k(u_{i,0n})\, dx \leq k \int_\Omega |u_{i,0n}|\, dx \leq k\|u_{i,0}\|_{L^1(\Omega)}. \tag{33}$$

For the remaining integral, we proceed by Young's inequality as follows, there exist a constant C_F:

$$\begin{aligned}
&\int_{Q_\sigma} F_{i,n} \nabla T_k(u_{i,n})\, dx\, dt \\
&= \frac{\alpha^2}{(\alpha+2)^2} \int_{Q_\sigma} \frac{(\alpha+2)^2}{\alpha^2} F_{i,n} \nabla T_k(u_{i,n})\, dx\, dt \\
&\leq \frac{\alpha^2}{(\alpha+2)^2} \int_{Q_\sigma} \overline{M}\Big(\frac{(\alpha+2)^2}{\alpha^2} F_{i,n}\, dx\, dt + \frac{\alpha^2}{(\alpha+2)^2} \int_{Q_\sigma} M\Big(\nabla T_k(u_{i,n})\Big)\, dx\, dt \\
&\leq C_F + \frac{\alpha^2}{(\alpha+2)^2} \int_{Q_\sigma} M\Big(\nabla T_k(u_{i,n})\Big)\, dx\, dt \text{ since } F_i \in (E_{\overline{M}}(Q_T))^N.
\end{aligned} \tag{34}$$

Combining (28), (30), (31), (32), (33) and (34) we get

$$\begin{aligned}
&\int_{Q_\sigma} a(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n})) \nabla T_k(u_{i,n})\, dx\, dt \\
&\leq \gamma_0 + k\overline{C} + C_k^\alpha + \frac{\alpha^2}{\alpha+2} \int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\, dx\, dt \\
&\quad + \frac{\alpha}{\alpha+1} \int_{Q_\sigma} M\Big(|\nabla T_k(u_{i,n})|\Big)\, dx\, dt + \frac{\alpha^2}{(\alpha+2)^2} \int_{Q_\sigma} M\Big(\nabla T_k(u_{i,n})\Big)\, dx\, dt,
\end{aligned} \tag{35}$$

where $\overline{C} = \|f_i\|_{L^1(\Omega)} + \|u_{i,0}\|_{L^1(\Omega)}$. Thanks to (H_3), we deduce

$$\int_{Q_\sigma} \Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1} - \frac{\alpha^2}{(\alpha+2)^2}\Big) M\Big(|\nabla T_k(u_{i,n})|\Big)\, dx\, dt \leq \gamma_0 + k\overline{C} + C_k^\alpha + C_F. \tag{36}$$

Since $\Big(\alpha - \frac{\alpha^2}{\alpha+2} - \frac{\alpha}{\alpha+1} - \frac{\alpha^2}{(\alpha+2)^2}\Big) = \frac{\alpha^3 + 2\alpha^2}{(\alpha+1)(\alpha+2)^3} > 0$, finally we have

$$\int_{Q_T} M\Big(|\nabla T_k(u_{i,n})|\Big)\, dx\, dt \leq (\gamma_0 + k\overline{C} + C_k^\alpha + C_F) \frac{(\alpha+1)(\alpha+2)^3}{\alpha^3 + 2\alpha^2} = C_k. \tag{37}$$

Now we prove (27), to this end, we use the integral Poincaré's type inequality in inhomogeneous Orlicz spaces with the constant δ and λ. Hence,

$$\begin{aligned} M\Big(\delta k\Big)\,\text{meas}\Big\{|u_{i,n}|>k\Big\} &= \int_{\{|u_{i,n}|>k\}} M\Big(\delta|T_k(u_{i,n})|\Big)\,dx\,dt \\ &\leq \lambda \int_{Q_T} M\Big(|\nabla T_k(u_{i,n}|)\Big)\,dx\,dt. \end{aligned}$$

Then, from (37) we get

$$\text{meas}\Big\{|u_{i,n}|>k\Big\} \leq \frac{\lambda C_k}{M\Big(\delta k\Big)} \quad \longrightarrow 0 \quad \text{as } k \longrightarrow \infty$$

which implies (27). ☑

Lemma 3.6. *Let $u_{i,n}$ be a solution of the approximate problem* (23)*, then:*

(i) $u_{i,n} \longrightarrow u_i$ a.e. in Q_T,

(ii) $\{a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\}_n$ *is bounded in* $(L_{\overline{M}}(Q_T))^N$.

Proof. To prove **(i)**, we proceed as in [14, 20], we take a $C^2(\mathbb{R})$ nondecreasing function Γ_k such that $\Gamma_k(s) = \begin{cases} s & \text{for } |s| \leq \dfrac{k}{2} \\ k & \text{for } |s| \geq k \end{cases}$ and multiplying the approximate problem (23) by $\Gamma'_k(u_{i,n})$ we obtain

$$\begin{aligned} &\frac{\partial \Gamma_k(u_{i,n})}{\partial t} - \text{div}\Big(a(x,t,u_{i,n},\nabla u_{i,n})\Gamma'_k(u_{i,n})\Big) + a(x,t,u_{i,n},\nabla u_{i,n})\Gamma''_k(u_{i,n})\nabla u_{i,n} \\ &\quad - \text{div}\Big(\Gamma'_k(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\Big) + \Gamma''_k(u_{i,n})\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} \\ &= f_{i,n}\Gamma'_k(u_n) - \text{div}\Big(\Gamma'_k(u_{i,n})F_{i,n}\Big) + \Gamma''_k(u_{i,n})F_{i,n}\nabla u_{i,n}. \end{aligned} \tag{38}$$

Since $\overline{M}^{-1} \circ M$ is an increasing function, $\gamma \in E_{\overline{M}}(Q_T)$, $supp(\Gamma'_k), supp(\Gamma''_k) \subset [-k,k]$ and using Young's inequality we get

$$\begin{aligned} &\Big|\int_{Q_T} \Gamma'_k \Phi_{i,n}(x,t,u_{i,n})\,dx\,dt\Big| \\ &\leq \|\Gamma'_k\|_{L^\infty}\Big(\int_{Q_T} |\gamma(x,t)|\,dx\,dt + \int_{Q_T} \overline{M}^{-1}(M(\delta|T_k(u_{i,n})|))\,dx\,dt\Big) \\ &\leq \|\Gamma'_k\|_{L^\infty}\Big(\int_{Q_T} \Big(\overline{M}(|\gamma(x,t)|) + M(1)\Big)\,dx\,dt + \int_{Q_T} \overline{M}^{-1}(M(\delta k))\,dx\,dt\Big) \\ &< C_{1,k} \end{aligned} \tag{39}$$

and (here, we use also (37))

$$\begin{aligned}
&\Big|\int_{Q_T}\Gamma_k''\Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n}dx\,dt\Big|\\
&\leq \|\Gamma_k''\|_{L^\infty}\Big(\int_{Q_T}|\gamma(x,t)|dx\,dt+\int_{Q_T}\overline{M}^{-1}(M(\delta|T_k(u_{i,n})|))|\nabla T_k(u_{i,n})|dx\,dt\Big)\\
&\leq \|\Gamma_k''\|_{L^\infty}\Big[\int_{Q_T}\Big(\overline{M}(|\gamma(x,t)|)+M(1)\Big)dx\,dt+\int_{Q_T}M(\delta k)dx\,dt\\
&\qquad\qquad+\int_{Q_T}M(|\nabla T_k(u_{i,n})|)dx\,dt\Big]\\
&< C_{2,k},
\end{aligned}\tag{40}$$

where $C_{1,k}$ and $C_{2,k}$ are two positive constants independent of n. Also, by Young's inequality and (37) we can deduce that $\int_{Q_T}\Gamma_k' F_{i,n}dx\ dt$ and $\int_{Q_T}\Gamma_k'' F_{i,n}\nabla u_{i,n}\,dx\,dt$ are bounded. Then (38), (39) and (40) imply that

$$\frac{\partial\Gamma_k(u_{i,n})}{\partial t}\text{ is bounded in }L^1(Q_T)+W^{-1,x}L_{\overline{M}}(Q_T).\tag{41}$$

Hence by Lemma 2.12 and using the same technics as in [18], we can deduce that there exists a measurable function $u_i\in L^\infty(0,T;L^1(\Omega))$ such that

$$u_{i,n}\longrightarrow u\text{ a.e. in }Q_T$$

and for every $k>0$,

$$T_k(u_{i,n})\rightharpoonup T_k(u_i)\text{ weakly in }W^{1,x}L_M(Q_T)\text{ for }\sigma(\Pi L_M,\Pi E_{\overline{M}}),\tag{42}$$

and

$$T_k(u_{i,n})\to T_k(u_i)\text{ strongly in }L^1(Q_T)\text{ and a.e. in }Q_T.\tag{43}$$

For **(ii)**, we use the Banach-Steinhaus theorem. Let $\phi\in(E_M(Q_T))^N$ be an arbitrary function. From (H_2) we can write

$$\Big(a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))-a(x,t,T_k(u_{i,n}),\phi)\Big)\cdot\Big(\nabla T_k(u_{i,n})-\phi\Big)\geq 0$$

which gives:

$$\begin{aligned}
&\int_{Q_T}a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\phi\,dx\\
&\leq\int_{Q_T}a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n})\,dx\\
&\quad+\int_{Q_T}a(x,t,T_k(u_{i,n}),\phi)(\phi-\nabla T_k(u_{i,n}))\,dx.
\end{aligned}\tag{44}$$

Let us denote by J_1 and J_2 the first and the second integral respectively in the right hand-side of (44), so that

$$J_1 = \int_{Q_T} a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n}))\nabla T_k(u_{i,n})\, dx.$$

Going back to (35), we obtain

$$\begin{aligned} J_1 \le \gamma_0 + k\overline{C} + C_k^{\alpha} + C_F + \frac{\alpha^2}{\alpha+2}\int_{Q_T} M\Big(|\nabla T_k(u_{i,n})|\Big)\, dx\, dt \\ + \frac{\alpha}{\alpha+1}\int_{Q_T} M\Big(|\nabla T_k(u_{i,n})|\Big)\, dx\, dt + \frac{\alpha^2}{(\alpha+2)^2}\int_{Q_T} M\Big(|\nabla T_k(u_{i,n})|\Big)\, dx\, dt, \end{aligned} \tag{45}$$

And thanks to (37), there exists a positive constant C_{J_1} independent of n such that

$$J_1 \le C_{J_1}. \tag{46}$$

Now we estimate the integral J_2. To this end, notice that

$$\begin{aligned} J_2 &= \int_{Q_T} a(x,t,T_k(u_{i,n}),\phi)(\phi - \nabla T_k(u_{i,n}))\, dx\, dt \\ &\le \int_{Q_T} |a(x,t,T_k(u_{i,n}),\phi)||\phi|\, dx\, dt + \int_{Q_T} |a(x,t,T_k(u_{i,n}),\phi)||\nabla T_k(u_{i,n})|\, dx\, dt. \end{aligned}$$

On the other hand, let η be large enough, from (H_1) and the convexity of $\overline{M}$, we get:

$$\begin{aligned} &\int_{Q_T} \overline{M}\Big(\frac{|a(x,t,T_k(u_{i,n}),\phi)|}{\eta}\Big)\, dx\, dt \\ &\le \int_{Q_T} \overline{M}\Big(\frac{\beta\Big[c(x,t) + k_1\overline{M}^{-1}(P(k_2|T_k(u_{i,n})|) + \overline{M}^{-1}(M(k_3|\phi|))\Big]}{\eta}\Big)\, dx\, dt \\ &\le \frac{\beta}{\eta}\int_{Q_T} \overline{M}(c(x,t))\, dx\, dt + \frac{\beta k_1}{\eta}\int_{Q_T} \overline{M}\Big(\overline{M}^{-1}(P(k_2|T_k(u_{i,n})|))\Big)\, dx\, dt \\ &\quad + \frac{\beta}{\eta}\int_{Q_T} \overline{M}\Big(\overline{M}^{-1}(M(k_3|\phi|))\Big)\, dx\, dt \\ &\le \frac{\beta}{\eta}\int_{Q_T} \overline{M}(c(x,t))\, dx\, dt + \frac{\beta k_1}{\eta}\int_{Q_T} P(k_2 k)\, dx\, dt \\ &\quad + \frac{\beta}{\eta}\int_{Q_T} M(k_3|\phi|)\, dx\, dt. \end{aligned} \tag{47}$$

Since $\phi \in (E_M(Q_T))^N$, $c(x,t) \in E_{\overline{M}}(Q_T)$, we deduce that $\{a(x,t,T_k(u_{i,n}),\phi)\}$ is bounded in $(L_{\overline{M}}(Q_T))^N$ and we have $\{\nabla T_k(u_{i,n})\}$ is bounded in $(L_M(Q_T))^N$, consequently, $J_2 \leq C_{J_2}$, where C_{J_2} is a positive constant not depending on n. Then we obtain

$$\int_{Q_T} a(x, T_k(u_{i,n}), \nabla T_k(u_{i,n}))\phi \, dx \, dt \leq C_{J_1} + C_{J_2}. \quad \text{for all } \phi \in (E_M(Q_T))^N. \tag{48}$$

Finally, $\{a(x,t,T_k(u_{i,n}), \nabla T_k(u_{i,n}))\}_n$ is bounded in $(L_{\overline{M}}(Q_T))^N$. ☑

Step 3: Almost everywhere convergence of the gradients. In this step, most parts of the proof of the following proposition are the same argument as in [14, Proposition 5.4], we give just those which are different.

Proposition 3.7. *Let $u_{i,n}$ be a solution of the approximate problem* (23). *Then, for all $k \geq 0$ we have (for a subsequence still denoted by $u_{i,n}$): as $n \to +\infty$,*

(i) $\nabla u_{i,n} \to \nabla u_i$ *a.e. in* Q_T,

(ii) $a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) \rightharpoonup a(x,t,T_k(u_i),\nabla T_k(u_i))$ *weakly in* $(L_{\overline{M}}(Q_T))^N$,

(iii) $M(|\nabla T_k(u_{i,n})|) \to M(|\nabla T_k(u_i)|)$ *strongly in* $L^1(Q_T)$.

Proof. Let $\theta_j \in \mathfrak{D}(Q_T)$ be a sequence such that $\theta_j \longrightarrow u$ in $W_0^{1,x}L_M(Q_T)$ for the modular convergence and let $\psi_i \in \mathfrak{D}(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Put $Z^{\mu}_{i,j} = T_k(\theta_j)_\mu + \mathrm{e}^{-\mu t}\, T_k(\psi_i)$ where $T_k(\theta_j)_\mu$ is the mollification with respect to time of $T_k(\theta_j)$, notice that $Z^{i}_{\mu,j}$ is a smooth function having the following properties:

$$\frac{\partial Z^{\mu}_{i,j}}{dt} = \mu(T_k(\theta_j) - Z^{\mu}_{i,j}), \quad Z^{\mu}_{i,j}(0) = T_k(\psi_i) \quad \text{and } |Z^{\mu}_{i,j}| \leq k,$$

$$Z^{\mu}_{i,j} \longrightarrow T_k(u)_\mu + \mathrm{e}^{-\mu t}\, T_k(\psi_i), \quad \text{in } W_0^{1,x}L_M(Q_T) \quad \text{modularly as } j \longrightarrow \infty,$$

$$T_K(u)_\mu + \mathrm{e}^{-\mu t}\, T_k(\psi_i) \longrightarrow T_k(u), \quad \text{in } W_0^{1,x}L_M(Q_T) \quad \text{modularly as } \mu \longrightarrow \infty.$$

Consider the function h_m defined on $\mathbb{R}$ for any $m \geq k$ by:

$$h_m(r) = \begin{cases} 1 & \text{if } |r| \leq m \\ -|r| + m + 1 & \text{if } m \leq |r| \leq m+1 \\ 0 & \text{if } |r| \geq m+1. \end{cases}$$

Put $E_m = \Big\{(x,t) \in Q_T : m \leq |u_{i,n}| \leq m+1\Big\}$ and testing the approximate problem (23) by the test function $\varphi^{\mu,i}_{n,j,m} = (T_k(u_{i,n}) - Z^{\mu}_{i,j})h_m(u_{i,n})$, we get

$$\begin{aligned}
&\Big\langle \frac{\partial u_{i,n}}{dt}, \varphi^{\mu,i}_{n,j,m}\Big\rangle + \int_{Q_T} a(x,t,u_{i,n},\nabla u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j})h_m(u_{i,n})\,dx\,dt \\
&\quad + \int_{Q_T} a(x,t,u_{i,n},\nabla u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\nabla u_{i,n} h'_m(u_{i,n})\,dx\,dt \\
&\quad + \int_{E_m} \Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} h'_m(u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\,dx\,dt \\
&\quad + \int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} h_m(u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j})\,dx\,dt \\
&\quad = \int_{Q_T} f_{i,n}\varphi^{\mu,i}_{n,j,m}\,dx\,dt + \int_{E_m} F_{i,n}\nabla u_{i,n} h'_m(u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\,dx\,dt \\
&\quad + \int_{Q_T} F_{i,n}\nabla u_{i,n} h_m(u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j})\,dx\,dt.
\end{aligned} \tag{49}$$

In order to simplify the notation, we will denote by $\epsilon(n,j,\mu,i)$ and $\epsilon(n,j,\mu)$ any quantities such that

$$\lim_{i\to+\infty}\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,\mu,i) = 0,$$

$$\lim_{\mu\to+\infty}\lim_{j\to+\infty}\lim_{n\to+\infty}\epsilon(n,j,\mu) = 0.$$

We have the following lemma which can be found in [14, Lemma 5.5].

Lemma 3.8. (cf. [14, Lemma 5.5]) *Let* $\varphi^{\mu,i}_{n,j,m} = (T_k(u_{i,n}) - Z^{\mu}_{i,j})h_m(u_{i,n})$, *then for any* $k \geq 0$ *we have:*

$$\Big\langle \frac{\partial u_{i,n}}{dt}, \varphi^{\mu,i}_{n,j,m}\Big\rangle \geq \epsilon(n,j,\mu,i), \tag{50}$$

where $<,>$ *denotes the duality pairing between* $L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T)$ *and* $L^\infty(Q_T) \cap W^{1,x}_0 L_M(Q_T)$.

To complete the proof of proposition 3.7, we establish the results below, for any fixed $k \geq 0$, we have:

(r_1) $\displaystyle\int_{Q_T} f_{i,n}\varphi^{\mu,i}_{n,j,m}dx\,dt = \epsilon(n,j,\mu)$.

(r_2) $\displaystyle\int_{Q_T} \Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} h_m(u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j})dx\,dt = \epsilon(n,j,\mu)$.

(r_3) $\displaystyle\int_{E_m} \Phi_{i,n}(x,t,u_{i,n})\nabla u_{i,n} h'_m(u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})dx\,dt = \epsilon(n,j,\mu)$.

(r_4) $\displaystyle\int_{Q_T} a(x,t,u_{i,n},\nabla u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j})\nabla u_{i,n} h'_m(u_{i,n}) dx\, dt \le \epsilon(n,j,\mu,m).$

(r_5) $\displaystyle\int_{Q_T} [a(x,t,T_k(u_{i,n}),\nabla T_k(u_{i,n})) - a(x,t,T_k(u_{i,n}),\nabla T_k(u_i)\chi_s)]$
$\times[\nabla T_k(u_{i,n}) - \nabla T_k(u_i)\chi_s] dx\, dt \le \epsilon(n,j,\mu,m,s).$

(r_6) $\displaystyle\int_{E_m} F_{i,n}\nabla u_{i,n} h'_m(u_{i,n})(T_k(u_{i,n}) - Z^{\mu}_{i,j}) dx\, dt = \epsilon(n,j,\mu).$

(r_7) $\displaystyle\int_{Q_T} F_{i,n}\nabla u_{i,n} h_m(u_{i,n})(\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j}) dx\, dt = \epsilon(n,j,\mu).$

The proof of (r_1), (r_3), (r_4) and (r_5) is the same as in [14, Proposition 5.4]).

We now prove (r_2). To this end, for $n \ge m+1$, we have

$$\Phi_{i,n}(x,t,u_{i,n})h_m(u_{i,n}) = \Phi_i(x,t,T_{m+1}(u_{i,n}))h_m(T_{m+1}(u_{i,n})) \text{ a.e in } Q_T.$$

put $P_{i,n} = \overline{M}\Big(\frac{|\Phi_i(x,t,T_{m+1}(u_{i,n})) - \Phi_i(x,t,T_{m+1}(u_i))|}{\eta}\Big)$. Since Φ_i is continuous with respect to its third argument and $u_{i,n} \longrightarrow u_i$ a.e in Q_T, then $\Phi_i(x,t,T_{m+1}(u_{i,n})) \to \Phi_i(x,t,T_{m+1}(u_i))$ a.e in Ω as n goes to infinity, besides $\overline{M}(0) = 0$, it follows

$$P_{i,n} \longrightarrow 0, \quad \text{a.e in } \Omega \text{ as } n \to \infty. \tag{51}$$

Using now the convexity of $\overline{M}$ and (H_4), we have for every $\eta > 0$ and $n \ge m+1$:

$$\begin{aligned}
P_{i,n} &= \overline{M}\Big(\frac{|\Phi_i(x,t,T_{m+1}(u_{i,n})) - \Phi_i(x,t,T_{m+1}(u_i))|}{\eta}\Big)\\
&\le \overline{M}\Big(\frac{2\gamma(x,t) + \overline{M}^{-1}(M(\delta|T_{m+1}(u_{i,n})|)) + \overline{M}^{-1}(M(\delta|T_{m+1}(u_i)|))}{\eta}\Big)\\
&\le \overline{M}\Big(\frac{2}{\eta}|\gamma(x,t)| + \frac{2}{\eta}\overline{M}^{-1}(M(\delta(m+1)))\Big)\\
&= \overline{M}\Big(\frac{1}{2}\frac{4}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{4}{\eta}\overline{M}^{-1}(M(\delta(m+1)))\Big)\\
&\le \frac{1}{2}\overline{M}(\frac{4}{\eta}|\gamma(x,t)|) + \frac{1}{2}\overline{M}(\frac{4}{\eta}\overline{M}^{-1}(M(\delta(m+1)))).
\end{aligned} \tag{52}$$

We put $C^{\eta}_m(x,t) = \frac{1}{2}\overline{M}(\frac{4}{\eta}|\gamma(x,t)|) + \frac{1}{2}\overline{M}(\frac{4}{\eta}\overline{M}^{-1}(M(\delta(m+1))))$. Since $\gamma \in E_{\overline{M}}(Q_T)$, we have $C^{\eta}_m \in L^1(Q_T)$, Then by Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty}\int_{Q_T} P_{i,n}\, dx\, dt = \int_{Q_T}\lim_{n\to\infty} P_{i,n}\, dx\, dt = 0. \tag{53}$$

This implies that $\{\Phi_i(x,t,T_{m+1}(u_{i,n}))\}$ converges modularly to $\Phi_i(x,t,T_{m+1}(u_i))$ as $n \to \infty$ in $(L_{\overline{M}}(Q_T))^N$. Moreover, $\Phi_i(x,t,T_{m+1}(u_{i,n}))$, $\Phi_i(x,t,T_{m+1}(u_i))$ lie in $(E_{\overline{M}}(Q_T))^N$, indeed, from (H_4) we have for every $\eta > 0$

$$
\begin{aligned}
&\int_{Q_T} \overline{M}\Big(\frac{|\Phi_i(x,t,T_{m+1}(u_{i,n}))|}{\eta}\Big)\,dx\,dt \\
&\leq \int_{Q_T} \overline{M}\Big(\frac{1}{\eta}|\gamma(x,t)| + \frac{1}{\eta}\overline{M}^{-1}(M(\delta|T_{m+1}(u_{i,n})|))\Big)\,dx\,dt \\
&\leq \int_{Q_T} \overline{M}\Big(\frac{1}{2}\frac{2}{\eta}|\gamma(x,t)| + \frac{1}{2}\frac{2}{\eta}\overline{M}^{-1}(M(\delta(m+1)))\Big)\,dx\,dt \\
&\leq \int_{Q_T} \frac{1}{2}\overline{M}(\frac{2}{\eta}|\gamma(x,t)|)\,dx\,dt + \int_{Q_T} \frac{1}{2}\overline{M}\Big(\frac{2}{\eta}\overline{M}^{-1}(M(\delta(m+1)))\Big)\,dx\,dt \\
&< \infty \text{ since } \gamma \in E_{\overline{M}}(Q_T) \text{ and } \Omega \text{ is bounded,}
\end{aligned}
$$

the same for $\Phi_i(x,t,T_{m+1}(u_i))$. Thanks to Lemma 2.5, we deduce that $\Phi_i(x,t,T_{m+1}(u_{i,n})) \longrightarrow \Phi_i(x,t,T_{m+1}(u_i))$ strongly in $(E_{\overline{M}}(\Omega))^N$. On the other hand, $\nabla T_k(u_{i,n}) \rightharpoonup \nabla T_k(u_i)$ weakly in $(L_M(Q_T))^N$ as n goes to infinity, it follows that

$$
\begin{aligned}
\lim_{n\to\infty} &\int_{Q_T} \Phi_i(x,t,u_{i,n})h_m(u_{i,n})[\nabla T_k(u_{i,n}) - \nabla Z^{\mu}_{i,j}]\,dx\,dt \\
&= \int_{Q_T} \Phi_i(x,t,u_i)h_m(u_i)[\nabla T_k(u_i) - \nabla Z^{\mu}_{i,j}]\,dx\,dt.
\end{aligned} \tag{54}
$$

Using the modular convergence of $Z^{\mu}_{i,j}$ as $j \longrightarrow \infty$ and then $\mu \longrightarrow \infty$, we get (r_2). Since $F_{i,n} \in (E_{\overline{M}}(Q_T))^N$ we can prove (r_6) and (r_7) as in the proof of (r_2). As a consequence of Lemma 3.1, the results of proposition 3.7 follow. ☑

Step 4: Passing to the limit. Now, we will pass to the limit. Let $v \in W^{1,x}L_M(Q_T) \cap L^\infty(Q_T)$ be such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(Q_T) + L^1(Q_T)$. From [7, lemma 5,theorem 3], there exists a prolongation $v_p = v$ on Q_T, $v_p \in W^1_x L_M(\Omega\times \mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R})$ and

$$
\frac{\partial v}{\partial t} \in W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R}).
$$

There exists also a sequence $(\omega_j) \subset \mathfrak{D}(\Omega\times\mathbb{R})$ such that

$$
\omega_j \to v_p \text{ in } W^{1,x}_0 L_M(\Omega\times\mathbb{R}), \text{ and } \frac{\partial \omega_j}{\partial t} \to \frac{\partial v_p}{\partial t} \text{ in } W^{-1,x}L_{\overline{M}}(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R}),
$$

for the modular convergence and $\|\omega_j\|_{\infty,Q_T} \leq (N+2)\|v\|_{\infty,Q_T}$.

Testing the approximate problem (23) by $T_k(u_{i,n}-\omega_j)\chi_{(0,\tau)}$ with $\tau\in[0,T]$, we get

$$\Big\langle\frac{\partial u_{i,n}}{dt},T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau}+\int_{Q_\tau}a(x,t,T_{k_0}(u_{i,n}),\nabla T_{K_0}(u_{i,n}))\nabla T_k(u_{i,n}-\omega_j)\,dx\,dt$$
$$+\int_{Q_\tau}\Phi_{i,n}(x,t,T_{k_0}(u_{i,n}))\nabla T_k(u_{i,n}-\omega_j)\,dx\,dt$$
$$=\int_{Q_\tau}f_{i,n}T_k(u_{i,n}-\omega_j)\,dx\,dt+\int_{Q_\tau}F_{i,n}\nabla T_k(u_{i,n}-\omega_j)\,dx\,dt, \tag{55}$$

where $k_0=k+(N+2)\|v\|_{\infty,Q_T}$. This implies, with $E_{n,j}:=Q_\tau\cap\{|u_{i,n}-\omega_j|\le k\}$, that

$$\Big\langle\frac{\partial u_{i,n}}{dt},T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau}+\int_{E_{n,j}}a(x,t,T_{k_0}(u_{i,n}),\nabla T_{k_0}(u_{i,n}))\nabla u_{i,n}\,dx\,dt$$
$$-\int_{E_{n,j}}a(x,t,T_{k_0}(u_{i,n}),\nabla T_{k_0}(u_{i,n}))\nabla\omega_j\,dx\,dt$$
$$+\int_{Q_\tau}\Phi_{i,n}(x,t,T_{k_0}(u_{i,n}))\nabla T_k(u_{i,n}-\omega_j)\,dx\,dt$$
$$=\int_{Q_\tau}f_{i,n}T_k(u_{i,n}-\omega_j)\,dx\,dt+\int_{Q_\tau}F_{i,n}\nabla T_k(u_{i,n}-\omega_j)\,dx\,dt. \tag{56}$$

Our aim here is to pass to the limit in each term in (56), let us start by the terms of the left-hand side:

We first consider the limit of the first term $\Big\langle\frac{\partial u_{i,n}}{dt},T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau}$. We have

$$\Big\langle\frac{\partial u_{i,n}}{dt},T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau}=\Big\langle\frac{\partial u_{i,n}}{dt}-\frac{\partial\omega_j}{dt},T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau}$$
$$+\Big\langle\frac{\partial\omega_j}{dt},T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau}$$
$$=\int_\Omega\widetilde{T}_k(u_{i,n}-\omega_j)\,dx+\Big\langle\frac{\partial\omega_j}{dt},T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau}$$
$$-\int_\Omega\widetilde{T}_k(u_{i,0n}-\omega_j(0))\,dx. \tag{57}$$

Since $u_{i,n}\longrightarrow u_i$ in $C([0,T],L^1(\Omega))$ (see [7]), by Lebesgue's theorem we have

$$\int_\Omega\widetilde{T}_k(u_{i,n}-\omega_j)\,dx\longrightarrow\int_\Omega\widetilde{T}_k(u_i-\omega_j)\,dx \text{ as } n\longrightarrow\infty.$$

Passing to the limit in (57), we get

$$\lim_{n\longrightarrow\infty}\Big\langle \frac{\partial u_{i,n}}{dt}, T_k(u_{i,n}-\omega_j)\Big\rangle_{Q_\tau} = \int_\Omega \widetilde{T}_k(u_i-\omega_j)\,dx + \Big\langle \frac{\partial \omega_j}{dt}, T_k(u_i-\omega_j)\Big\rangle_{Q_\tau} - \int_\Omega \widetilde{T}_k(u_{i,0}-\omega_j(0))\,dx.$$

For the second and the third terms of (56), we have from (ii) of proposition 3.7

$$a(x,t,T_{k_0}(u_{i,n}),\nabla T_{k_0}(u_{i,n})) \rightharpoonup a(x,t,T_{k_0}(u_i),\nabla T_{k_0}(u_i)) \text{ weakly in } (L_{\overline{M}}(Q_T))^N,$$

thus Fatou's lemma allows us to get

$$\begin{aligned} \liminf_{n\to\infty}\Big(\int_{E_{n,j}} & a(x,t,T_{k_0}(u_{i,n}),\nabla T_{k_0}(u_{i,n}))\nabla u_{i,n}\,dx\,dt \\ & - \int_{E_{n,j}} a(x,t,T_{k_0}(u_{i,n}),\nabla T_{k_0}(u_{i,n}))\nabla \omega_j\,dx\,dt\Big) \\ \geq & \int_{E_{n,j}} a(x,t,T_{k_0}(u_i),\nabla T_{k_0}(u_i))\nabla u\,dx\,dt \\ & - \int_{E_{n,j}} a(x,t,T_{k_0}(u_i),\nabla T_{k_0}(u_i))\nabla \omega_j\,dx\,dt. \end{aligned} \tag{58}$$

Concerning the fourth term of the left-hand side of (56), we proceed as in (52) to get

$$\Phi_i(x,t,T_{k_0}(u_{i,n})) \to \Phi_i(x,t,T_{k_0}(u_i)) \text{ as } n\to\infty$$

and since

$$\nabla T_k(u_{i,n}-\omega_j) \rightharpoonup \nabla T_k(u_i-\omega_j) \text{ in } L_M(Q_T) \text{ as } n\to\infty,$$

we can deduce

$$\begin{aligned} & \int_{Q_\tau} \Phi_{i,n}(x,t,T_{k_0}(u_{i,n}))\nabla T_k(u_{i,n}-\omega_j)\,dx\,dt \\ & \quad \to \int_{Q_\tau} \Phi_i(x,t,T_{k_0}(u_i))\nabla T_k(u_i-\omega_j)\,dx\,dt \end{aligned}$$

and

$$\begin{aligned} & \int_{Q_\tau} F_{i,n}\nabla T_k(u_{i,n}-\omega_j)\,dx\,dt \\ & \quad \to \int_{Q_\tau} F_i(x,t,T_{k_0}(u_i))\nabla T_k(u_i-\omega_j)\,dx\,dt, \end{aligned}$$

Finally, we turn to see the right-hand side of (56), since

$$T_k(u_{i,n}-\omega_j) \to T_k(u_i-\omega_j) \text{ weakly* in } L^\infty \text{ as } n\to\infty,$$

we obtain

$$\int_{Q_\tau} f_{i,n} T_k(u_{i,n} - \omega_j)\, dx\, dt \to \int_{Q_\tau} f_i T_k(u_i - \omega_j)\, dx\, dt.$$

Now, we are ready to pass to the limit as $n \to \infty$ in each term of (56) to conclude that

$$\begin{aligned}
&\int_\Omega \widetilde{T}_k(u_i - \omega_j)\, dx + \Big\langle \frac{\partial \omega_j}{dt}, T_K(u_i - \omega_j)\Big\rangle_{Q_\tau} \\
&\quad + \int_{Q_\tau} a(x,t,u_i,\nabla u_i)\nabla T_k(u_i - \omega_j)\, dx\, dt \\
&\qquad + \int_{Q_\tau} \Phi_i(x,t,u_i)\nabla T_k(u_i - \omega_j)\, dx\, dt \\
&\qquad\quad \le \int_\Omega \widetilde{T}_k(u_{i,0} - \omega_j(0))\, dx + \int_{Q_\tau} f_i T_k(u_i - \omega_j)\, dx\, dt \\
&\qquad\qquad + \int_{Q_\tau} F_i \nabla T_k(u_i - \omega_j)\, dx\, dt.
\end{aligned} \tag{59}$$

Now, we pass to the limit in (59) as $j \longrightarrow \infty$, we obtain

$$\begin{aligned}
&\int_\Omega \widetilde{T}_k(u_i - v)\, dx + \Big\langle \frac{\partial v}{dt}, T_k(u_i - v)\Big\rangle_{Q_\tau} \\
&\quad + \int_{Q_\tau} a(x,t,u_i,\nabla u_i)\nabla T_k(u_i - v)\, dx\, dt \\
&\qquad + \int_{Q_\tau} \Phi_i(x,t,u_i)\nabla T_k(u_i - v)\, dx\, dt \\
&\qquad\quad \le \int_\Omega \widetilde{T}_k(u_{i,0} - v(0))\, dx + \int_{Q_\tau} f_i T_k(u_i - v)\, dx\, dt \\
&\qquad\qquad + \int_{Q_\tau} F_i \nabla T_k(u_i - v)\, dx\, dt.
\end{aligned} \tag{60}$$

It remains to show that u_i satisfies the initial condition of (23). Recall that, $\dfrac{\partial u_{i,n}}{\partial t}$ is bounded in $L^1(Q_T) + W^{-1,x}L_{\overline{M}}(Q_T)$. As a consequence, an Aubin's type Lemma (cf [21] , Corollary 4) and (lemma 2.8) implies that $u_{i,n}$ lies in a compact set of $C^0([0,T]; L^1(\Omega))$. It follows that, $u_{i,n}(x,t=0) = u_{i,0n}$ converges to $u_i(x,t=0)$ strongly in $L^1(\Omega)$. Then we conclude that $u_i(x,t=0) = u_{i,0}(x)$ in Ω.

That is the full proof of the main result.

References

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Department of Mathematics
Faculty of Sciences
Tetouan, Abdelmalek Essaadi University,
BP 2121, Tetouan, Morocco.
e-mail: jabir1984@gmail.com

Department of Mathematics and Statistics
Laboratory LAMA, Faculty of Sciences Dhar el Mahraz,
Sidi Mohamed Ben Abdellah University,
PB 1796 Fez-Atlas, Fez, Morocco.
e-mail: mohamedbourahma@gmail.com

Department of Mathematics
Faculty of Sciences
Tetouan, Abdelmalek Essaadi University,
BP 2121, Tetouan, Morocco.
e-mail: hjiajhassane@yahoo.fr

Department of Mathematics and Statistics
Laboratory LAMA, Faculty of Sciences Dhar el Mahraz,
Sidi Mohamed Ben Abdellah University,
PB 1796 Fez-Atlas, Fez, Morocco.
e-mail: jbennouna@hotmail.com

Department of Mathematics and Statistics
Laboratory LAMA, Faculty of Sciences Dhar el Mahraz,
Sidi Mohamed Ben Abdellah University,
PB 1796 Fez-Atlas, Fez, Morocco.
e-mail: abd.benkirane@gmail.com