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k-Pell numbers written as a product of two Narayana numbers

Números de Pell de tipo k que se representan como producto de dos números de Narayana

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Abstract. For an integer $k \geq 2$, let $\{P_n^{(k)}\}_n$ be the k-generalized Pell sequence which starts with $0, \dots, 0, 1$(k terms) and each term afterwards is the sum of k preceding terms. The purpose of this paper is to determine all k-Pell numbers, which are the product of two of Narayana's numbers. More precisely, we study the Diophantine equation

$$P_n^{(k)} = N_m N_l$$

in positive integers (n, k, m, l) with $k \geq 2$, where $\{N_m\}_m$ is the Narayana's cows sequence.

Key words and phrases. k-Pell numbers, Narayana's cows sequence, linear forms in logarithms, reduction method.

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Resumen. Para cada entero $k \geq 2$, sea $\{P_n^{(k)}\}_n$ la k-sucesión de Pell que comienza en $0, \dots, 0, 1$ (los primeros k términos) y los términos siguientes se calculan como la suma de los k términos anteriores. El objetivo de este artículo es determinar cuáles números de la k-sucesión de Pell se pueden escribir como el producto de dos números de Narayana. De forma más precisa, estudiamos la ecuación Diofantina

$$P_n^{(k)} = N_m N_l$$

para enteros (n, k, m, l) con $k \geq 2$, donde $\{N_m\}_m$ es la sucesión de Narayana.

Palabras y frases clave. Números de Pell, sucesión de Narayana, formas lineares en logaritmos, método de reducción.

1. Introduction

The Pell sequence $\{P_n\}_{n\geq 0}$ is a binary recurrence sequence given by

$$P_{n+2} = 2P_{n+1} + P_n \quad \text{for } n \geq 0,$$

with initials $P_0 = 0$ and $P_1 = 1$.

Let $k \geq 2$ be an integer. We consider a generalization of the Pell sequence known as the k-generalized Pell sequence, $\{P_n^{(k)}\}_{n\geq -(k-2)}$ is given by the recurrence

$$P_n^{(k)} = 2P_{n-1}^{(k)} + P_{n-2}^{(k)} + \cdots + P_{n-k}^{(k)} \text{ for all } \quad n \geq 2, \tag{1}$$

with initials $P_{-(k-2)}^{(k)} = P_{-(k-3)}^{(k)} = \cdots = P_0^{(k)} = 0$ and $P_1^{(k)} = 1$. We shall refer to $P_n^{(k)}$ as the nth k-Pell number. This generalization is a family of sequences, with each new choice of k producing a unique sequence. For example, if $k = 2$, we get $P_n^{(2)} = P_n$, the nth Pell number.

The Narayana's cows sequence $\{N_m\}_{m\geq 0}$ is a ternary recurrent sequence given by

$$N_{m+3} = N_{m+2} + N_m \quad \text{for } m \geq 0,$$

with initials $N_0 = N_1 = N_2 = 1$. It is derived from the book "Ganita Kaumud" and named after an Indian mathematician Narayana Pandit. It is the sequence A000930 in the OEIS and holds a significant place due to its properties and relationships with other mathematical sequences and their important applications in other various fields, such as cryptography, coding theory, and graph theory. The first few terms of the Narayana's cows sequence are

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, \ldots$$

In recent times, several researchers have intensely studied the problems of the terms of linear recurrence sequences written as a product of two other sequences. For example, Ddamulira et al. [10] studied the question consisting of describing those Fibonacci numbers that are products of two Pell numbers and those Pell numbers which are products of two Fibonacci numbers. Following that, Şiar [9] looked into Lucas numbers, which are products of two balancing numbers. Erduvan and Keskin in [13], searched for Fibonacci numbers which are products of two Jacobsthal numbers. Later, Alan and Alan [3] searched Mersenne numbers, which are products of two Pell numbers. In [1], Adédji et al. found Padovan and Perrin numbers, which are products of two generalized Lucas numbers. Erduvan and Keskin [14], in their work, obtained Fibonacci numbers, which are products of two Jacobsthal-Lucas numbers. Recently, Rihane searched for all k-Fibonacci numbers expressible as products of two Balancing or Lucas-Balancing numbers and k-Fibonacci and k-Lucas numbers written as a product of two Pell numbers in [18, 19] respectively.

Especially in the last decade, the study of Diophantine equations involving Narayana's cows sequence has been a source of attraction for many authors. For further details, we refer the reader to [2, 7, 17, 21]. In this paper, motivated by the above works, we study the problem of determining k-Pell numbers as the product of two Narayana numbers. To accomplish this, we solve the Diophantine equation

$$P_n^{(k)} = N_m N_l. \tag{2}$$

In particular, our main result is the following.

Theorem 1.1. *All the solutions of the Diophantine equation* (2) *in positive integers* $n, m, l,$ *and* k *with* $k \geq 2$ *and* $2 \leq m \leq l$ *are given by*

$$P_1^{(k)} = N_2^2, k \geq 2, \qquad P_2^{(k)} = N_2 N_3, k \geq 2, \qquad P_4^{(k)} = N_2 N_8, k \geq 3,$$

$$P_4^{(2)} = N_3 N_6 = N_4 N_5, \quad P_6^{(4)} = N_2 N_{13}, \quad P_7^{(2)} = N_8^2 \quad \text{and} \quad P_6^{(3)} = N_4 N_{10}.$$

We specify the condition $l \geq m \geq 2$, in the above theorem, is only to avoid k-Pell numbers, which are the product of two Narayana's numbers. Theorem 1.1 allows us to deduce the following statement.

Corollary 1.2. *All the solutions of the Diophantine equation*

$$P_n^{(k)} = N_m \tag{3}$$

in positive integers with $k \geq 2$ *are given by*

$$P_1^{(k)} = N_0 = N_1 = N_2, \quad P_2^{(k)} = N_3, \quad \textit{and} \quad P_6^{(4)} = N_{13}$$

except in the cases $k \geq 3$ *which we can additionally have* $P_4^{(k)} = N_8$.

To establish the proof of Theorem 1.1, we first find an upper bound for n in terms of k by applying Matveev's result on linear forms in logarithms [16]. When k is small, we use a reduction algorithm due to de Weger to reduce the upper bounds to a size that can be easily handled. When k is large, we use the fact that the dominant root of the k-generalized Pell sequence is exponentially close to ϕ^2 (see [6], Lemma 3.2) where $\phi = \frac{1+\sqrt{5}}{2}$. So, we use this estimation in our further calculation with linear forms in logarithms to obtain absolute upper bounds for n and k, which can be reduced by using reduction algorithm due to de Weger [11]. In this way, we complete the proof of our main result. Our proof relies on a few preliminary results, which are discussed in the next section.

2. A few preliminaries

This section is devoted to gathering several definitions, notations, properties, and results that will be used in the rest of this study.

2.1. Properties of k-generalized Pell sequence

The characteristic polynomial of the k-generalized Pell sequence is

$$\Phi_k(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - x - 1.$$

The above polynomial is irreducible over $\mathbb{Q}[x]$ and it has one positive real root that is $\gamma := \gamma(k)$ which is located between $\phi^2(1-\phi^{-k})$ and ϕ^2, lies outside the unit circle (see [6]). The other roots are firmly contained within the unit circle. To simplify the notation, we will omit the dependence on k of γ whenever no confusion may arise.

The Binet formula for $P_n^{(k)}$ found in [6] is

$$P_n^{(k)} = \sum_{i=1}^{k} g_k(\gamma_i)\gamma_i^n, \tag{4}$$

where γ_i represents the roots of the characteristic polynomial $\Phi_k(x)$ and the function g_k is given by

$$g_k(z) := \frac{z-1}{(k+1)z^2 - 3kz + k - 1}, \tag{5}$$

for an integer $k \geq 2$. Additionally, it is also shown in [6, Theorem 3.1] that the roots located inside the unit circle have a very minimal influence on the formula (4), which is given by the approximation

$$\left| P_n^{(k)} - g_k(\gamma)\gamma^n \right| < \frac{1}{2} \quad \text{holds for all} \quad n \geq 2-k. \tag{6}$$

Therefore, for $n \geq 1$ and $k \geq 2$, we have

$$P_n^{(k)} = g_k(\gamma)\gamma^n + e_k(n), \quad \text{where} \quad |e_k(n)| \leq \frac{1}{2}. \tag{7}$$

Furthermore, it is shown in [6, Theorem 3.1] that the inequality

$$\gamma^{n-2} \leq P_n^{(k)} \leq \gamma^{n-1} \text{ holds for all } n \geq 1. \tag{8}$$

The following result was proved by Bravo and Hererra [5].

Lemma 2.1. ([5], Lemma 1). *Let $k \geq 2$ be an integer. Then we have*

$$0.276 < g_k(\gamma) < 0.5 \text{ and } |g_k(\gamma_i)| < 1 \quad for \quad 2 \leq i \leq k.$$

Furthermore, they showed that the logarithmic height of $g_k(\gamma)$ satisfies

$$h(g_k(\gamma)) < 4k \log \phi + k \log(k+1) \quad \text{for all} \quad k \geq 2. \tag{9}$$

Lemma 2.2. ([5], Lemma 2). *If $k \geq 30$ and $n \geq 1$ are integers that satisfies $n < \phi^{k/2}$, then*

$$g_k(\gamma)\gamma^n = \frac{\phi^{2n}}{\phi+2}(1+\xi), \quad \text{where} \quad |\xi| < \frac{4}{\phi^{k/2}}. \tag{10}$$

2.2. Properties of Narayana's cows sequence

The characteristic polynomial of Narayana's cows sequence is

$$f(x) = x^3 - x^2 - 1.$$

This polynomial is irreducible in $\mathbb{Q}[x]$ and has only real root α which has absolute value > 1, while the other two conjugate complex roots are β and $\bar{\beta}$ with $|\beta| = |\bar{\beta}| < 1$. The Binet formula for the Narayana's cows sequence is given by

$$N_m = C_\alpha \alpha^m + C_\beta \beta^m + C_{\bar{\beta}} \bar{\beta}^m. \tag{11}$$

From the three initial values of the Narayana sequence and using Vieta's theorem, one has

$$C_x = \frac{x^2}{x^3+2}, \quad x \in \{\alpha, \beta, \bar{\beta}\}.$$

The coefficient C_α has the minimal polynomial $31x^3 - 31x^2 + 10x - 1$ over $\mathbb{Z}$ and all the zeros of this polynomial lie strictly inside the unit circle. In fact, $\alpha \approx 1.46557$, and it is easy to see that the contribution of the complex conjugates $\beta, \bar{\beta}$, in the right-hand side of (11), is very small. In particular setting

$$\Pi(m) = N_m - C_\alpha \alpha^m = C_\beta \beta^m + C_{\bar{\beta}} \bar{\beta}^m, \quad \text{then} \quad |\Pi(m)| < \frac{1}{\alpha^{m/2}}, \tag{12}$$

for $m \geq 1$. Using induction, it can seen that

$$\alpha^{m-2} \leq N_m \leq \alpha^{m-1} \quad \text{hold for} \quad m \geq 1. \tag{13}$$

2.3. Linear forms in logarithms

Let γ be an algebraic number of degree d with a minimal primitive polynomial

$$f(Y) := b_0 Y^d + b_1 Y^{d-1} + \cdots + b_d = b_0 \prod_{j=1}^{d} (Y - \gamma^{(j)}) \in \mathbb{Z}[Y],$$

where the b_j's are relatively prime integers, $b_0 > 0$, and the $\gamma^{(j)}$'s are conjugates of γ. Then the *logarithmic height* of γ is given by

$$h(\gamma) = \frac{1}{d}\left(\log b_0 + \sum_{j=1}^{d} \log\left(\max\{|\gamma^{(j)}|, 1\}\right)\right).$$

With the above notation, Matveev (see [16] or [8, Theorem 9.4]) proved the following result.

Theorem 2.3. *Let $\eta_1, \ldots, \eta_s$ be positive real algebraic numbers in a real algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$. Let $a_1, \ldots, a_s$ be non-zero integers such that*

$$\Lambda := \eta_1^{a_1} \cdots \eta_s^{a_s} - 1 \neq 0.$$

Then

$$-\log|\Lambda| \leq 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) \cdot B_1 \cdots B_s,$$

where

$$D \geq \max\{|a_1|, \ldots, |a_s|\},$$

and

$$B_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \text{ for all } j = 1, \ldots, s.$$

2.4. The de Weger reduction algorithm

Here, we present a variant of the reduction method of Baker and Davenport [4] (and improved by Dujella and Pethő [12]) due to de Weger [11].

Let $\vartheta_1, \vartheta_2, \delta \in \mathbb{R}$ be given and let $x_1, x_2 \in \mathbb{Z}$ be unknowns. Let

$$\Lambda = \delta + x_1\vartheta_1 + x_2\vartheta_2. \tag{14}$$

Set $X = \max\{|x_1, |x_2|\}$. Let X_0, Y be positive. Assume that

$$|\Lambda| < c \cdot \exp(-\rho Y) \tag{15}$$

and

$$Y \leq X \leq X_0, \tag{16}$$

where c, ρ be positive constants.

When $\delta = 0$ in (14), we get

$$\Lambda = x_1\vartheta_1 + x_2\vartheta_2.$$

Put $\vartheta = -\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by

$$[a_0, a_1, a_2, \ldots]$$

and let the kth convergent of ϑ be p_k/q_k for $k = 0, 1, 2, \ldots$. We may assume without loss of generality that $|\vartheta_1| < |\vartheta_2|$ and that $x_1 > 0$. We have the following results.

Lemma 2.4. ([11], Lemma 3.1) *If* (15) *and* (16) *hold for x_1, x_2 with $X \geq 1$ and $\delta = 0$, then $(-x_2, x_1) = (p_k, q_k)$ for an index k that satisfies*

$$k \leq -1 + \frac{\log(1 + X_0\sqrt{5})}{\log\left(\frac{1+\sqrt{5}}{2}\right)} := Y_0.$$

Lemma 2.5. ([11], Lemma 3.2) *Let*

$$A = \max_{0\le k\le Y_0} a_{k+1}.$$

If (15) *and* (16) *holds for x_1, x_2 with $X \ge 1$ and $\delta = 0$, then*

$$Y < \frac{1}{\rho}\log\left(\frac{c(A+2)}{|\vartheta_2|}\right) + \frac{1}{\rho}\log X < \frac{1}{\rho}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\delta \neq 0$ in (14)*, put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \delta/\vartheta_2$. Then we have*

$$\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2.$$

Let p/q be a convergent of ϑ with $q > X_0$. For a real number x we let $\|x\| = \min\{|x-n| : n \in \mathbb{Z}\}$ be the distance from x to the nearest integer. We have the following result.

Lemma 2.6. ([11], Lemma 3.3) *Suppose that*

$$\|q\psi\| > \frac{2X_0}{q}.$$

Then, the solutions of (15) *and* (16) *satisfy*

$$Y < \frac{1}{\rho}\log\left(\frac{q^2 c}{|\vartheta_2|X_0}\right).$$

2.5. Other useful lemmas

We conclude this section by recalling two lemmas that we will need in this work.

Lemma 2.7. ([11], Lemma 2.7) *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a}\cdot|x|$$

and

$$|x| < \frac{a}{1-e^{-a}}\cdot|e^x - 1|.$$

Lemma 2.8. ([20], Lemma 7) *If $m \ge 1$, $S \ge (4m^2)^m$ and $\frac{x}{(\log x)^m} < S$, then $x < 2^m S(\log S)^m$.*

3. Proof of Theorem 1.1

In this section, we give all the details about the proof of Theorem 1.1. For this, two cases will be considered according to the values of n.

3.1. The case $1 \leq n \leq k+1$

It is known that from [15] that for $1 \leq n \leq k+1$, we have

$$P_n^{(k)} = F_{2n-1}$$

where F_n is the nth Fibonacci number. So the equation (2) becomes

$$F_{2n-1} = N_m N_l.$$

By using Theorem 1.1 of [22], we deduce that the solutions of (2) for $1 \leq n \leq k+1$ are

$$P_1^{(k)} = N_2^2, k \geq 2, \qquad P_2^{(k)} = N_2 N_3, k \geq 2 \qquad \text{and} \qquad P_4^{(k)} = N_2 N_8, k \geq 3.$$

3.2. The case $n \geq k+2$

We start this subsection by assuming that $n \geq k+2$. we have the following result, which gives us the bounds of $m+l$ in terms of n.

Lemma 3.1. *If integers m, n, k and l with $m \leq l$ satisfy the Diophantine equation (2) for $k \geq 2$, then we have the following inequalities*

$$1.25n - 0.5 < m + l < 2.52n + 1.48. \tag{17}$$

Proof. Combining the estimates (8) and (13) with the equation (2), we have

$$\alpha^{m+l-4} < N_m N_l = P_n^{(k)} < \gamma^{n-1}$$

and

$$\gamma^{n-2} < P_n^{(k)} = N_m N_l = \alpha^{m+l-2}.$$

This leads us to the inequalities

$$(n-2)\frac{\log \gamma}{\log \alpha} + 2 < m + l < (n-1)\frac{\log \gamma}{\log \alpha} + 4.$$

In addition to this, using the fact $\phi^2(1-\phi^{-k}) < \gamma(k) < \phi^2$ for $k \geq 2$, we deduce that

$$1.25n - 0.5 < m + l < 2.52n + 1.48.$$

This completes the proof. ☑

Next, we get the following result, which gives an upper bound of n in terms of k with the help of Lemma 3.1.

Lemma 3.2. *If (k, l, m, n) is a solution of in positive integers of (2) with $k \geq 2$ and $n \geq k+2$, then the inequality*

$$n < 4.18 \times 10^{33} k^8 \log^5 k \tag{18}$$

hold.

Proof. Using (2) and (12), we have

$$C_\alpha^2 \alpha^{m+l} = P_n^{(k)} - C_\alpha \alpha^m \Pi(l) - C_\alpha \alpha^l \Pi(m) - \Pi(m)\Pi(l). \tag{19}$$

Inserting (7) in (19), yields

$$\begin{aligned} \left| C_\alpha^2 \alpha^{m+l} - g_k(\gamma)\gamma^n \right| &\le \frac{1}{2} + C_\alpha \alpha^m |e(l)| + C_\alpha \alpha^l |\Pi(m)| + |\Pi(m)||\Pi(l)| \\ &\le \frac{3}{2} + C_\alpha \alpha^m + C_\alpha \alpha^l. \end{aligned} \tag{20}$$

Dividing across by $C_\alpha^2 \alpha^{m+l}$ gives us

$$\left| g_k(\gamma)\gamma^n C_\alpha^{-2} \alpha^{-(m+l)} - 1 \right| \le \frac{1}{C_\alpha \alpha^l} + \frac{1}{C_\alpha \alpha^m} + \frac{1.5}{C_\alpha^2 \alpha^{m+l}} < \frac{13.5}{\alpha^{2m}}.$$

Let

$$\Lambda_1 := g_k(\gamma)\gamma^n C_\alpha^{-2} \alpha^{-(m+l)} - 1. \tag{21}$$

From (20), we have

$$|\Lambda_1| < 13.5 \cdot \alpha^{-2m}. \tag{22}$$

To apply Theorem 2.3, we need to show that $\Lambda_1 \neq 0$. Indeed, $\Lambda_1 = 0$ implies

$$g_k(\gamma) = C_\alpha^2 \alpha^{m+l} \gamma^{-n}.$$

Hence, $g_k(\gamma)$ is an algebraic integer, which is impossible. Thus, $\Lambda_1 \neq 0$. Therefore, we apply Theorem 2.3 to get a lower bound for Λ_1 given by (21) with the parameters:

$$\eta_1 := C_\alpha^{-2} g_k(\gamma), \quad \eta_2 := \gamma, \quad \eta_3 := \alpha,$$

and

$$a_1 := 1, \quad a_2 := n, \quad a_3 := -(m+l).$$

Note that the algebraic numbers η_1, η_2, η_3 belongs to the field $\mathbb{L} := \mathbb{Q}(\gamma, \alpha)$, so we can assume $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] \le 3k$. Since $h(\eta_2) = (\log \gamma)/k < 2 \log \phi / k$ and $h(\eta_3) = (\log \alpha)/3$, it follows that

$$\max\{3kh(\eta_2), |\log \eta_2|, 0.16\} = 6 \log \phi := B_2$$

and

$$\max\{3kh(\eta_3), |\log \eta_3|, 0.16\} = k \log \alpha := B_3.$$

Since $h(C_a) = \frac{\log 31}{3}$. Therefore, by the estimate (9) and the properties of logarithmic height, it follows that for all $k \ge 2$

$$\begin{aligned} h(\eta_1) &\le h(C_\alpha^2) + h(g_k(\gamma)) \\ &< 2\left(\frac{\log 31}{3}\right) + 4k \log \phi + k \log(k+1) \\ &< 12.1 \log k. \end{aligned}$$

Thus, we obtain

$$\max\{3kh(\eta_1), |\log \eta_1|, 0.16\} = 36.3k \log k := B_1.$$

Finally, by inequalities (17) and $2.52n + 1.48 < 2.9n$, which holds for $n \geq 4$, it is seen that we can take $D := 2.9n$. Then by Theorem 2.3, we have

$$\log |\Lambda_1| > -1.432 \times 10^{11}(3k)^2(1 + \log 3k)(1 + \log(2.9n))(36.3k \log k) (6 \log \phi)(k \log \alpha). \quad (23)$$

From the comparison of lower bound (23) and upper bound (22) of $|\Lambda_1|$ gives us

$$m \log \alpha - \log 13.5 < 5.17 \times 10^{13} k^4 \log k(1 + \log 3k)(1 + \log(2.9n).$$

Using the facts $1 + \log 3k < 4.1 \log k$ for all $k \geq 2$ and $1 + \log(2.9n) < 2.5 \log n$ for all $n \geq 4$, we conclude that

$$m \log \alpha < 5.31 \times 10^{14} k^4 \log^2 k \log n. \quad (24)$$

In order to apply Theorem 2.3 a second time, we go back to (2) and we express it as

$$C_\alpha \alpha^l = \frac{P_n^{(k)}}{N_m} - \Pi(l).$$

From this, it follows

$$\left| C_\alpha \alpha^l - \frac{g_k(\gamma)\gamma^n}{N_m} \right| = \left| \frac{P_n^{(k)} - g_k(\gamma)\gamma^n}{N_m} + \Pi(l) \right| \leq 2.5 \quad (25)$$

If we divide through by $C_\alpha \alpha^l$, we obtain

$$\left| \frac{g_k(\gamma)\gamma^n}{N_m C_\alpha \alpha^l} - 1 \right| < \frac{3.6}{\alpha^l}. \quad (26)$$

Then, from (26), we have

$$|\Lambda_2| < 3.6 \cdot \alpha^{-l} \quad (27)$$

where

$$\Lambda_2 := \frac{g_k(\gamma)}{C_\alpha N_m} \gamma^n \alpha^{-l} - 1. \quad (28)$$

In a similar manner used to show that $\Lambda_1 \neq 0$, one can verify that $\Lambda_2 \neq 0$. Now, let us apply Theorem 2.3 with

$$(\eta_1, a_1) := \left(\frac{g_k(\gamma)}{C_\alpha N_m}, 1 \right), \qquad (\eta_2, a_2) := (\gamma, n), \quad \text{and} \quad (\eta_3, a_3) := (\alpha, -l).$$

The number field containing η_1, η_2, η_3 is $\mathbb{L} := \mathbb{Q}(\gamma, \alpha)$, which has degree $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 3k$. As calculated before, we take

$$B_2 = 6 \log \phi, \quad B_3 = k \log \alpha, \quad D = 2.9n.$$

But we must calculate $h(\eta_1)$ and then B_1. Using logarithmic properties and (25), we deduce that for all $k \geq 2$,

$$\begin{aligned} h(\eta_1) &\leq h\left(\frac{g_k(\gamma)}{C_\alpha N_m}\right) \\ &\leq h(g_k(\gamma)) + h(C_\alpha) + h(N_m) \\ &< 4k \log \phi + k \log(k+1) + \frac{\log 31}{3} + (m-1) \log \alpha \\ &< 5.32 \times 10^{14} k^4 \log^2 k \log n. \end{aligned}$$

Thus, we take $B_1 = 1.59 \times 10^{15} k^5 \log^2 k \log n$. Applying Theorem 2.3 and comparing the resulting inequality with (27), we obtain

$$l < 6.1 \times 10^{28} k^8 \log^3 k \log^2 n$$

where we have used the facts $1 + \log 3k < 4.1 \log k$ and $1 + \log(2.9n) < 2.5 \log n$ for all $k \geq 2$ and $n \geq 4$. By the inequality (17), the last inequality becomes

$$\frac{n}{\log^2 n} < 9.84 \times 10^{28} k^8 \log^3 k. \tag{29}$$

Thus, putting $S := 9.84 \times 10^{28} k^8 \log^3 k$ in (29) and using Lemma 2.8 with the fact $66.75 + 8 \log k + 3 \log(\log k) < 103 \log k$ for all $k \geq 2$, gives

$$\begin{aligned} n &< 2^2 \left(9.84 \times 10^{28} k^8 \log^3 k\right) \left(\log \left(9.84 \times 10^{28} k^8 \log^3 k\right)\right)^2 \\ &< 4.18 \times 10^{33} k^8 \log^5 k. \end{aligned}$$

This establishes (18) and finishes the proof of Lemma 3.1. ☑

Subsequently, we will discuss the cases in which the value of k is small or large.

3.2.1. The case when $2 \leq k \leq 760$

To reduce the above bound on n, we first set

$$\Gamma_1 := \log(\Lambda_1 + 1) = n \log \gamma - (m + l) \log \alpha + \log\left(C_\alpha^{-2} g_k(\gamma)\right). \tag{30}$$

Therefore, (22) implies that $|\Lambda_1| < 0.64$ for $m \geq 4$. Choosing $a := 0.64$, we obtain the inequality

$$|\Gamma_1| < \frac{-\log(1 - 0.64)}{0.64} \cdot \frac{13.5}{\alpha^{2m}} < 21.56 \cdot \exp(-0.76 \cdot m). \tag{31}$$

by Lemma 2.7. In view to apply Lemma 2.6 to Γ_1, we set

$$c := 21.56, \quad \rho := 0.76, \quad \psi := \frac{\log\left(C_\alpha^{-2} g_k(\gamma)\right)}{\log \gamma}$$

and

$$\vartheta := \frac{\log \alpha}{\log \gamma}, \quad \vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log \gamma, \quad \delta := \log\left(C_\alpha^{-2} g_k(\gamma)\right).$$

For each $k \in [2, 760]$, we find a good approximation of φ and a convergent p_l/q_l of the continued fraction of ϑ such that $q_l > X_0$, where $X_0 = \lfloor 4.18 \times 10^{33} k^8 \log^5 k \rfloor$, which is an upper bound of $\max\{n, m+l\}$ from Lemma 3.1. After doing this, we use Lemma 2.6 on inequality (31). A computer search with *Mathematica* revealed that if $k \in [2, 760]$, then the maximum value of $\left\lfloor \frac{1}{\rho} \log\left(q^2 c/|\vartheta_2| X_0\right) \right\rfloor$ is 390.465, which is according to Lemma 2.6, is an upper bound on m.

Now, we fix $2 \leq m \leq 390$ and let

$$\Gamma_2 := \log(\Lambda_2 + 1) = n \log \gamma - l \log \alpha + \log\left(\frac{g_k(\gamma)}{C_\alpha N_m}\right). \tag{32}$$

Since $l \geq 4$, then by (26) we have $|\Lambda_2| < 0.79$. Choosing $a := 0.78$, we obtain the inequality

$$|\Gamma_2| < \frac{-\log(1 - 0.79)}{0.79} \cdot \frac{3.6}{\alpha^l} < 7.12 \cdot \exp(-0.38 \cdot l) \tag{33}$$

by Lemma 2.7. To apply Lemma 2.6 to Γ_2, we take

$$c := 7.12, \quad \rho := 0.38, \quad \psi := \frac{\log\left(\frac{g_k(\gamma)}{C_\alpha N_m}\right)}{\log \gamma},$$

and

$$\vartheta := \frac{\log \alpha}{\log \gamma}, \quad \vartheta_1 := -\log \alpha, \quad \vartheta_2 := \log \gamma, \quad \delta := \log\left(\frac{g_k(\gamma)}{C_\alpha N_m}\right).$$

Again, for each $(k, l) \in [2, 760] \times [2, 390]$, we find a good approximation of γ and a convergent p_l/q_l of the continued fraction of ϑ such that $q_l > X_0$, where $X_0 = \lfloor 4.18 \times 10^{33} k^8 \log^5 k \rfloor$, which is an upper bound of $\max\{n, m+l\}$ from Lemma 3.1. After doing this, we use Lemma 2.6 on inequality (33). A computer search with *Mathematica* revealed that the maximum value of $\left\lfloor \frac{1}{\rho} \log\left(q^2 c/|\vartheta_2| X_0\right) \right\rfloor$ over all $k \in (k, l) \in [2, 760] \times [2, 390]$ is 813.969, which is according to Lemma 2.6, is an upper bound on l.

Hence, we deduce that the possible solutions (k, l, m, n) of equation (2) for which $k \in [2, 760]$ have $2 \leq m \leq l \leq 813$, therefore, we use inequalities (17) to obtain $n \leq 1301$.

Finally, we used Mathematica to compare P_n^k and $N_m N_l$ for the range $4 \leq n \leq 1301$ and $2 \leq l \leq m \leq 813$, with $m + l < 2.52n + 1.48$ and checked that the only solution of the equation (2) are

$$P_4^{(2)} = N_3 N_6 = N_4 N_5, \quad P_6^{(4)} = N_2 N_{13}, \quad P_7^{(2)} = N_8^2 \quad \text{and} \quad P_6^{(3)} = N_4 N_{10}.$$

3.2.2. The case when $k > 760$

In this case, we need to show that the Diophantine equation (2) has no solution. We have the following lemmas.

Lemma 3.3. *If* (k, l, m, n) *is a solution of the Diophantine equation* (2) *with* $k > 760$ *and* $n \geq k + 2$*, then* k *and* n *are bounded as*

$$k < 1.51 \times 10^{37} \quad \text{and} \quad n < 5.62 \times 10^{340}.$$

Proof. For $k > 760$, the following inequalities hold

$$n < 4.18 \times 10^{33} k^8 \log^5 k < \phi^{k/2}.$$

Inserting (10) in (20), we obtain

$$\begin{aligned} \left| C_\alpha^2 \alpha^{m+l} - \frac{\phi^{2n}}{\phi + 2} \right| &\leq \frac{\phi^{2n}}{\phi + 2} |\zeta| + C_\alpha \alpha^m |\Pi(l)| + C_\alpha \alpha^l |\Pi(m)| + |\Pi(l)||\Pi(m)| \\ &< \frac{\phi^{2n}}{(\phi + 2)\phi^{k/2}} + C_\alpha \alpha^m + C_\alpha \alpha^l + 1. \end{aligned}$$

The above inequality and the facts that $\alpha^{m+l-4} < \gamma^{n-1}$ together with $n \geq k+2$ show that

$$\left| C_\alpha^2 \alpha^{m+l} \frac{\phi + 2}{\phi^{2n}} - 1 \right| < \frac{4}{\phi^{k/2}} + \frac{11.8}{\alpha^{2m}} < \frac{15.8}{\phi^\lambda}, \tag{34}$$

where $\lambda = \min\{k/2, \theta m\}$ and $\theta = 2 \log \alpha / \log \phi$. To apply Theorem 2.3 in (34), set

$$\Lambda_3 := C_\alpha^2 \alpha^{m+l} \frac{\phi + 2}{\phi^{2n}} - 1.$$

In contrast to this, assume that $\Lambda_3 = 0$, then we would get

$$\frac{\phi^{2n}}{\phi + 2} = C_\alpha^2 \alpha^{m+l}.$$

Using the $\mathbb{Q}$-automorphism (α, β) of the Galois extension $\mathbb{Q}(\phi, \alpha, \beta)$ over $\mathbb{Q}$ we get that $12 < \phi^{2n}/\phi + 2 < |C_\beta^2||\beta|^{m+l} < 1$, which is impossible. Therefore, $\Lambda_3 \neq 0$. We take $s := 3$,

$$\eta_1 := C_\alpha^2(\phi + 2), \quad \eta_2 := \phi, \quad \eta_3 := \alpha$$

and

$$a_1 := 1, \quad a_2 := 2n, \quad a_3 := m + l.$$

Note that $\mathbb{L} := \mathbb{Q}(\alpha, \sqrt{5})$ contains η_1, η_2, η_3 and has $d_{\mathbb{L}} := 6$. Since $m+l < 2.9n$, we deduce that $D := \max\{|a_1|, |a_2|, |a_3|\} = 2.9n$. Moreover, since $h(\eta_2) = \frac{\log \phi}{2}$, $h(\eta_3) = \frac{\log \alpha}{3}$ and

$$\begin{aligned} h(\eta_1) &\le 2h\left(C_\alpha\right) + h(\phi) + h(2) + \log 2 \\ &\le 2\left(\frac{\log 31}{3}\right) + h(\phi) + h(2) + \log 2 \\ &\le 2\left(\frac{\log 31}{3}\right) + 2\log 2 + \frac{\log \phi}{2}. \end{aligned}$$

Therefore, we may take

$$B_1 := 4\log 31 + 3\log \phi + 12\log 2, \quad B_2 := 3\log \phi \quad \text{and} \quad B_3 := 2\log \alpha.$$

As before, applying Theorem 2.3, we have

$$-\log|\Lambda_3| < 9.34 \times 10^{14} \log n, \tag{35}$$

where $1 + \log(2.9n) < 2.5\log n$ holds for all $n \ge 4$. When we compare (35) with (34), it yields

$$\lambda < 1.95 \times 10^{14} \log n. \tag{36}$$

Now, we distinguish two cases according to λ.

Case 1: $\lambda = \frac{k}{2}$. In this case, it results from (36)

$$k < 3.9 \times 10^{14} \log n.$$

Inserting (18) into the above result, it yields

$$k < 1.95 \times 10^{14} \log\left(4.18 \times 10^{33} k^8 \log^5 k\right).$$

Further using Lemma 2.8 and Lemma 3.1 it leads to

$$k < 1.75 \times 10^{18} \quad \text{and} \quad n < 4.81 \times 10^{187}. \tag{37}$$

Case 2 If $\lambda = \theta m$. In this case it result from (36) that

$$m < 1.23 \times 10^{14} \log n. \tag{38}$$

Using inequality (10), we can rearrange (25) as

$$\left| C_\alpha \alpha^l - \frac{\phi^{2n}}{(\phi+2)N_m} \right| < 1.5 + \frac{\phi^{2n}}{\phi+2}|\xi|,$$

where $|\xi| < \frac{4}{\phi^{k/2}}$. Dividing $\frac{\phi^{2n}}{(\phi+2)N_m}$ on both sides of the above inequality and using the facts $n \geq k+2$ and $\theta m < \frac{k}{2}$, we get

$$\begin{aligned}\left|\alpha^l \phi^{-2n} C_\alpha N_m(\phi+2) - 1\right| &< \frac{1.5N_m(\phi+2)}{\phi^{2n}} + |\xi| \\ &< \frac{1.42\alpha^m}{\phi^{2k}} + \frac{4}{\phi^{k/2}} < \frac{5.42}{\phi^{k/2}}. \end{aligned} \tag{39}$$

Now apply Theorem 2.3 to (39) with the data $s := 3$ and

$$\Lambda_4 := \alpha^l \phi^{-2n} C_\alpha N_m(\phi+2) - 1,$$

where

$$\eta_1 := \alpha, \quad \eta_2 := \phi, \quad \eta_3 := C_\alpha N_m(\phi+2),$$

and

$$a_1 := l, \quad a_2 := -2n, \quad a_3 := 1.$$

Here $\Lambda_4 \neq 0$. If it were, then we would get a contradiction just as Λ_3. Thus $\Lambda_4 \neq 0$. As previously calculated, we can take

$$d_{\mathbb{L}} := 6, \quad B_1 := 2\log\alpha, \qquad B_2 := 3\log\phi \quad \textit{and} \quad D := 2.9n.$$

Moreover, by using (38), we have

$$\begin{aligned} h(\eta_3) &= h\left(C_\alpha N_m(\phi+2)\right) \\ &\leq h(C_\alpha) + h(\phi+2) + h(N_m) \\ &\leq \frac{\log 31}{3} + 2\log 2 + (m-1)\log\alpha + \frac{\log\phi}{2} \\ &< 4.72 \times 10^{13}\log n. \end{aligned}$$

Hence, we can take $B_3 := 2.84 \times 10^{14}\log n$. According to Theorem 2.3 and (39) we obtain

$$\frac{k}{2}\log\phi - \log 5.42 < 1.13 \times 10^{28}(\log n)^2$$

and hence

$$k < 4.74 \times 10^{28}(\log n)^2.$$

Further using Lemma 2.8 and Lemma 3.1 this leads to

$$k < 1.51 \times 10^{37} \quad \text{and} \quad n < 5.2 \times 10^{340}. \tag{40}$$

So, in all cases, inequalities (40) hold. Whence the result. ☑

The bounds obtained for n and k from the above lemma are very large. Now, our next goal is to reduce this upper bound to a reasonable range for which we will prove the following lemma.

Lemma 3.4. *The equation* (2) *has no solutions for* $k > 760$ *and* $n \geq k+2$.

Proof. Here, we attempt to reduce the upper bound on n and k. To do so, we first consider

$$\Gamma_3 := (l+m)\log\alpha - 2n\log\phi + \log\left(C_\alpha^2(\phi+2)\right).$$

Since $m \geq 4$, then from inequality (34), we get $|\Lambda_3| < 0.75$. Choosing $a := 0.75$, we obtain the inequality

$$|\Gamma_3| = |\log(\Lambda_3+1)| < \frac{-\log(1-0.75)}{0.75}\cdot\frac{15.8}{\phi^\lambda} < 29.21\cdot\exp(-0.48\cdot\lambda). \quad (41)$$

We apply Lemma 2.6 with the data

$$c := 29.21, \quad \rho := 0.48, \quad \psi := -\frac{\log\left(C_\alpha^2(\phi+2)\right)}{\log\phi}$$

and

$$\vartheta := \frac{\log\alpha}{\log\phi}, \quad \vartheta_1 := \log\alpha, \quad \vartheta_2 := -\log\phi, \quad \delta := \log\left(C_\alpha^2(\phi+2)\right).$$

Further, since $m+l < 2.9n$ and $n < 5.62\times10^{340}$ by Lemma 3.2, then we can take $X_0 := 1.63\times10^{341}$. Let

$$[a_0, a_1, a_2, \dots] := [0,1,3,1,6,3,1,3,2,1,16,1,4,1,2,1,3,1,1,1,3,8,1,14,\dots]$$

be the continued fraction expansion of ϑ. By Lemma 2.4, we have

$$Y_0 = -1 + \frac{\log\left(1+\sqrt{5}\times1.63\times10^{341}\right)}{\log\left(\frac{1+\sqrt{5}}{2}\right)} = 1633.363.$$

With the help of *Mathematica*, we find that

$$\max_{0\leq k\leq Y_0} a_{k+1} = 784 := A.$$

Then by Lemma 2.5, we have

$$\lambda < \frac{1}{0.48}\cdot\log\left(\frac{29.21\cdot786\cdot1.63\cdot10^{341}}{\log\phi}\right) < 1659.26$$

Case 1: $\lambda = \frac{k}{2}$. In this case, we get that

$$k \leq 3318. \quad (42)$$

Case 2: $\lambda = \theta m$. In this case, we obtain that $m \leq 1044$. Now, let

$$\Gamma_4 := l\log\alpha - 2n\log\phi + \log\left(C_\alpha N_m(\phi+2)\right).$$

Since $k > 760$, then from (39) we have $|\Lambda_4| < 0.01$. Choosing $a := 0.01$, we obtain the inequality

$$|\Gamma_4| = |\log(\Lambda_4 + 1)| < \frac{-\log(1-0.01)}{0.01} \cdot \frac{5.42}{\phi^{k/2}} < 5.45 \cdot \exp(-0.24 \cdot k). \quad (43)$$

For $2 \le m \le 1044$, we apply Lemma 2.6 with the Parameters

$$c := 5.45, \quad \rho := 0.24, \quad \psi := -\frac{\log\left(C_\alpha N_m(\phi+2)\right)}{\log\phi}$$

and

$$\vartheta := \frac{\log\alpha}{\log\phi}, \quad \vartheta_1 := \log\alpha, \quad \vartheta_2 := -\log\phi, \quad \delta := \log\left(C_\alpha N_m(\phi+2)\right).$$

Further, Lemma 3.2 implies that we can take $X_0 := 1.63 \times 10^{341}$. Using *Mathematica*, it is seen that q_{662} satisfies the hypotheses of Lemma 2.8. Furthermore, from Lemma 2.8, we obtain $k < 3364$. Therefore, $k < 3364$ holds always. With this new upper bound on k, we obtain

$$n < 2.43 \times 10^{66}.$$

We now proceed as we did before but with $X_0 := 7.1 \times 10^{66}$, we obtain that $k < 747$, which contradicts our assumption $k > 760$. This completes the proof of Theorem 1.1. ☑

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