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A Decomposition Theorem with an Application to Dirac Spinors

Un teorema de descomposición con una aplicación a los espinores de Dirac

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Abstract. We prove a decomposition theorem for a linear transformation using only linear algebra techniques. This result leads to a completely different proof of a well-known decomposition theorem from representation theory, a result that is known to have an application to the Dirac spinor representation.

Key words and phrases. complex conjugate vector space, complexification, realification, Dirac equation.

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Resumen. Probamos un teorema de descomposición para transformaciones lineales usando sólo herramientas de álgebra lineal. Este resultado provee una prueba nueva de un resultado bien establecido de teoría de representaciones, que tiene aplicaciones en la teoría de representaciones de spinores de Dirac.

Palabras y frases clave. espacio vectorial complejo conjugado, complejificación, realificación, ecuación de Dirac.

1. Introduction

In this paper we show how a linear transformation on a finite-dimensional complex vector space, when first considered as a real transformation and then complexified is isomorphic to the direct sum of two linear transformations. As a corollary, we obtain a well-known decomposition result (Theorem 3.2) for a complex representation of a group that is completely different from the usual proof that relies on results from representation theory. When the group above is the double covering of the Lorentz group, this theorem has an application to the Dirac spinor representation.

Throughout this paper, V will denote a finite-dimensional complex vector space.

2. Transformations

2.1. Definitions

(1) Given a scalar product on V represented by

$$\alpha v, \alpha \in \mathbb{C}, v \in V$$

we define *the complex conjugate vector space* of V, denoted by $\bar{V}$. Elements of $\bar{V}$ are the same as elements of V and addition is defined the same as addition of elements of V, but scaler multiplication on $\bar{V}$ is given by $\alpha \times v = \bar{\alpha} v$ for $v \in \bar{V}, \alpha \in \mathbb{C}$.

(2) A linear transformation $T : V \to V$ defines a linear transformation $\bar{T} : \bar{V} \to \bar{V}$. If T has complex matrix representation [T] with respect to a basis β, then $\bar{T}$ is represented by the matrix $[\bar{T}]$ in this basis whose matrix entries are the respective complex conjugates of entries in matrix [T].

(3) We construct the direct sum of V and $\bar{V}$, denoted $V \bigoplus \bar{V}$, as the set of elements $(C, D), C \in V, D \in \bar{V}$. Addition is defined component-wise, and multiplication by $\alpha \in \mathbb{C}$ is defined by $\alpha(C, D) = (\alpha C, \bar{\alpha} D)$. The linear transformation $T \bigoplus \bar{T} : V \bigoplus \bar{V} \to V \bigoplus \bar{V}$, is defined by $(T \bigoplus \bar{T})(C, D) = (T(C), \bar{T}(D))$. Moreover, with respect to the basis of $V \bigoplus \bar{V}$ defined by β, the matrix representation of $[T \bigoplus \bar{T}]$ is $\begin{bmatrix} [T] & 0 \\ 0 & [\bar{T}] \end{bmatrix}$.

(4) Let V be complex vector space of dimension n and consider V as a real vector space of dimension $2n$, denoted by $V_{\mathbb{R}}$. The transformation T gives rise to a real transformation $T_{\mathbb{R}} : V_{\mathbb{R}} \to V_{\mathbb{R}}$, the *realification* of T, simply: $T_{\mathbb{R}}(u) = T(u)$. Note, $T_{\mathbb{R}}(iu) = T(iu) = iT(u)$.

(5) We will denote the complexification, $V_{\mathbb{R}} \otimes \mathbb{C}$, of $V_{\mathbb{R}}$, by $(V_{\mathbb{R}})^{\mathbb{C}}$, which equals $V_{\mathbb{R}} \times V_{\mathbb{R}}$ as a set. An element of $(V_{\mathbb{R}})^{\mathbb{C}}$ is an ordered pair (u, v) where $u, v \in V_{\mathbb{R}}$, which we write as $u + iv$. Addition on $(V_{\mathbb{R}})^{\mathbb{C}}$ is defined by $(u_1 + iv_1) + (u_1 + iv_1) = (u_1 + v_1) + i(u_2 + v_2)$. Complex scalar multiplication on $(V_{\mathbb{R}})^{\mathbb{C}}$ is defined by $(a + bi)(u + iv) = (au - bv) + i(av + bu)$.

Given a basis $\beta = \{u_1, u_2, \ldots, u_n\}$ for V, $\tilde{\beta} = \{u_1, iu_1, \ldots u_n, iu_n\}$ is a real basis for $V_{\mathbb{R}}$ and a complex basis for $(V_{\mathbb{R}})^{\mathbb{C}}$. The transformation, $T_{\mathbb{R}} : V_{\mathbb{R}} \to V_{\mathbb{R}}$, can be extended to a linear transformation $(T_{\mathbb{R}})^C : (V_{\mathbb{R}})^{\mathbb{C}} \to (V_{\mathbb{R}})^C$ by allowing $(T_{\mathbb{R}})^{\mathbb{C}}$ to now act on complex vectors by $(T_{\mathbb{R}})^{\mathbb{C}}(iv) = iT(v)$. With respect to the basis $\tilde{\beta}$, the matrix representation of $(T_{\mathbb{R}})^{\mathbb{C}}$ is the same as the matrix representation of $T_{\mathbb{R}}$ with respect to $\tilde{\beta}$.

Lemma 2.1. *The map* $F : (V_{\mathbb{R}})^{\mathbb{C}} \to V \bigoplus \bar{V}$ *given by*

$$F(u + iv) = ((u + iv), (u - iv))$$

is an isomorphism.

Proof. Both vector spaces are of dimension $2n$. The map F is clearly one-to-one and linear over the real numbers. We show that multiplication by $i = \sqrt{-1}$ is preserved.

Since $i(u+iv) = -v + iu \in (V_{\mathbb{R}})^{\mathbb{C}}$, we have

$$F(i(u+iv)) = F(-v+iu) = ((-v+iu), (-v-iu)).$$

On the other hand,

$$iF(u+iv) = i((u+iv), (u-iv)) = (i(u+iv), (-i)(u-iv)) = ((-v+iu), (-v-iu)).$$

This shows that $F(i(u+iv)) = iF(u+iv)$. ☑

The inverse of F is

$$F^{-1} : V \bigoplus \bar{V} \to (V_{\mathbb{R}})^{\mathbb{C}} \text{ given by } F^{-1}(C,D) = \frac{1}{2}(C+D) + \frac{i}{2}(iD - iC)$$

Definition 2.2. Two linear transformations $T : V \to V$ and $S : W \to W$ are said to be isomorphic, denoted $T \sim S$, if there exists an isomorphism $F : V \to W$ such that $F \circ T \circ F^{-1} = S$.

Theorem 2.3. *Given a linear transformation, $T : V \to V$, of a complex n-dimensional vector space V, $(T_{\mathbb{R}})^{\mathbb{C}} \sim T \bigoplus \bar{T}$.*

Before proving the theorem above, for clarity, we provide examples in two very special cases.

Example: One-Dimensional Case

Let $\beta = \{u\}$ be a basis for the one-dimensional complex vector space V. $V_{\mathbb{R}}$ as a real vector space has dimension two with basis $\tilde{\beta} = \{u, iu\}$. The complexification, $(V_{\mathbb{R}})^{\mathbb{C}}$, has complex dimension two with this same basis $\tilde{\beta}$. With respect to basis $\tilde{\beta} = \{u, iu\}$ for $(V_{\mathbb{R}})^{\mathbb{C}}$ and basis $\{(u,0),(0,u)\}$ for $V \bigoplus \bar{V}$, we have the following matrix representations

$$[F] = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \text{ and } [F]^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$$

Let $T(u) = \alpha u, \alpha = a + bi \in \mathbb{C}$ with the matrix representation $[T] = [\alpha] = [a+bi]$.

We let $iu = \tilde{u}$ and calculate $T_{\mathbb{R}}$.

$$\begin{aligned}
T_{\mathbb{R}}(u) = T(u) &= \alpha u = (a+bi)u \\
&= au + b(iu) \\
&= au + b\tilde{u}, \\
T_{\mathbb{R}}(\tilde{u}) = T_{\mathbb{R}}(iu) &= iT(u) = i(a+ib)u \\
&= (ia-b)u \\
&= -bu + i(au) \\
&= -bu + a\tilde{u}.
\end{aligned}$$

The matrix representation of $T_{\mathbb{R}}$ with respect to the basis $\tilde{\beta}$ is $[T_{\mathbb{R}}] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. We extend $T_{\mathbb{R}}$ to $(V_{\mathbb{R}})^{\mathbb{C}}$, to get the map $(T_{\mathbb{R}})^{\mathbb{C}}$. The matrix of $(T_{\mathbb{R}})^{\mathbb{C}}$ with respect to the basis $\tilde{\beta}$ is given by $[(T_{\mathbb{R}})^{\mathbb{C}}] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, and the matrix representation of $F \circ (T_{\mathbb{R}})^{\mathbb{C}} \circ F^{-1} : V \bigoplus \bar{V} \to V \bigoplus \bar{V}$ is $[F][(T^{\mathbb{R}})_{\mathbb{C}}][F]^{-1} = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$. Therefore, $(T^{\mathbb{R}})_{\mathbb{C}} \sim T \bigoplus \tilde{T}$.

Example: Two-Dimensional Case

Let V be of complex dimension two with basis $\beta = \{u_1, u_2\}$. Then, the real vector space $V_{\mathbb{R}}$ has dimension four with basis $\tilde{\beta} = \{u_1, iu_1, u_2, iu_2\}$. We denote $iu_1 = \tilde{u_1}, iu_2 = \tilde{u_2}$. $(V^{\mathbb{R}})_{\mathbb{C}}$ has complex dimension four. With respect to corresponding bases of $(V_{\mathbb{R}})^{\mathbb{C}}$ and $V \bigoplus \bar{V}$, we have the matrix representations

$$[F] = \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \end{bmatrix} \text{ and } [F]^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -i & 0 & i \end{bmatrix}$$

Let $T : V \to V$ be a linear transformation with matrix representation $[T] = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{2,1} & \alpha_{22}, \end{bmatrix}$ where $\alpha_{lk} = a_{lk} + ib_{lk}$.

$$
\begin{aligned}
T(u_k) &= \sum_{l=1}^{2} \alpha_{lk} u_l \\
T_{\mathbb{R}}(u_1) &= T(u_1) = \alpha_{11}u_1 + \alpha_{21}u_2 \\
&= (a_{11} + ib_{11})u_1 + (a_{21} + ib_{21})u_2 \\
&= a_{11}u_1 + b_{11}\tilde{u_1} + a_{21}u_2 + b_{21}\tilde{u_2} \\
T_{\mathbb{R}}(iu_1) &= iT(u_1) = i(\alpha_{11}u_1 + \alpha_{21}u_2) \\
&= i(a_{11} + ib_{11})u_1 + i(a_{21} + ib_{21})u_2) \\
&= -b_{11}u_1 + a_{11}(iu_1) - b_{21}u_2 + a_{21}(iu_2) \\
&= -b_{11}u_1 + a_{11}\tilde{u}_1 - b_{21}u_2 + a_{21}\tilde{u}_2 \\
T_{\mathbb{R}}(u_2) &= T(u_2) = a_{12}u_1 + b_{12}\tilde{u}_1 + a_{22}u_2 + b_{22}\tilde{u}_2 \\
T_{\mathbb{R}}(iu_2) &= iT(u_2) = -b_{12}u_1 + a_{12}\tilde{u}_1 - b_{22}u_2 + a_{22}\tilde{u}_2
\end{aligned}
$$

Expanding $T_{\mathbb{R}}$ to $(T_{\mathbb{R}})^{\mathbb{C}}$, we find the matrix representation of $(T_{\mathbb{R}})^{\mathbb{C}}$ with respect to $\tilde{\beta}$ is $[T_{\mathbb{R}})^{\mathbb{C}}] = [T_{\mathbb{R}}] = \begin{bmatrix} a_{11} & -b_{11} & a_{12} & -b_{12} \\ b_{11} & a_{11} & b_{12} & a_{12} \\ a_{21} & -b_{21} & a_{22} & -b_{22} \\ b_{21} & a_{21} & b_{22} & a_{22} \end{bmatrix}$.

Thus,

$$
[F][T_{\mathbb{R}})^{\mathbb{C}}][F^{-1}] = \tfrac{1}{2} \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \\ 1 & -i & 0 & 0 \\ 0 & 0 & 1 & -i \end{bmatrix} \begin{bmatrix} a_{11} & -b_{11} & a_{12} & -b_{12} \\ b_{11} & a_{11} & b_{12} & a_{12} \\ a_{21} & -b_{21} & a_{22} & -b_{22} \\ b_{21} & a_{21} & b_{22} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ -i & 0 & i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -i & 0 & i \end{bmatrix},
$$

resulting in $\begin{bmatrix} \alpha_{11} & \alpha_{12} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & 0 & 0 \\ 0 & 0 & \bar{\alpha}_{11} & \bar{\alpha}_{12} \\ 0 & 0 & \bar{\alpha}_{21} & \bar{\alpha}_{22} \end{bmatrix} = \begin{bmatrix} [T] & \mathbf{0} \\ \mathbf{0} & [\bar{T}] \end{bmatrix}$ where $\mathbf{0}$ is the 2×2 zero matrix, proving $(T_{\mathbb{R}})^{\mathbb{C}} \sim T \bigoplus \bar{T}$.

Proof of Theorem 2.3

Let $\mathcal{B} = \{u_1, u_2, \dots, u_n\}$ be a basis for V. For $k = 1, 2, \dots, n$, the linear transformation T is given by:

$$
T(u_k) = \sum_{l=1}^{n} \alpha_{lk} u_l, \quad \alpha_{lk} = a_{lk} + ib_{lk}.
$$

Similarly, the transformation $\bar{T}$ is defined by:

$$
\bar{T}(u_k) = \sum_{l=1}^{n} \bar{\alpha}_{lk} u_l, \quad \bar{\alpha}_{lk} = a_{lk} - ib_{lk}.
$$

Let:

$$(C,D)=\left(\sum_{k=1}^{n}c_ku_k,\sum_{k=1}^{n}d_ku_k\right)\in V\oplus\bar{V}.$$

Then,

$$\begin{aligned}(T\oplus\bar{T})(C,D)&=\left(\sum_{l,k=1}^{n}\alpha_{lk}c_ku_l,\sum_{l,k=1}^{n}\bar{\alpha}_{lk}d_ku_l\right)\\&=\left(\sum_{l,k=1}^{n}(a_{lk}+ib_{lk})c_ku_l,\sum_{l,k=1}^{n}(a_{lk}-ib_{lk})d_ku_l\right)\end{aligned}$$

We can decompose this to write:

$$(T\oplus\bar{T})(C,D)=\left(\sum_{l,k=1}^{n}(a_{lk}+ib_{lk})c_ku_l,0\right)+\left(0,\sum_{l,k=1}^{n}(a_{lk}-ib_{lk})d_ku_l\right)\quad(*)$$

Next we consider $(T_{\mathbb{R}})^{\mathbb{R}}$:

The set $\tilde{\beta}=\{u_1,iu_1,\ldots,u_n,iu_n\}$ is a basis for $(V_{\mathbb{R}})^C$. We let $iu_k=\tilde{u}_k$. We have the following:

$$\begin{aligned}(T_{\mathbb{R}})^{\mathbb{C}}(u_k)&=T_{\mathbb{R}}(u_k)=T(u_k)\\&=\sum_{l=1}^{n}\alpha_{lk}u_l=\sum_{l=1}^{n}(a_{lk}+ib_{lk})u_l.\\&=\sum_{l=1}^{n}(a_{lk}u_l+b_{lk}(iu_l))\\&=\sum_{l=1}^{n}(a_{lk}u_l+b_{lk}\tilde{u}_l)\\(T_{\mathbb{R}})^{\mathbb{C}}(iu_k)&=T_{\mathbb{R}}(iu_k)=iT(u_k)\\&=i\sum_{l=0}^{n}\alpha_{lk}u_l=\sum_{l=0}^{n}(a_{lk}+ib_{lk})(iu_l)=\sum_{l=0}^{n}(-b_{lk}u_l+a_{lk}(iu_l))\\&=\sum_{l=0}^{n}(-b_{lk}u_l+a_{lk}\tilde{u}_l).\end{aligned}$$

We will show that $F\circ(T_{\mathbb{R}})^{\mathbb{C}}\circ F^{-1}=T\bigoplus\bar{T}$.

$$F^{-1}(C,D)=\frac{1}{2}(C+D)+\frac{i}{2}(iD-iC)$$

can be simplified to get

$$F^{-1}(C,D)=\sum_{k=1}^{n}\frac{1}{2}(c_k+d_k)u_k+\sum_{k=1}^{n}\frac{1}{2}(id_k-ic_k)(iu_k).$$

Replacing iu_k by $\tilde{u_k}$ and applying $(T_{\mathbb{R}})^{\mathbb{C}}$ we have

$$(T_{\mathbb{R}})^{\mathbb{C}}\circ F^{-1}(C,D)=\frac{1}{2}\sum_{l,k=1}^{n}\left((c_k+d_k)a_{lk}+(ic_k-id_k)b_{lk}\right)u_l$$

$$+\frac{1}{2}\sum_{l,k=1}^{n}\left((c_k+d_k)b_{lk}+(id_k-ic_k)a_{lk}\right)\tilde{u}_l$$

Finally, applying F we get:

$F\circ(T_{\mathbb{R}})^{\mathbb{C}}\circ F^{-1}=$

$\frac{1}{2}\left(\sum_{l,k=1}^{n}\left((c_k+d_k)a_{lk}+(ic_k-id_k)b_{lk}\right)u_l+\left((c_k+d_k)b_{lk}+(id_k-ic_k)a_{lk}\right)\tilde{u}_l,0\right)$

$+\frac{1}{2}\left(0,\sum_{l,k=1}^{n}\left((c_k+d_k)a_{lk}+(ic_k-id_k)b_{lk}\right)u_l-\left((c_k+d_k)b_{lk}+(id_k-ic_k)a_{lk}\right)\tilde{u}_l\right)$

Replacing $\tilde{u}_l$ with iu_l and using scalar multiplcation of V and $\bar{V}$, we can further simplify to arrive at

$F\circ(T_{\mathbb{R}})^{\mathbb{C}}\circ F^{-1}(C,D)=$

$\left(\sum_{k,l=1}^{n}(a_{lk}+ib_{lk})c_k u_l,0\right)+\left(0,\sum_{k,l=1}^{n}(a_{lk}-ib_{lk})d_k u_l\right)$

By (*), this equals $(T\bigoplus\bar{T})(C,D)$.

3. Representations

Given a group G and a representation into a finite-dimensional complex vector space $V,\rho:G\to GL(V)$, we define the following representations:

(1) $\rho_{\mathbb{R}}:G\to GL(V_{\mathbb{R}})$ given by $\rho_{\mathbb{R}}(g)=(\rho(g))_{\mathbb{R}},g\in G$.

(2) $(\rho_{\mathbb{R}})^{\mathbb{C}}:G\to GL((V_{\mathbb{R}})^{\mathbb{C}})$ defined by $(\rho_{\mathbb{R}})^{\mathbb{C}}(g)=((\rho(g)_{\mathbb{R}})^{\mathbb{C}}$.

(3) $\bar{\rho}:G\to GL(\bar{V})$ given by $\bar{\rho}(g)=\overline{\rho(g)}$.

Definition 3.1. Two reprentations of a group G, $\rho:G\to GL(V)$ and $\rho':G\to GL(W)$ are said to be isomorphic, denoted $\rho\sim\rho'$, if there exists an isomorphism $\phi:V\to W$ such that $\rho'(g)\circ\phi=\phi\circ\rho(g)$ for all $g\in G$.

Theorem 3.2. *Let $\rho:G\to GL(V)$ be a representation. Then,*

$(\rho_{\mathbb{R}})^{\mathbb{C}}\sim\rho\bigoplus\bar{\rho}$.

Proof. For an arbitrary $g \in G$, let $T = \rho(g) : V \to V$ be a linear transformation as in Theorem 2.3. The proof of that theorem shows that the isomorphism $F : (V_{\mathbb{R}})^{\mathbb{C}} \to V \oplus \bar{V}$ for which $F \circ T_{\mathbb{R}} \circ F^{-1} = T \oplus \bar{T}$ only depends on the dimension n of V and not on the transformation T.

Thus, for each dimension n,

$$F \circ (\rho_{\mathbb{R}})^{\mathbb{C}}(g) = (\rho \oplus \bar{\rho})(g) \circ F, \text{ proving the theorem.}$$

☑

In terms of matrices, there exists a single basis such that for every $g \in G$,

$$[F][(\rho_{\mathbb{R}})^{\mathbb{C}}](g)[F^{-1}] = [(\rho \oplus \bar{\rho})(g)]$$

is a block diagonal matrix of dimension $2n \times 2n$.

4. Dirac Spinors

In 1924-25, while studying atomic spectra and the periodic table, Austrian physicist Wolfgang Pauli proposed that electrons had a"two-valuedness not describable classically", a property we now call the *spin* of the electron, an intrinsic angular momentum that allows only two states: either "spin up (1/2 state)" or "spin down (-1/2 state)". [2, p. 165-166]. If we include, spin 0, spin has since been shown to be a property of all fundamental particles - half-integer spin particles (1/2, 3/2, ...) are called *fermions*, while whole-integer spin particles $(0, 1, 2, \ldots)$ are called *bosons* (The Higgs boson has spin 0 and the photon has spin 1.) Although particles of spin greater than one have been proposed in theory, no elementary particle with spin greater than one has been observed to date.

The solution to Schrodinger's equation determines the quantum state of a particle by providing the probability of finding a particle at a specific location in space at a given time. However, it does not give any information about the particle's spin state. For information about the relativistic spin state of an electron, one considers Paul Dirac's equation:

$$(i\gamma^u \partial_u - m)\Psi = 0$$

where *Dirac's* 4×4 *gamma matrices* are given by

$\gamma^0 = \begin{bmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{bmatrix}$, $\gamma^i = \begin{bmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{bmatrix}$ for $i = 1, 2, 3$ where I_2 is the 2×2 zero matrix.

Here, the *Pauli matrices* are

$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma^2 = \begin{bmatrix} 0 & -i \\ I & 0 \end{bmatrix}$, $\sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ [3, p. 41-42]

Dirac found both negative energy (relating to the electron) and positive energy (relating to the *antielectron or positron*) solutions. [1, p. 28], The current belief is that for every particle there is a corresponding antiparticle having the same mass but with opposite charge.

Solutions to the Dirac equation are four component *Dirac spinors.* $\Psi = \begin{bmatrix}\Psi_L \\ \Psi_R\end{bmatrix}$.

The two-component spinors Ψ_L and Ψ_R are respectively called the *left and right Weyl spinors.*

The Lorentz group $SO(1,3)$, which preserves the Minkowski metric $\eta_{\mu\nu} = diag(1,-1,-1,-1)$, is a symmetry for the Dirac equation. [3, p. 42], The connected component of the Lorentz group, denoted $SO^+(1,3)$ has double cover $SL(2,\mathbb{C})$, the set of all 2×2 complex matrices of determinant one. A solution $\Psi = \begin{bmatrix}\Psi_L \\ \Psi_R\end{bmatrix}$ transforms under $SL(2,C)$, as follows:

For $M \in SL(2,C), M\Psi = \begin{bmatrix}M\Psi_L \\ \overline{M}\Psi_R\end{bmatrix}$. [4, Chapter 4], and [3, pp. 41-44],

$\Psi_L \to M\Psi_L$, is the *fundamental representation of SL(2,C)*, denoted by $(\frac{1}{2},0)$, and $\Psi_R \to \overline{M}\Psi_R$, is the *complex conjugate of this representation* denoted by $(0,\frac{1}{2})$. [4], (p 91)

Let $\rho_1 = \left(\frac{1}{2},0\right)$ be the fundamental two-dimensional irreducible representation of $SL(2,\mathbb{C})$. Consider its realification to obtain the four-dimensional real representation $(\rho_1)_{\mathbb{R}}$, and then complexify this representation to arrive at the four-dimensional complex representation $(\rho_1)_{\mathbb{R}})^{\mathbb{C}}$.

By Theorem 3.2,

$$((\rho_1)_{\mathbb{R}})^{\mathbb{C}} \sim \rho_1 \oplus \bar{\rho}_1 \sim \left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right),$$

is the representation that transforms a four-component solution of the Dirac equation.

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