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On strongly Elliptic problem with weak monotonicity in anisotropic weighted Sobolev spaces

Sobre un problema elíptico con monotonía débil en espacios de Sobolev ponderados anisotrópicos

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Abstract. In this study, we prove the existence of solutions to the nonlinear elliptic boundary value problem described by the equation

$$-\operatorname{div} a(x, u, \nabla u) + \Psi(x, u) = f$$

where f, $\Psi(x, u)$ are elements of $L^1(\Omega)$, and where no monotonicity condition will be supposed on the function $a(x, s, \xi)$.

Key words and phrases. Weighted Sobolev, elliptic problem, Truncation, L1-data, weak monotonicity.

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Resumen. En este artículo, probamos la existencia de soluciones de la ecuación elíptica con condiciones de frontera dada por

$$-\operatorname{div} a(x, u, \nabla u) + \Psi(x, u) = f$$

donde f, $\Psi(x, u)$ son elementos de $L^1(\Omega)$ y la función $a(x, s, \xi)$ satisface una condición de no-monotonicidad.

Palabras y frases clave. Espacios de Sobolev, problema elíptico, Truncation, espacio de datos de tipo L1, monotonicidad débil.

1. Introduction

The concept of anisotropic weighted Sobolev spaces introduces a new framework that incorporates directional derivatives with distinct weights (see Section 2.1). In our setting, we consider Ω as an open, bounded subset of $\mathbb{R}^N$ with $N \geq 2$ and we let p_i be $N+1$ different exponents, where $1 < p_i < N$ for $i \in \{0, \dots, N\}$. This paper aims to study the existence of solutions for problems of the form

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + \Psi(x,u) = f \text{ in } \Omega, \\ u = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $w = \{w_i, 0 \leq i \leq N\}$ is a collection of weight functions on $\Omega, w^* = \left\{w_i^{1-p_i'}, 0 \leq i \leq N\right\}$, and $f \in L^1(\Omega)$.

Recent studies have produced results regarding the existence, qualitative properties, and regularity of solutions to nonlinear anisotropic elliptic equations where the data belongs to L^1 spaces. For example, the author's work on the problem described in (1) in the framework of the space $W_0^{1,p}(\Omega)$ is detailed in [5].

Additionally, the paper by B. El Haji et al. ([11]) provides insights into the existence of solutions within weighted Orlicz spaces. Similarly, Y. Akdim et al ([1]) have studied the following equation in weighted Orlicz-Sobolev spaces

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the second term $F \in W^{-1,p'}(\Omega, w^*)$.

The aim of this work is to investigate elliptic problems where the methods presented in [13] and [7] are not applicable. Instead, we will consider cases where the strict monotonicity conditions are not assumed on the function a. To overcome the lack of strict monotonicity which prevents the almost everywhere convergence of the gradient of u_n, we utilize a tool involving techniques based on Minty's lemma [13]. Given that $f \in L^1(\Omega)$, the pseudo-monotonicity approach is not applicable. In [6], it was shown that u_n is bounded in the Marcinkiewicz space to ensure the almost everywhere convergence of Δu_n. In contrast, our current study focuses on establishing the local convergence in measure of u_n, as detailed in Section 4.2.2.

Extensive research has been devoted to examining the existence of solutions for parabolic and elliptic problems under different sets of hypotheses. For a comprehensive overview, readers can refer to the extensive studies and publications available on this subject (see [12, 10, 9, 2]).

This paper is organized in the following manner: We begin by introducing our work, with defining the new anisotropic weighted Sobolev spaces and present some technical results in Section 2, which will be utilized in Section 4.

Section 3 outlines the key assumptions necessary for the existence of solutions. Finally, Section 4 provides a detailed presentation of our main results and their proofs.

2. Preliminaries

2.1. Basic tools

Let Ω be a bounded open subset of $\mathbb{R}^N$. Consider $p_0, p_1, \ldots, p_N$ as $N+1$ exponents, where $1 < p_i < \infty$ for $i = 0, 1, \ldots, N$. Let $w = \{w_i(x)\}_{i=0}^N$ denote a vector of weight functions, where each $w_i(x)$ is a measurable function that is strictly positive almost everywhere in Ω. Additionally, we will assume the following hypothesis

$$w_i \in L^1_{\text{loc}}(\Omega), \tag{2}$$

$$w_i^{\frac{-1}{p_i-1}} \in L^1_{\text{loc}}(\Omega), \tag{3}$$

The Space $L^{p_i}(\Omega, \gamma)$, where γ represents a weight function, can be defined as follows:

$$L^{p_i}(\Omega, \gamma) = \left\{ u = u(x), u\gamma^{\frac{1}{p_i}} \in L^{p_i}(\Omega) \right\}$$

and equipped with the norm

$$\|u\|_{L^{p_i}(\Omega,\gamma)} = \|u\|_{p_i,\gamma} = \left(\int_\Omega |u(x)|^{p_i} \gamma(x) dx \right)^{\frac{1}{p_i}}.$$

We set

$$(p_i) = (p_0, \ldots, p_N), \quad D^0 u = u \quad \text{and} \quad D^i u = \frac{\partial u}{\partial x_i} \quad \text{for } i = 1, \ldots, N,$$

and we define

$$\underline{p_i} = \min\{p_0, p_1, \ldots, p_N\} \quad \text{then} \quad \underline{p_i} > 1. \tag{4}$$

and

$$\|u\|_{1,(p_i),w} = \|u\|_{p_0,w_0} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i,w_i}. \tag{5}$$

The anisotropic weighted Sobolev space $W^{1,(p_i)}(\Omega, w)$ is defined as follows:

$$W^{1,(p_i)}(\Omega, w) = \{u \in L^{p_0}(\Omega, w_0) \mid \nabla u \in L^{p_i}(\Omega, w_i) \text{ for } i = 1, \ldots, N\}$$

where $L^{p_0}(\Omega, w_0)$ consists of functions u such that $u \cdot w_0^{\frac{1}{p_0}} \in L^{p_0}(\Omega)$, and ∇u denotes the gradient of u. For each component of the gradient ∇u, the condition $\nabla u \in L^{p_i}(\Omega, w_i)$ holds, meaning that each component of the gradient of u, when weighted by w_i, is in the space $L^{p_i}(\Omega)$.

The hypothesis (2) ensures that $C_0^\infty(\Omega)$ is a subspace of $W^{1,(p_i)}(\Omega, w)$. Consequently, we can define the subspace $V = W_0^{1,(p_i)}(\Omega, w)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm given in (5).

Furthermore, from (3), we can deduce that both $W^{1,(p_i)}(\Omega, w)$ and $W_0^{1,(p_i)}(\Omega, w)$ are reflexive Banach spaces.

The dual space of weighted Sobolev spaces $W_0^{1,(p_i)}(\Omega, w)$ is identified with $W^{-1,(p_i')}(\Omega, w^*)$, where $w^* = \left\{ w_i^* = w_i^{1-p_i'}, i = 0, \dots, N \right\}$ and $(p_i') = (p_0', p_1', \dots, p_N')$, where p_i' is the complementary of p_i; i.e., $p_i' = \frac{p_i}{p_i - 1}$, (see [8] for the isotropic case).

Let us consider, for $k > 1$ and s in $\mathbb{R}$, the function:

$$T_k(s) = \begin{cases} s \text{ if } |s| \leq k \\ k\dfrac{s}{|s|} \text{ if } |s| > k. \end{cases}$$

Lemma 2.1. *(See lemma 8 in [4]) Let $(u_n)_n$ be a bounded sequence in $W_0^{1,(p_i)}$ (Ω, ω). If $u_n \rightharpoonup u$ weakly in $W_0^{1,(p_i)}(\Omega, \omega)$, therefore $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,(p_i)}(\Omega, \omega)$ for any $k > 0$.*

We assume that

$$\|u\| = \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} w_i(x) dx \right)^{\frac{1}{p_i}} \tag{6}$$

is equivalent to the norm defined in (5). Additionally, there exists a weight function $\omega_0(x)$ on Ω and $1 < q_i < \infty$ such that the following Hardy inequality is satisfied

$$\left(\int_\Omega |u(x)|^{q_i} \omega_0(x) dx \right)^{\frac{1}{q_i}} \leq c \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^{p_i} w_i(x) dx \right)^{\frac{1}{p_i}} \tag{7}$$

for every $u \in W_0^{1,(p_i)}(\Omega, w)$ with a constant $c > 0$ independent of u.

Proof of Hardy inequality (7) **:**

According to the Poincaré inequality (see [4]) we have

$$\|u\|_{L^{p_i}(\Omega, \omega_i)} \leqslant c \left\| D^i u \right\|_{L^{p_i}(\Omega, \omega_i)}.$$

Therefore, for all $1 \leqslant q_i \leqslant p_i$, we have

$$\|u\|_{L^{q_i}(\Omega, \omega_i)} \leqslant c \left\| D^i u \right\|_{L^{p_i}(\Omega, \omega_i)},$$

then

$$\left(\int_\Omega |u|^{q_i}\omega_i dx\right)^{\frac{1}{q_i}} \leqslant c\left(\int_\Omega \left|D^i u\right|^{p_i}\omega_i dx\right)^{\frac{1}{p_i}},$$

and since $\omega_0 \leqslant \omega_i$ for all i, then

$$\begin{aligned}\left(\int_\Omega |u|^{q_i}\omega_0 dx\right)^{\frac{1}{q_i}} &\leqslant \left(\int_\Omega |u|^{q_i}\omega_i dx\right)^{\frac{1}{q_i}}\\ &\leqslant c\left(\int_\Omega \left|D^i u\right|^{p_i}\omega_i dx\right)^{\frac{1}{p_i}}\\ &\leqslant c\sum_{i=1}^N\left(\int_\Omega \left|D^i u\right|^{p_i}\omega_i dx\right)^{\frac{1}{p_i}},\end{aligned}$$

for all $1 \leqslant q_i \leqslant p_i$, with $i = 1, \dots, N$.

Furthermore, the embedding,

$$W_0^{1,(p_i)}(\Omega, w) \hookrightarrow\hookrightarrow L^{q_i}(\Omega, w_0) \text{ is compact .} \tag{8}$$

3. Statements of results

Our aim is to study the problem:

$$\begin{cases} -\operatorname{div}(a(x,u,\nabla u)) + \Psi(x,u) = f \text{ in } \Omega, \\ \qquad u = 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{9}$$

Let $a(x,s,\xi) : \Omega \times \boldsymbol{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function which satisfies

For $i = 1, \dots, N$

$$|a_i(x,s,\xi)| \leq \beta w_i^{\frac{1}{p_i}}(x)\left[k(x) + w_0^{\frac{1}{p_i}}|s|^{\frac{q_i}{p_i'}} + \sum_{j=1}^N w_j^{\frac{1}{p_i'}}(x)\,|\xi_j|^{p_i-1}\right], \tag{10}$$

for a.e., $x \in \Omega$, all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^N$, $k(x) \in L^{p_i'}(\Omega)\left(\frac{1}{p_i} + \frac{1}{p_i'} = 1\right)$ and $\beta > 0$. Here w_0 and q_i are as in (7).

$$\langle a(x,s,\xi) - a(x,s,\eta), \xi - \eta\rangle \geq 0 \text{ for all } (\xi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{11}$$

$$\langle a(x,s,\xi), \xi\rangle \geq \alpha\sum_{i=1}^N w_i\,|\xi_i|^{p_i}, \alpha > 0, \tag{12}$$

$$\Psi(x,s)s \geq 0, \tag{13}$$

$$\sup_{|s|\leq n}|\Psi(x,s)| = h_n(x) \in L^1(\Omega), \tag{14}$$

$$f \in L^1(\Omega). \tag{15}$$

Definition 3.1. A function u is called an entropy solution (9) if $T_k(u)$ belongs in $W_0^{1,(p_i)}(\Omega, w)$ for every $k > 0$ and satisfies the following condition

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\Phi]\rangle\, dx + \int_\Omega \Psi(x,u) T_k[u-\Phi] dx = \int_\Omega f T_k[u-\Phi] dx$$

for every $\Phi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$.

Theorem 3.2. Let Ω be a bounded open subset of $\mathbb{R}^N, N \geq 2$ and *assume that the hypothesises* (10)-(15) *holds, then u is an entropy solution of* (9) *in the sense of the definition 3.1.*

4. Proof of Existence theorem 3.2

4.1. The key Lemma

Lemma 4.1. Let u be a measurable function with $T_k(u)$ belongs to $W_0^{1,(p_i)}(\Omega, w)$ for every $k > 0$. Then

$$\int_\Omega \langle a(x,u,\nabla \Phi), \nabla T_k[u-\Phi]\rangle\, dx + \int_\Omega \Psi(x,u) T_k[u-\Phi] dx \leq \int_\Omega f T_k[u-\Phi] dx$$

is equivalent to

$$\int_\Omega \langle a(x,u,\nabla u), \nabla T_k[u-\Phi]\rangle\, dx + \int_\Omega \Psi(x,u) T_k[u-\Phi] dx = \int_\Omega f T_k[u-\Phi] dx$$

for every Φ in $W^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$.

Proof The proof follows a procedure similar to the one presented in [9].

4.2. Proof of the main result

4.2.1. Approximation

Let us assume that the sequence of functions f_n belonging to the space $L^\infty(\Omega)$ converge strongly to the limit f in $L^1(\Omega)$, with the condition that $\|f_n\|_{L^1} \leq \|f\|_{L^1}$, and let u_n be a solution in $W_0^{1,(p_i)}(\Omega, w)$ of the problem

$$\begin{cases} -\operatorname{div} a\left(x, u_n, \nabla u_n\right) + \Psi_n\left(x, u_n\right) = f_n \text{ in } \Omega \\ u_n = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{16}$$

where

$$\Psi_n(x,s) = \frac{\Psi(x,s)}{1+\frac{1}{n}|\Psi(x,s)|}\varpi_n(x)\ ,\ \varpi_n(x) = T_{\frac{1}{n}}\left(w_0^{\frac{1}{q_i}}(x)\right),$$

similarly as in [3], Lemma 4.2 we show the existence of weak solution for the approximate problem.

Now testing the problem (16) by the test function $T_k(u_n)$, we may get

$$\int_\Omega \langle a(x,u_n,\nabla u_n), \nabla T_k(u_n)\rangle\, dx + \int_\Omega \Psi_n(x,u_n)\, T_k(u_n)\, dx = \int_\Omega f_n T_k(u_n)\, dx$$

using $\nabla T_k(u_n) = \nabla u_n \chi_{\{|u_n|\le k\}}$ and according to hypothesis (12), we get

$$\int_\Omega \langle a(x,u_n,\nabla u_n), \nabla T_k(u_n)\rangle\, dx \ge \alpha \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p_i} dx,$$

and since $\Psi_n(x,u_n)\, T_k(u_n) \ge 0$ we have,

$$\alpha \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p_i} dx \le k\|f\|_{L^1}.$$

Young's inequality implies that

$$\alpha \sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p_i} dx \le k\|f\|_{L^1} + \frac{\alpha}{2}\sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p_i} dx.$$

Then,

$$\frac{\alpha}{2}\sum_{i=1}^N \int_\Omega w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p_i} dx \le k(\|f\|_{L^1}).$$

For $k > 1$, this implies that

$$\sum_{i=1}^N \left(\int_\Omega \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p_i} w_i(x) dx\right)^{\frac{1}{p_i}} \le c k^{\frac{1}{p_i}}. \tag{17}$$

4.2.2. Locally convergence of u_n in measure

Let $k > 0$ large enough, by using (8), we obtain

$$\begin{aligned} k \text{ meas } (\{|u_n| > k\} \cap B_R) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\, dx \le \int_{B_R} |T_k(u_n)|\, dx \\ &\le \left(\int_\Omega |T_k(u_n)|^{p_i} w_0 dx\right)^{\frac{1}{p_i}} \cdot \left(\int_{B_R} w_0^{1-p_i'} dx\right)^{\frac{1}{p_i'}} \\ &\le c_R \sum_{i=1}^N \left(\int_\Omega \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^{p_i} w_i(x) dx\right)^{\frac{1}{p_i}} \\ &\le c_1 k^{\frac{1}{p_i}}. \end{aligned}$$

which gives

$$\text{meas}(\{|u_n| > k\} \cap B_R) \le \frac{c_1}{k^{1-\frac{1}{p_i}}} \quad \forall k > 1. \tag{18}$$

We obtain, for every $\delta > 0$,

$$\begin{aligned} \operatorname{meas}\left(\left\{\left|u_n-u_r\right|>\delta\right\} \cap B_R\right) \leq \operatorname{meas}\left(\left\{\left|u_n\right|>k\right\} \cap B_R\right) \\ +\operatorname{meas}\ \left(\left\{\left|u_r\right|>k\right\} \cap B_R\right)+\ \operatorname{meas}\ \left\{\left|T_k\left(u_n\right)-T_k\left(u_r\right)\right|>\delta\right\}. \end{aligned} \tag{19}$$

Since $T_k\left(u_n\right)$ is bounded in $W_0^{1,(p_i)}(\Omega, w)$, there exists some $v_k \in W_0^{1,(p_i)}(\Omega, w)$, such that

$$\left\{\begin{array}{c} T_k\left(u_n\right) \rightharpoonup v_k \quad \text{ in } W_0^{1,(p_i)}(\Omega, w) \quad (\text{ Weak convergence }), \\ T_k\left(u_n\right) \rightarrow v_k \quad \text{ in } L^{q_i}\left(\Omega, w_0\right) \text{ and a.e. in } \Omega \quad (\text{ Strong convergence }). \end{array}\right.$$

Hence, we can deduce that $T_k\left(u_n\right)$ is a Cauchy sequence in measure in Ω. Let $\varepsilon > 0$. By equations (18) and (19), there exists a $k(\varepsilon) > 0$ such that $\operatorname{meas}\ \left(\left\{\left|u_n-u_r\right|>\ \delta\right\} \cap B_R\right)<\varepsilon$ for all $n, r \geq n_0(k(\varepsilon), \delta, R)$ for all $n, r \geq n_0(k(\varepsilon), \delta, R)$. This establishes that $\left(u_n\right)$ constitutes a Cauchy sequence in measure in B_R, thus converging almost everywhere to some measurable function u. Consequently,

$$\left\{\begin{array}{c} T_k\left(u_n\right) \rightharpoonup T_k(u) \text{ in } W_0^{1,(p_i)}(\Omega, w), \\ T_k\left(u_n\right) \rightarrow T_k(u) \quad \text{ in } L^{q_i}\left(\Omega, w_0\right) \text{ and a.e in } \Omega. \end{array}\right. \tag{20}$$

4.2.3. Equi-integrability of $\Psi_n\left(x, u_n\right)$

We verify that

$$\Psi_n\left(x, u_n\right) \rightarrow \Psi(x, u) \text{ strongly in } L^1(\Omega). \tag{21}$$

In order to prove (21) it is enough to show the equi-integrable of $\Psi_n\left(x, u_n\right)$. To do this, testing the problem (16) by the function test $T_{k'+1}\left(u_n\right)-T_k'\left(u_n\right)$, we may get

$$\begin{aligned} & \int_{\Omega}\left\langle a\left(x, u_n, \nabla u_n\right), \nabla\left(T_{k'+1}\left(u_n\right)-T_{k'}\left(u_n\right)\right)\right\rangle d x+ \\ & \int_{\Omega} \Psi_n\left(x, u_n\right)\left(T_{k'+1}\left(u_n\right)-T_{k'}\left(u_n\right)\right) d x \\ & =\int_{\Omega} f\left(T_{k'+1}\left(u_n\right)-T_{k'}\left(u_n\right)\right) d x . \end{aligned}$$

Consequently,

$$\begin{aligned} & \int_{\left\{k' \leq\left|u_n\right| \leq k'+1\right\}}\left\langle a\left(x, u_n, \nabla u_n\right), \nabla u_n\right\rangle d x+\int_{\left\{\left|u_n\right| \geq k'+1\right\}}\left|\Psi_n\left(x, u_n\right)\right| d x \\ & \leq c \int_{\left\{\left|u_n\right| \geq k'\right\}}|f| d x . \end{aligned}$$

Therefore, by (12), we obtain

$$\int_{\{|u_n|\geq k'+1\}} |\Psi_n(x,u_n)|\,dx \leq c\int_{\{|u_n|\geq k'\}} |f_n|\,dx.$$

Let $\varepsilon > 0$, then there exist $k'(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n|>k'(\varepsilon)\}} |\Psi_n(x,u_n)|\,dx \leq \frac{\varepsilon}{2}. \tag{22}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}\int_E |\Psi_n(x,u_n)|\,dx &\leq \int_{E\cap\{|u_n|\leq k'(\varepsilon)\}} |\Psi_n(x,u_n)|\,dx + \int_{E\cap\{|u_n|>k'(\varepsilon)\}} |\Psi_n(x,u_n)|\,dx\\ &\leq \int_E \left|h_{k'(\varepsilon)}(x)\right|dx + \int_{E\cap\{|u_n|>k'(\varepsilon)\}} |\Psi_n(x,u_n)|\,dx.\end{aligned}$$

In view to (14) there exist $\eta(\varepsilon) > 0$ such that

$$\int_E \left|h_{k'(\varepsilon)}(x)\right|dx \leq \frac{\varepsilon}{2} \tag{23}$$

for all sets E with $meas(E) < \eta(\varepsilon)$.

By using (22) and (23), we show that $\int_E |\Psi_n(x,u_n)|\,dx \leq \varepsilon$, for all E such that meas $(E) < \eta(\varepsilon)$.

4.2.4. An intermediate Inequality

Here, we shall prove that for $\Psi \in W_0^{1,(p_i)}(\Omega,w) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega \langle a(x,u_n,\nabla\Psi), \nabla T_k[u_n-\Phi]\rangle\,dx + \int_\Omega \Psi_n(x,u_n)\,T_k[u_n-\Phi]\,dx\\ \leq \int_\Omega f_n T_k[u_n-\Phi]\,dx.\end{aligned} \tag{24}$$

Furthermore, testing the equation (16) by the test function $T_k(u_n-\Phi)$, with $\Phi \in W_0^{1,(p_i)}(\Omega,w) \cap L^\infty(\Omega)$, we can obtain

$$\begin{aligned}\int_\Omega \langle a(x,u_n,\nabla u_n), \nabla T_k[u_n-\Phi]\rangle\,dx + \int_\Omega \Psi_n(x,u_n)\,T_k[u_n-\Phi]\,dx\\ = \int_\Omega f_n T_k[u_n-\Phi]\,dx.\end{aligned}$$

Adding and subtracting the term $\int_\Omega \langle a\left(x, u_n, \nabla \Psi\right), \nabla T_k\left[u_n-\Phi\right]\rangle dx$ i.e.,

$$
\begin{aligned}
& \int_\Omega \langle a\left(x, u_n, \nabla u_n\right), \nabla T_k\left[u_n-\Phi\right]\rangle dx+\int_\Omega \langle a\left(x, u_n, \nabla \Psi\right), \nabla T_k\left[u_n-\Phi\right]\rangle dx \\
& -\int_\Omega \langle a\left(x, u_n, \nabla \Psi\right), \nabla T_k\left[u_n-\Phi\right]\rangle dx+\int_\Omega \Psi_n\left(x, u_n\right) T_k\left[u_n-\Phi\right] dx \\
& =\int_\Omega f_n T_k\left[u_n-\Phi\right] dx.
\end{aligned} \tag{25}
$$

Referring to (11) and with the aid of the truncation function, we may obtain

$$
\int_\Omega \langle\left[a\left(x, u_n, \nabla u_n\right)-a\left(x, u_n, \nabla \Psi\right)\right], \nabla T_k\left[u_n-\Phi\right]\rangle dx \geq 0. \tag{26}
$$

As a result of (25) and (26), we have (24).

4.2.5. Passing to the limit

Our next aim is to prove that for $\Phi \in W_0^{1,(p_i)}(\Omega, w) \cap L^{\infty}(\Omega)$. We get

$$
\int_\Omega \langle a(x, u, \nabla \Psi), \nabla T_k[u-\Phi]\rangle dx+\int_\Omega \Psi(x, u) T_k[u-\Phi] dx \leq \int_\Omega f T_k[u-\Phi] dx.
$$

Now, we claim that

$$
\int_\Omega \langle a\left(x, u_n, \nabla \Psi\right), \nabla T_k\left[u_n-\Phi\right]\rangle dx \rightarrow \int_\Omega \langle a(x, u, \nabla \Psi), \nabla T_k[u-\Phi]\rangle dx \text{ as } n \rightarrow+\infty.
$$

Since $T_{\mathbb{M}}\left(u_n\right) \rightharpoonup T_{\mathbb{M}}(u)$ in $W_0^{1,(p_i)}(\Omega, w)$(weak convergence), with $\mathbb{M}=k+\|\Psi\|_{\infty}$, then according to the Lemma 2.1, we can obtain

$$
T_k\left(u_n-\Phi\right) \rightharpoonup T_k(u-\Phi) \text{ in } W_0^{1,(p_i)}(\Omega, w), \tag{27}
$$

which gives

$$
\frac{\partial T_k}{\partial x_i}\left(u_n-\Phi\right) \rightharpoonup \frac{\partial T_k}{\partial x_i}(u-\Phi) \text{ in } L^{p_i}\left(\Omega, w_i\right) \forall i=1, .., N. \text{ (weak convergence).} \tag{28}
$$

Secondly, we prove that

$$
a_i\left(x, T_{\mathbb{M}}\left(u_n\right), \nabla \Psi\right) \rightarrow a_i\left(x, T_{\mathbb{M}}(u), \nabla \Psi\right) \text{ strongly in } L^{p_i^{\prime}}\left(\Omega, w_i^*\right).
$$

By using the hypothesis (10), we obtain

$$\begin{aligned}|a_i(x,T_{\mathbb{M}}(u_n),\nabla\Psi)|^{p'_i} w_i^{\frac{-p'_i}{p_i}} &\leq \beta\left[k(x)+|T_{\mathbb{M}}(u_n)|^{\frac{q^i}{p'_i}} w_0^{\frac{1}{p'_i}}+\sum_{j=1}^{N}\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i-1} w_i^{\frac{1}{p'_i}}\right]^{p'_i}\\ &\leq \gamma\left[k(x)^{p'_i}+|T_{\mathbb{M}}(u_n)|^{q_i} w_0+\sum_{j=1}^{N}\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i} w_i\right],\end{aligned} \tag{29}$$

with $\beta,\gamma>0$. Since $T_{\mathbb{M}}(u_n)\rightharpoonup T_{\mathbb{M}}(u)$ in $W_0^{1,(p_i)}(\Omega,w)$ (weak convergence) and $W_0^{1,(p_i)}(\Omega,w)\hookrightarrow\hookrightarrow L^q(\Omega,w_0)$, then $T_{\mathbb{M}}(u_n)\to T_{\mathbb{M}}(u)$ in $L^q(\Omega,w_0)$ and a.e. in Ω (strong convergence), hence

$$|a_i(x,T_{\mathbb{M}}(u_n),\nabla\Psi)|^{p'_i} w_i^{*}\to|a_i(x,T_{\mathbb{M}}(u),\nabla\Psi)|^{p'_i} w_i^{*} \text{ a.e in } \Omega.$$

and

$$\gamma\left[k(x)^{p'_i}+|T_{\mathbb{M}}(u_n)|^{q_i} w_0+\sum_{j=1}^{N}\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i} w_i\right]\to$$

$$\gamma\left[k(x)^{p'_i}+|T_{\mathbb{M}}(u)|^{q_i} w_0+\sum_{j=1}^{N}\left|\frac{\partial\Psi}{\partial x_i}\right|^{p_i} w_i\right] \text{ a.e. in } \Omega.$$

Then, we deduce

$$a_i(x,T_{\mathbb{M}}(u_n),\nabla\Psi)\to a_i(x,T_{\mathbb{M}}(u),\nabla\Psi) \text{ in } L^{p'_i}(\Omega,w_i^{*}), \text{ as } n\to+\infty. \tag{30}$$

By using (27) and (28), we may obtain

$$\int_\Omega\langle a(x,u_n,\nabla\Psi),\nabla T_k[u_n-\Phi]\rangle\,dx\to\int_\Omega\langle a(x,u,\nabla\Psi),\nabla T_k[u-\Phi]\rangle\,dx, \text{ as } n\to+\infty. \tag{31}$$

Finally, our final goal is to establish

$$\int_\Omega f_nT_k[u_n-\Phi]\,dx\to\int_\Omega fT_k[u-\Phi]dx. \tag{32}$$

We have $f_nT_k[u_n-\Phi]\to fT_k[u-\Phi]$ a.e. in Ω and $|f_nT_k[u_n-\Phi]|\leq k|f_n|$, and $k|f_n|\to k|f|$ in $L^1(\Omega)$, consequently Vitali's theorem gives (32).

Similarly using the hypothesis (21) we prove that

$$\int_\Omega\Psi_n(x,u_n)T_k[u_n-\Phi]\,dx\to\int_\Omega\Psi(x,u)T_k[u-\Phi]dx \text{ as } n\to\infty. \tag{33}$$

Based on (31) and (33), we can take the limit in (24), then for all $\Phi \in W_0^{1,(p_i)}(\Omega, w) \cap L^\infty(\Omega)$, we deduce

$$\int_\Omega a(x, u, \nabla\Phi).\nabla T_k[u-\Phi]dx \int_\Omega \Psi(x, u)T_k[u-\Phi]dx \leq \int_\Omega fT_k[u-\Phi]dx.$$

Based on the key lemma, it follows that u is a solution for the problem (9) in the sense of the notion given in definition 3.1.

5. Statements and Declarations

There is no conflict of interest statement within the manuscript.

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