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Existence results for some anisotropic degenerate parabolic problems with lower-order term and source

Resultados de existencia para algunos problemas parabólicos degenerados anisotrópicos con términos y funciones de fuente de orden inferior

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Abstract. In this study, we establish the existence and regularity of weak solutions for the anisotropic degenerate parabolic equation, namely.

$$\partial_t u - \sum_{i=1}^{N}(D_i(a_i(t,x,u)|D_i u|^{p_i-2}D_i u) + |u|^{\tau p_i-2}u|D_i u|^{p_i}) = |u|^{r-2}u,$$

if $1 \leq r < \tau p^- + 1$ and $\tau \geq \frac{1}{p^-}$ with $p^- = \min_{1\leq i\leq N} p_i$, then there exists a non-negative weak solution for every positive initial data in L^1.

Key words and phrases. L^∞ estimates, nonlinear anisotropic parabolic equations, degenerate coercivity.

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Resumen. En este artículo, probamos la existencia y regularidad de las soluciones de la ecuación parabólica anisotrópica no lineal

$$\partial_t u - \sum_{i=1}^{N}(D_i(a_i(t,x,u)|D_i u|^{p_i-2}D_i u) + |u|^{\tau p_i-2}u|D_i u|^{p_i}) = |u|^{r-2}u,$$

si $1 \leq r < \tau p^- + 1$ y $\tau \geq \frac{1}{p^-}$ con $p^- = \min_{1\leq i\leq N} p_i$. Probamos la existencia de una solución débil cuando los datos iniciales son funciones en L^1.

Palabras y frases clave. Estimados L^∞, ecuaciones anisotrópicas parabólicas no lineales, coercitividad degenerada.

1. Introduction and preliminaries

Our main objective is to prove the existence and regularity of solutions to the following nonlinear anisotropic degenerate parabolic problem:

(P)

$$\begin{cases} \partial_t u - \sum_{i=1}^{N}(D_i(a_i(t,x,u)|D_iu|^{p_i-2}D_iu) + |u|^{\tau p_i-2}u|D_iu|^{p_i}) = |u|^{r-2}u & \text{in } Q_T, \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_T. \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ ($N > 2$) with Lipschitz boundary denoted by $\partial\Omega$, T is a positive constant, $u_0 \in L^1(\Omega)$, $Q_T = (0,T) \times \Omega$ with the lateral boundary $\Gamma_T = (0,T) \times \partial\Omega$, and under the hypotheses

$$\tau \geq \frac{1}{p^-}, \quad 1 \leq r < \tau p^- + 1. \tag{1}$$

The function $a_i : (0,T) \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and satisfy for almost every $(t,x) \in Q_T$, $\forall s \in \mathbb{R}$, the following:

$$\frac{c_1}{(1+|s|)^{\varrho_i}} \leq a_i(t,x,s) \leq c_2, \tag{2}$$

where c_1, c_2 are strictly positive real numbers and $\varrho_i \geq 0$. We recall that if (2) holds true, the differential operator

$$u \mapsto -\sum_{i=1}^{N}(D_i(a_i(t,x,u)|D_iu|^{p_i-2}D_iu)),$$

is not coercive as u becomes large. Degenerate coercivity suggests that when $|u|$ gets large, $\frac{c_1}{(1+|u|)^{\varrho_i}}$ goes towards zero. To address the issue of non-coercivity, the approach consists of approximating the operator by using truncations in a_i, the function that influences the coefficients of the differential operator. This approximation is aimed at making the operator coercive. The original problem (P) is approximated by a sequence of non-degenerate problems (P_n). This approximation likely simplifies the analysis by replacing the original problem with a series of related, easier-to-handle problems. The next step involves proving a priori estimates on the sequence of approximate solutions obtained from solving the (P_n) problems. These estimates provide bounds on the solutions, which are crucial for passing to the limit in the subsequent step. After establishing the a priori estimates, the passage to the limit is performed. This involves showing that as n tends to infinity, the solutions of the approximate problems converge to a solution of the original problem (P). This step requires careful analysis to ensure convergence. Depending on the initial data u_0, the existence of solutions to (P) is proven using different techniques, which are new compared to those in [2]-[16]. For $u_0 \in L^\infty(\Omega)$, the passage to the limit from the sequence of non-degenerate problems is employed. For $u_0 \in L^1(\Omega)$ is approximated by

$u_{0n} \in L^{\infty}(\Omega)$, and the results from the first case are used to establish the existence of solutions

$$u \in L^{q^-}(0,T;W_0^{1,q_i}(\Omega)), \quad 1 \leq q_i < p_i - \frac{N}{N+1},\ i = 1,\dots,N.$$

In the non degenerate isotropic case $p_i = p = 2, \quad \varrho_i = 0$ and for positive initial data, the existence of solutions to problem (P) is proved in [5] and if $\tau = 1/2$ on this problem

$$\begin{cases} \partial_t u - \triangle u + u^{2\tau-1}|\nabla u|^q = u^{r-1} & \text{in } Q_T = (0,T)\times\Omega \\ u(0,x) = u_0(x) \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_T = (0,T)\times\partial\Omega \end{cases} \tag{3}$$

Chipot and Weissler [10] examined the presence of non-global positive classical solutions, specifying conditions for blow-up given certain assumptions about p, q, N and Ω. In the work by Andreu [4],the global existence of non-negative initial data was demonstrated for the scenario where $q \geq p > 1$. However, in reference [19] it was observed that problem (1) does not have global classical solutions when $r > 2, \tau \geq 1/2$ and $2\tau + 2 < r$.

Mecheter and Mokhtari [14] studied degenerate coercivity with a singular lower-order term depending on the gradient, but their work was limited to isotropic operators. In contrast, handling anisotropic diffusion (the p_i-Laplacian) and proving regularity under weaker assumptions (Theorem 2.3) requires the application of the anisotropic Sobolev inequality (5).

In [12]–[16], the authors investigated similar anisotropic problems, but without lower-order terms. The inclusion of the term $|u|^{\tau p_i - 2}u|D_i u|^{p_i}$ necessitates new gradient estimates (Lemma 4.2) and a modified limit passage (Section 4.1).

Mecheter [13] focused on an isotropic weighted elliptic problem with variable exponents and L^1 data. However, the present work addresses the parabolic anisotropic case, introducing time-dependent truncations and compactness arguments (Lemma 3.4) that were unnecessary in the elliptic setting (as in [1]).

In recent years, research on anisotropic problems has grown significantly. This area is important because it has many real-world applications. Materials like liquid crystals, wood, and the Earth's crust often have different properties in different directions. Problem (P) helps model these natural behaviors. For instance, it can describe how fluids move in materials that conduct differently in various directions. In biology, it can model how diseases spread in uneven environments (see, for example, [6]-[8]).

The paper is structured as follows: In Section 2, we define the weak solution and focus on presenting the main results of the paper. Then, in Section 3, we demonstrate the existence of weak solutions u in the first case, where $u_0 \in L^{\infty}(\Omega)$. Section 4 examines (P) with initial data of $u_0 \in L^1(\Omega)$.

We assume here that the exponents q_i satisfy $\overline{q} < N$. Then

$$\overline{q}^* = \frac{N\overline{q}}{N-\overline{q}}, \quad \text{where} \quad \frac{1}{\overline{q}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{q_i}$$

and there are continuous embedding $W_0^{1,q_i}(\Omega) \hookrightarrow L^s(\Omega)$ for all $s \leq \overline{q}^*$, which turn out to be compact only when $s < \overline{q}^*$. Possible references on the theory of anisotropic Sobolev spaces are [3, 20]. For the convenience of the reader, let us recall some basic fact about some anisotropic Sobolev spaces.

Let Ω be a bounded smooth open subset of $\mathbb{R}^N$ and $p_i \geq 1$ for $i = 1, ..., N$. We introduce the anisotropic space

$$W^{1,p_i}(\Omega) = \{u \in L^{p_i}(\Omega) | D_i u \in L^{p_i}(\Omega)\},$$

which is a Banach space under the norm

$$\|u\|_{W^{1,p_i}(\Omega)} = \|u\|_{L^{p_i}(\Omega)} + \|D_i u\|_{L^{p_i}(\Omega)}, \; i = 1, ..., N. \tag{4}$$

We define also $W_0^{1,p_i}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm (4), its dual is denoted by $W^{-1,p_i'}(\Omega)$. Then one has the following Lemmas.

Lemma 1.1. [15] *There is $c_i > 0$ such that:*

$$\forall \varphi \in W_0^{1,p_i}(\Omega), \quad \|\varphi\|_{L^{p_i}} \leq c_i \|D_i \varphi\|_{L^{p_i}}$$

Lemma 1.2. [20] *There exists a positive constant C, depending only on Ω, such that for $u \in W_0^{1,(p_i)}(\Omega)$, $\overline{p} < N$, we have*

$$\|u\|_{L^s(\Omega)} \leq C \prod_{i=1}^{N} \|D_i u\|_{L^{p_i}(\Omega)}^{\frac{1}{N}}, \quad \forall s \in [1, \overline{p}^*] \tag{5}$$

where $s = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ with $\overline{p}$ given by $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$, if $\overline{p} \geq N$, the inequality (5) *is true for all $s \geq 1$, and C depends on s and $|\Omega|$.*

2. Statement of Main Results

Given a positive real number j and r, we define the functions $T_j(r)$ as $\max(-j, \min(r, j))$ and its primitive $\Theta_j : \mathbb{R} \to \mathbb{R}^+$ as follows:

$$\Theta_j(r) = \int_0^r T_j(t)dt = \begin{cases} \frac{r^2}{2}, & \text{if;} |r| \leq j \\ j|r| - \frac{j^2}{2}, & \text{if.} |r| > j \end{cases}$$

we will then use the following results

$$j|r| - \frac{j^2}{2} \leq \Theta_j(r) \leq j|r|, \quad \forall r \in \mathbb{R}. \tag{6}$$

Definition 2.1. Assume that (2),(1) hold true. We will call a weak solution of (P) a function $u \in L^1(0,T;W_0^{1,1}(\Omega)) \cap L^\infty(Q_T) \cap C([0,T];L^1(\Omega))$ such that $u^{\tau p_i - 1}|D_i u|^{p_i}$ belongs to $L^1(Q_T)$, $i = 1, \dots, N$, $u^{r-1} \in L^1(Q_T)$ and

$$\begin{aligned}
&-\int_0^T \int_\Omega u \partial_t \varphi dx dt - \int_\Omega \varphi(0,x) u_0(x) dx \\
&\quad + \sum_{i=1}^N \int_0^T \int_\Omega a_i(t,x,u)|D_i u|^{p_i-2} D_i u D_i \varphi dx dt \\
&\quad + \sum_{i=1}^N \int_0^T \int_\Omega u^{\tau p_i - 1}|D_i u|^{p_i} \varphi dx dt = \int_0^T \int_\Omega u^{r-1} \varphi dx dt,
\end{aligned} \tag{7}$$

for every $\varphi \in L^{p^-}(0,T;W_0^{1,p_i}(\Omega)) \cap L^\infty(Q_T)$ such that $\partial_t \varphi \in L^{p'^-}(0,T;W^{-1,p_i'}(\Omega)) + L^1(Q_T)$.

The first result deals with a given $u_0 \in L^\infty(\Omega)$.

Theorem 2.2. *Let $u_0 \in L^\infty(\Omega)$ and suppose that (1)-(2) hold true. Then, the problem (P) has at least one non-negative weak solution $u \in L^{p^-}(0,T;W_0^{1,p_i}(\Omega))$ $\cap L^\infty(Q_T) \cap C([0,T];L^1(\Omega))$ in the sense of Definition 2.1.*

The next result considers the case where $u_0 \in L^1(\Omega)$.

Theorem 2.3. *Let $u_0 \in L^1(\Omega)$ and suppose that (1)-(2) hold true. Then, the problem (P) has at least one non-negative weak solution u, i.e., a function $u \in L^{q^-}(0,T;W_0^{1,q_i}(\Omega)) \cap C([0,T];L^1(\Omega))$ in the sense of Definition 2.1, where*

$$1 \leq q_i < p_i - \frac{N}{N+1}, \quad i = 1, ..., N. \tag{8}$$

3. Proof of Theorems 2.2

Let $n \in \mathbb{N}$ be arbitrary, let us consider the following approximated problem (P_n^*)

$$\begin{cases}
\partial_t u_n - \sum_{i=1}^N (D_i(a_i(t,x,T_n(u_n))|D_i u_n|^{p_i-2} D_i u_n) + |u_n|^{\tau p_i - 2} u_n |D_i u_n|^{p_i}) \\
\quad = T_n(|u_n|^{r-2} u_n), & \text{in } Q_T \\
u_n(0,x) = u_0(x), & \text{in } \Omega \\
u_n = 0, & \text{on } \Gamma_T.
\end{cases}$$

Note that by (2) and $|T_n(u_n)| \leq n$, we have

$$a_i(t,x,T_n(u_n)) \geq \frac{c_1}{(1+n)^{\varrho^+}}, \text{ where } \varrho^+ = \max_{1 \leq i \leq N} \varrho_i,$$

so that the operator $B : v \mapsto \operatorname{div}(a_i(t,x,T_n(u_n))|D_i v_n|^{p_i-2} D_i v_n)$ is coercive. For the existence of the solution u_n in $L^{p^-}(0,T;W_0^{1,p_i}(\Omega)) \cap C([0,T];L^2(\Omega))$ of problem (P_n^*) is classical, see [11] for instance.

***Remark* 3.1.** *According to [9], and given the inequality* $|T_n(|u_n|^{r-2}u_n)| \leq n$*, where* $u_0 \in L^\infty(\Omega)$*, it can be easily verified that* $\psi(t) = nt + \|u_0\|_{L^\infty}$ *is a supersolution and* $\varphi(t) = 0$ *is a subsolution of problem* (P_n^*)*. As a result, there exists a weak solution* u_n *of* (P_n^*) *such that* $\varphi \leq u_n \leq \psi$ *in* Q_T*, and consequently,* u_n *belongs to* $L^\infty(Q_T)$ *and* $u_n \geq 0$*. Therefore,* u_n *solves the problem.*

$$\begin{cases} \partial_t u_n - \sum_{i=1}^N (D_i(a_i(t,x,T_n(u_n))|D_iu_n|^{p_i-2}D_iu_n) + u_n^{\tau p_i -1}|D_iu_n|^{p_i}) & \\ \quad = T_n(u_n^{r-1}), & \text{in } Q_T \\ u_n(0,x) = u_0(x), & \text{in } \Omega \\ u_n = 0, & \text{on } \Gamma_T. \end{cases} \tag{9}$$

We will use specific constants C or C_k that rely solely on the structure of d, p_i, τ, r, T, u_0, $|Q_T|$, for each k in the set of natural numbers.

Lemma 3.2. *Let* u_n *be a solution of problem* (9) *and suppose that* (1) *holds true with* $p_i \geq 1$, $i = 1, \ldots, N$*. Then, we have*

- *the sequence* $\{u_n\}$ *is bounded in* $L^\infty(Q_T)$*,*
- *the sequence* $\{T_j(u_n)\}$ *is bounded in* $L^{p^-}(0,T; W_0^{1,p_i}(\Omega))$*.*

Moreover, there exist a positive constant C *such that*

$$\|u_n^{\tau p_i -1}|D_iu_n|^{p_i}\|_{L^1(Q_T)} \leq C,\ i = 1, ..., N.$$

***Proof*.** Choosing $\varphi = u_n$ as test function in the approximated problem (9), we obtain, using (2)

$$\begin{aligned} \int_0^T \langle \partial_t u_n, u_n\rangle dt + c_1 \sum_{i=1}^N \int_0^T \int_\Omega \frac{|D_iu_n|^{p_i}}{(1+|T_n(u_n)|)^{\varrho_i}} dxdt & \\ + \sum_{i=1}^N \int_0^T \int_\Omega u_n^{\tau p_i}|D_iu_n|^{p_i} dxdt & \\ \leq \int_0^T \int_\Omega |T_n(u_n^{r-1})|u_n dxdt. & \end{aligned} \tag{10}$$

Because that $|T_n(u_n^{r-1})|u_n \leq u_n^r$ and

$$\int_0^T \langle \partial_t u_n, u_n\rangle dt = \frac{1}{2}\int_\Omega (u_n(T,x))^2 dx - \frac{1}{2}\int_\Omega (u_n(0,x))^2 dx,$$

Applying Lemma 1.1 and dropping positive term, we get

$$\begin{aligned} C_1 \int_0^T \int_\Omega u_n^{p^-(\tau+1)} dxdt &\le \sum_{i=1}^N \int_0^T \int_\Omega u_n^{p_i \tau} |D_i u_n|^{p_i} dxdt \\ &\le \int_0^T \int_\Omega u_n^r dxdt + \frac{1}{2} \int_\Omega u_0^2 dx, \end{aligned}$$

Since $r < \tau p^- + 1 \le p^-(\tau+1)$, it follows from Young inequality that

$$\int_0^T \int_\Omega u_n^{p^-(\tau+1)} dxdt \le C_2,$$

which implies that the boundedness of (u_n) in $L^{p^-(\tau+1)}(Q_T)$. Now, we prove that the sequence $(u_n^{r-1})_n$ is bounded in $L^1(Q_T)$. To this end, we choose $u_n^{\nu+1}$ as test function in (9) where $\nu > 0$, we find

$$\begin{aligned} c_1(\nu+1) \sum_{i=1}^N \int_0^T \int_\Omega \frac{|D_i u_n|^p}{(1+|T_n(u_n)|)^{\varrho_i}} u_n^\nu dxdt + \sum_{i=1}^N \int_0^T \int_\Omega u_n^{\tau p_i + \nu} |D_i u_n|^{p_i} dxdt \\ \le \int_0^T \int_\Omega |T_n(u_n^{r-1})| u_n^{\nu+1} dxdt + C. \end{aligned}$$

Dropping the non-negative term and by Lemma 1.1, we obtain

$$\int_0^T \int_\Omega u_n^{p^-(\tau+1)+\nu} dxdt \le \int_0^T \int_\Omega u_n^{r+\nu} dxdt + C. \tag{11}$$

Now, we can choose $\nu = p^-(\tau+1) - r$ ($\nu > 0$ since (1)), so $p^-(\tau+1) + \nu = 2p^-(\tau+1) - r$. Then by (11), we obtain that u_n is bounded in $L^{2p^-(\tau+1)-r}(Q_T)$. Consequently, an iterating procedure gives us that (u_n) is bounded in $L^m(Q_T)$ for all $m < +\infty$. Indeed, if we consider $\nu_1 > 0$ such that $r + \nu_1 = 2p^-(\tau + 1) - r$, by (11) and the fact that (u_n) is bounded in $L^{2p^-(\tau+1)-r}(Q_T)$, then it is bounded in $L^{4p^-(\tau+1)-3r}$. Now consider $\nu_2 > 0$ such that $r + \nu_2 = 4p^-(\tau+1) - 3r$ and deduce that (u_n) is bounded in $L^{8p^-(\tau+1)-7r}(Q_T)$. Hence we can obtain that (u_n) is bounded in $L^{2^s p^-(\tau+1)-(2^s-1)r}(Q_T)$ for all $s \in \mathbb{N}$. Since

$$2^s p^-(\tau+1) - (2^s - 1)r = 2^s(p^-(\tau+1) - r) + r \to +\infty, \quad \text{as } s \to +\infty,$$

we deduce that (u_n) is bounded in $L^m(Q_T)$ for all $m < +\infty$. Because there is $s' > 0$ such that $\frac{s'(p^-(\tau+1)-r)+r}{r} > \frac{N}{p^-} + 1$,

$$(u_n^r) \text{ is bounded in } L^m(Q_T) \text{ for some } m > \frac{N}{p^-} + 1, \tag{12}$$

witch implies that (u_n^r) is bounded in $L^1(Q_T)$. Using the nonnegativity of the lower order gradient term and solely the principal part of the operator, standard parabolic estimates (see, for instance, [7]) imply that $(u_n)_n$ is bounded in $L^\infty(Q_T)$. To prove that the sequence $(T_j(u_n))_n$ is bounded in $L^{p^-}(0,T;W_0^{1,p_i}(\Omega))$, we choose $T_j(u_n)$ as test function in (9) and the fact that $|T_n(u_n^{r-1})| \leq u_n^{r-1}$, we obtain

$$\begin{aligned}\int_\Omega \Theta_j(u_n)(T)dx + \sum_{i=1}^N \int_0^T \int_\Omega a_i(t,x,T_n(u_n))|D_iu_n|^{p_i-2}D_iu_nD_i(T_j(u_n))dxdt \\ + \sum_{i=1}^N \int_0^T \int_\Omega u_n^{\tau p_i-1}|D_iu_n|^{p_i}T_j(u_n)dxdt \\ \leq \int_0^T \int_\Omega u_n^{r-1}T_j(u_n)dxdt + \int_\Omega \Theta_j(u_n)(0)dx. \qquad (13)\end{aligned}$$

Since $\Theta_j \geq 0$ and $u_n^{\tau p_i-1}T_j(u_n) \geq 0$, so after dropping non-negative terms and using (6), we obtain

$$\begin{aligned}\sum_{i=1}^N \int_0^T \int_\Omega a_i(t,x,T_n(u_n))|D_iu_n|^{p_i-2}D_iu_nD_i(T_j(u_n))dxdt \\ \leq \int_0^T \int_\Omega u_n^{r-1}|T_j(u_n)|dxdt + j\|u_0\|_{L^1(\Omega)}.\end{aligned}$$

According to the conditions (2) and for $n > j > 0$, we get

$$\frac{c_1}{(1+j)^{\rho^+}} \sum_{i=1}^N \int_0^T \int_\Omega |D_iT_j(u_n)|^{p_i}dxdt \leq j\int_0^T \int_\Omega u_n^{r-1}dxdt + j\|u_0\|_{L^1(\Omega)}.$$

Using (12) and the fact that $u_0 \in L^\infty(\Omega)$, there exist a positive constant C such that

$$\|T_j(u_n)\|_{L^{p^-}(0,T;W_0^{1,p_i}(\Omega))} \leq C,\ i = 1, ..., N. \qquad (14)$$

Therefore by the bounded of u_n in $L^\infty(Q_T)$, the estimate (14) implies that

$$\|u_n\|_{L^{p^-}(0,T;W_0^{1,p_i}(\Omega))} \leq C, \qquad (15)$$

Moreover, if we take $\frac{T_j(u_n)}{j}$ as test function in (9) and dropping the non-negative terms, we obtain

$$\begin{aligned}\sum_{i=1}^N \int_0^T \int_\Omega u_n^{\tau p_i-1}|D_iu_n|^{p_i}\frac{T_j(u_n)}{j}dxdt \\ \leq \int_0^T \int_\Omega u_n^{r-1}\left|\frac{T_j(u_n)}{j}\right|dxdt + \frac{1}{j}\int_\Omega |\Theta_j(u_n)(0)|dx.\end{aligned}$$

Using $\left|\frac{T_j(u_n)}{j}\right| \leq 1$ and (6), we have

$$\int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \frac{T_j(u_n)}{j} dxdt \leq \int_0^T \int_\Omega u_n^{r-1} dxdt + \|u_0\|_{L^1(\Omega)}.$$

Letting j tend to 0 and by Fatous Lemma, we deduce

$$\|u_n^{\tau p_i - 1} |D_i u_n|^{p_i}\|_{L^1(Q_T)} \leq C. \tag{16}$$

☑

If we take n to be sufficiently large, we obtain $T_n(u_n^{r-1}) = u_n^{r-1}$ and $T_n(u_n) = u_n$, so we conclude that u_n is a weak solution of
(P_n)

$$\begin{cases} \partial_t u_n - \sum_{i=1}^N (D_i(a_i(t,x,u_n)|D_i u_n|^{p_i-2} D_i u_n) + u_n^{\tau p_i - 1} |D_i u_n|^{p_i}) = u_n^{r-1}, & \text{in } Q_T \\ u_n(0,x) = u_0(x) \geq 0, & \text{in } \Omega \\ u_n = 0, & \text{on } \Gamma_T. \end{cases}$$

Then, we have the weak formulation of (P_n) as follows

$$\begin{aligned} &-\int_0^T \int_\Omega u_n \partial_t \varphi dxdt - \int_\Omega \varphi(0,x) u_{0n}(x) dx \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega a_i(t,x,u_n) |D_i u_n|^{p_i-2} D_i u_n D_i \varphi \\ &\quad + \sum_{i=1}^N \int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \varphi dxdt = \int_0^T \int_\Omega u^{r-1} \varphi dxdt, \end{aligned} \tag{17}$$

for every $\varphi \in L^{p^-}(0,T; W_0^{1,p_i}(\Omega)) \cap L^\infty(Q_T)$. with $\partial_t \varphi \in L^{p'^-}(0,T; W^{-1,p_i'}(\Omega))$ $+L^1(Q_T)$.

***Remark* 3.3.** *We observe that Theorems 2.2-2.3 still hold even when the coefficients $a_i(t,x,u)$ are discontinuous in x, provided they are either piecewise Carathéodory or bounded measurable functions approximated by continuous ones. The crucial point is that the truncation technique employed in Lemma 3.2 does not rely on global continuity-measurability and boundedness suffice for the argument.*

Now, to pass to the limit, we have to prove the convergence almost everywhere of $D_i u_n$ to $D_i u$.

Lemma 3.4. *Let $u_n \in L^{p^-}(0,T; W_0^{1,p_i}(\Omega))$ is a weak solutions of problem* (9). *Then,*

$$D_i u_n \longrightarrow D_i u \text{ a.e. in } Q_T. \tag{18}$$

Proof. By Lemma 3.2, we have the sequence $\{u_n\}$ is bounded in the space $L^\infty(Q_T) \cap L^{p^-}(0,T;W_0^{1,p_i}(\Omega))$. Then, there exist a function $u \in L^\infty(Q_T) \cap L^{p^-}(0,T;W_0^{1,p_i}(\Omega))$ and a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^{p^-}(0,T;W_0^{1,p_i}(\Omega)). \tag{19}$$

Moreover, we have

$$\partial_t u_n \text{ is bounded in } L^{p'^-}(0,T;W^{-1,p'_i}(\Omega)) + L^1(Q_T), \quad p'_i = \frac{p_i}{p_i - 1},$$

using compactness argument in ([18] Corollary 4), we obtain that

$$u_n \to u \quad \text{strongly in} \quad L^1(Q_T), \text{ and a.e. in } Q_T. \tag{20}$$

Now, we prove that

$$D_i T_j(u_n) \to D_i T_j(u) \quad \text{strongly in} \quad L^{p_i}(Q_T) \quad \text{for all} \quad k \in \mathbb{N}. \tag{21}$$

We present a new time truncation regularization. We will approximate $T_j(u)$ by using the sequence $(T_j(u))_\nu$. When $\nu > 0$, the time regularization of the function $T_j(u)$ is defined by the following equation:

$$(T_j(u))_\nu(t,x) := \nu \int_{-\infty}^{t} e^{\nu(s-t)} T_j(u(s,x))ds + e^{-\nu t} T_j(u_0), \tag{22}$$

where $T_j(u(s,x))$ is the zero extension of u for $s < 0$ (See [21]). This regularization has the following properties:

- $|((T_j(u))_\nu)| \leq j$
- $(D_i T_j(u))_\nu \longrightarrow D_i T_j(u) \quad \text{strongly in} \quad L^{p_i}(Q_T) \text{ as } \nu \longrightarrow +\infty.$

Let $l > j$ and let us take

$$\varpi_n = T_{2j}(u_n - T_l(u_n) + T_j(u_n) - (T_j(u))_\nu),$$

as test function in (P_n), we have

$$\int_0^T \langle \partial_t u_n, \varpi_n \rangle dt + \sum_{i=1}^{N} \int_0^T \int_\Omega a_i(t,x,u_n)(|D_i u_n|^{p_i-2} D_i u_n) D_i \varpi_n dxdt$$
$$+ \sum_{i=1}^{N} \int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \varpi_n dxdt = \int_0^T \int_\Omega u_n^{r-1} \varpi_n dxdt. \tag{23}$$

We will also denote by $\tau(n,\nu,l)$ any quantity I such that

$$\lim_{l \longrightarrow +\infty} \lim_{\nu \longrightarrow +\infty} \lim_{n \longrightarrow +\infty} I = 0.$$

By the same technique of [17], we can obtain that

$$\int_0^T \langle \partial_t u_n, \varpi_n \rangle dt \geq \tau(n, \nu, l).$$

From the estimate above, we can express the inequality (23) as

$$\sum_{i=1}^{N} \int_0^T \int_\Omega a_i(t,x,u_n)|D_i u_n|^{p_i-2} D_i u_n D_i \varpi_n dxdt \\ + \sum_{i=1}^{N} \int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \varpi_n dxdt \leq \int_0^T \int_\Omega u_n^{r-1} \varpi_n dxdt - \tau(n,\nu,l).$$

Since $|T_j(u)| \leq j$, ϖ_n can be written as

$$\varpi_n = T_{l+j}(u_n - (T_j(u))_\nu) - T_{l-j}(u_n - T_j(u_n)).$$

By splitting the integral on the left side into the sets where $|u_n| \leq j$ and where $|u_n| > j$ and discarding some nonnegative terms, we can derive the following

$$\sum_{i=1}^{N} \int_0^T \int_\Omega a_i(t,x,u_n)|D_i u_n|^{p_i-2} D_i u_n D_i \varpi_n dxdt \\ \geq \frac{c_1}{(1+j)^{\varrho^+}} \sum_{i=1}^{N} \int_0^T \int_{\{|u_n| \leq j\}} |D_i T_j(u_n)|^{p_i-2} D_i T_j(u_n) D_i (T_j(u_n) - (T_j(u))_\nu) dxdt \\ - c_2 \sum_{i=1}^{N} \int_0^T \int_{\{|u_n| > j\}} ||D_i u_n|^{p_i-2} D_i u_n||D_i (T_j(u))_\nu| dxdt.$$

The inequality above indicates that

$$\frac{c_1}{(1+j)^{\varrho^+}} \sum_{i=1}^{N} \int_0^T \int_{\{|u_n| \leq j\}} |D_i T_j(u_n)|^{p_i-2} D_i T_j(u_n) D_i (T_j(u_n) - (T_j(u))_\nu) dxdt \\ + \sum_{i=1}^{N} \int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \varpi_n dxdt \leq c_2 \\ \sum_{i=1}^{N} \int_0^T \int_{\{|u_n| > j\}} |D_i u_n|^{p_i-1} |D_i (T_j(u))_\nu| dxdt \\ + \int_0^T \int_\Omega u_n^{r-1} \varpi_n dxdt - \tau(n,\nu,l).$$

Furthermore, we have

$$
\begin{aligned}
\frac{c_1}{(1+j)^{\varrho^+}} \sum_{i=1}^{N} \int_0^T \int_\Omega & [|D_i T_j(u_n)|^{p_i-2} D_i T_j(u_n) - |D_i T_j(u)|^{p_i-2} D_i T_j(u)] \\
& \times D_i(T_j(u_n) - (T_j(u))_\nu) dxdt \\
& \leq c_2 \sum_{i=1}^{N} \int_0^T \int_{\{|u_n|>j\}} |D_i u_n|^{p_i-1} |D_i (T_j(u))_\nu| dxdt \\
& - \frac{c_1}{(1+j)^{\varrho^+}} \sum_{i=1}^{N} \int_0^T \int_\Omega |D_i T_j(u)|^{p_i-2} D_i T_j(u) D_i (T_k(u_n) - (T_j(u))_\nu) dxdt \\
& - \sum_{i=1}^{N} \int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \varpi_n dxdt + \int_0^T \int_\Omega u_n^{r-1} \varpi_n dxdt - \tau(n,\nu,l) \\
& = I_1 + I_2 + I_3 + I_4 - \tau(n,\nu,l).
\end{aligned} \tag{24}
$$

Limit of I_1. By ensuring that $|D_i u_n|^{p_i-1}$ remains bounded in $L^{p'_i}(Q_T)$, and by invoking the dominated convergence theorem, we have

$$
\chi_{\{|u_n|>j\}} |D_i (T_j(u))_\nu| \longrightarrow \chi_{\{|u|>j\}} |D_i T_j(u)| \quad \text{strongly in} \quad L^{p_i}(Q_T),
$$

which is zero, as $n, \nu \to +\infty$. Thus, we obtain

$$
\lim_{\nu \longrightarrow +\infty} \lim_{n \longrightarrow +\infty} I_1 = 0. \tag{25}
$$

Limit of I_2. Since (14), we obtain $D_i T_j(u_n) \rightharpoonup D_i T_j(u)$ weakly in $L^{p_i}(Q_T)$ and the boundedness of $|D_i T_j(u)|^{p_i-2} D_i T_j(u)$ in $L^{p'_i}(Q_T)$, we obtain

$$
\lim_{\nu \longrightarrow +\infty} \lim_{n \longrightarrow +\infty} I_2 = 0. \tag{26}
$$

Limit of I_3 we have

$$
\begin{aligned}
-\sum_{i=1}^{N} \int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \varpi_n dxdt & \leq -\int_0^T \int_{\{0 \leq u_n \leq j\}} u_n^{\tau p_i - 1} |D_i T_j(u_n)|^{p_i} \varpi_n dxdt \\
& \leq C_{j,\tau}^{p_i} \int_0^T \int_\Omega |D_i T_j(u_n)|^{p_i} |\varpi_n| dxdt,
\end{aligned} \tag{27}
$$

where $C_{j,\tau}^{p_i}$ be a positive constant, such that $C_{j,\tau}^{p_i} = \max_{u_n \in [0,j]} u_n^{\tau p_i - 1}$. Given (14) and the inequality

$$
|D_i T_j(u_n)|^{p_i} |\varpi_n| \leq 2j \|D_i T_j(u_n)\|_{L^{p_i}(Q_T)} \in L^1(Q_T),
$$

Using (20), the definition of $(T_j(u))_\nu$ and the Lebesgue dominated convergence theorem, we have

$$
\lim_{l \longrightarrow +\infty} \lim_{\nu \longrightarrow +\infty} \lim_{n \longrightarrow +\infty} I_3 \leq C_{j,\tau}^{p_i} \lim_{l \longrightarrow +\infty} \int_0^T \int_\Omega |D_i T_j(u_n)|^{p_i} |T_{2j}(u - T_l(u))| dxdt = 0.
$$

Limit of I_4 By the properties of $(T_j(u))_\nu$ and (20), we find

$$|u_n^{r-1}\varpi_n| \leq 2j\|u_n^{r-1}\|_{L^\infty(Q_T)} \in L^1(Q_T),$$

also, by Lebesgue dominated convergence theorem,

$$\lim_{l\longrightarrow+\infty}\lim_{\nu\longrightarrow+\infty}\lim_{n\longrightarrow+\infty} I_4 = 0.$$

Now, passing to the limits in (24) as n, ν, l tend to infinity, we deduce that

$$\lim_{n\longrightarrow+\infty} A_{ni}^j = 0,$$

where

$$A_{ni}^j = \int_0^T\!\!\int_\Omega [|D_iT_j(u_n)|^{p_i-2}D_iT_j(u_n) - |D_iT_j(u)|^{p_i-2}D_iT_j(u)]D_i(T_j(u_n)-T_j(u))dxdt.$$

Using the following inequality $(|\xi|^{p_i-2}\xi - |\eta|^{p_i-2}\eta)(\xi-\eta) \geq 2^{2-p_i}|\xi-\eta|^p$ if $p_i \geq 2$ we find,

$$A_{ni}^j \geq 2^{2-p^+}\int_0^T\int_{\{x\in\Omega, p\geq 2\}} |D_i(T_j(u_n)-T_j(u))|^{p_i}dxdt.$$

If $1 < p_i < 2$ we have $(|\xi|^{p_i-2}\xi - |\eta|^{p_i-2}\eta)(\xi-\eta) \geq (p_i-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p_i}}$, so

$$\begin{aligned}
&\int_0^T\int_\Omega |D_i(T_j(u_n)-T_j(u))|^{p_i}\,dxdt \\
&= \int_0^T\int_\Omega \frac{|D_i(T_j(u_n)-T_j(u))|^{p_i}}{(|D_iT_j(u_n)|+|D_iT_j(u)|)^{\frac{p_i(2-p_i)}{2}}} \times (|D_iT_j(u_n)|+|D_iT_j(u)|)^{\frac{p_i(2-p_i)}{2}}\,dxdt \\
&\leq \Big\|\frac{|D_i(T_j(u_n)-T_j(u))|^{p_i}}{(|D_iT_j(u_n)|+|D_iT_j(u)|)^{\frac{p_i(2-p_i)}{2}}}\Big\|_{L^{\frac{2}{p_i}}([0,T]\times\Omega)} \\
&\qquad\qquad \times \|(|D_iT_j(u_n)|+|D_iT_j(u)|)^{\frac{p_i(2-p_i)}{2}}\|_{L^{\frac{2}{2-p_i}}([0,T]\times\Omega)} \\
&\leq \Big(\int_0^T\int_\Omega \frac{|D_i(T_j(u_n)-T_j(u))|^2}{\big(|D_iT_j(u_n)|+|D_iT_j(u)|\big)^{2-p_i}}\,dxdt\Big)^{\frac{p_i}{2}} \\
&\qquad\qquad \Big(\int_0^T\int_\Omega \big(|D_iT_j(u_n)|+|D_iT_j(u)|\big)^{p_i}\,dxdt\Big)^{\frac{2-p_i}{2}} \\
&\leq (p_i-1)^{-\frac{p_i}{2}}(A_{ni}^j)^{\frac{p_i}{2}}\Big(\int_0^T\int_\Omega \big(|D_iT_j(u_n)|+|D_iT_j(u)|\big)^{p_i}\,dxdt\Big)^{\frac{2-p_i}{2}}.
\end{aligned} \tag{28}$$

Since $A_{ni}^j \longrightarrow 0$ as $n \longrightarrow +\infty$ and $(T_j(u_n))_n$ is bounded in $L^{p^-}(0,T;W_0^{1,p_i}(\Omega))$ (see lemma 3.2), we obtain (21). So, we conclude that (up to subsequences)

$$D_iu_n \longrightarrow D_iu \quad \text{almost everywhere in} \quad Q_T. \tag{29}$$

☑

Lemma 3.5. *We have*

$$u_n^{\tau p_i-1}|D_iu_n|^{p_i} \longrightarrow u^{\tau p_i-1}|D_iu|^{p_i} \quad \text{strongly in} \quad L^1(Q_T). \tag{30}$$

Proof. In view of (18) and (20), we already know that this sequence converges a.e. in Q_T. So, on account of Vitali theorem, only the proof of the equi-integrability is necessary. To this end, we take again $\varphi = T_j(u_n)$ as a test function in problems (P_n) and after dropping the non-negative terms, we derive

$$\begin{aligned}\int_0^T\int_\Omega u_n^{\tau p_i-1}|D_iu_n|^{p_i}T_j(u_n)dxdt &\leq \sum_{i=1}^N\int_0^T\int_\Omega u_n^{\tau p_i-1}|D_iu_n|^{p_i}T_j(u_n)dxdt\\ &\leq \int_0^T\int_\Omega u_n^{r-1}|T_j(u_n)|dxdt + \int_\Omega \Theta_j(u_n)(0)dx.\end{aligned}$$

Taking into account that for any $K>0$, $0\leq |T_j(s)| \leq K + j\mathbf{1}_{s>K}$, $s\in\mathbb{R}^+$, we have

$$\int_0^T\int_\Omega u_n^{r-1}|T_j(u_n)|dxdt \quad \leq \quad K.C\|u_n\|^{r-1}_{L^{p_i}(Q_T)} + j\int_0^T\int_{u_n>K} u_n^{r-1}dxdt,$$

Consequently, since (15), we have

$$\begin{aligned}&\int_0^T\int_{u_n\geq j} u_n^{\tau p_i-1}|D_iu_n|^{p_i}dxdt = \frac{1}{j}\int_0^T\int_{u_n\geq j} u_n^{\tau p_i-1}|D_iu_n|^{p_i}T_j(u_n)dxdt\\ &\leq \frac{1}{j}\left(K.C + j\int_0^T\int_{u_n>K} u_n^{r-1}dxdt + j\int_\Omega \chi_{\{u_n>k\}}u_{0n}dx\right),\\ &\leq C\frac{k}{j} + \int_0^T\int_\Omega \chi_{\{u_n>K\}}u_n^{r-1}dxdt + \int_\Omega \chi_{\{u_n>k\}}u_{0n}dx.\end{aligned}$$

We take $k=\sqrt{j}$, we obtain

$$\int_0^T\int_{u_n\geq j} u_n^{\tau p_i-1}|D_iu_n|^{p_i}dxdt \rightarrow^{j\rightarrow+\infty} 0 \quad \text{uniformly with respect to } n, \tag{31}$$

then, there exist $j_0>1$, such that

$$\int_0^T\int_{u_n\geq j} u_n^{\tau p_i-1}|D_iu_n|^{p_i}dxdt \leq \frac{\varepsilon}{2} \quad \forall j\geq j_0,\ \forall n\in\mathbb{N}. \tag{32}$$

Let E be a measurable subset of Ω, then

$$\begin{aligned}\int_0^T\int_E u_n^{\tau p_i-1}|D_iu_n|^{p_i}dxdt \leq &\int_0^T\int_{E\cap\{u_n\geq j\}} u_n^{\tau p_i-1}|D_iu_n|^{p_i}dxdt\\ &+ C_{j,\tau}^{p_i}\int_0^T\int_{E\cap\{u_n\leq j\}}|D_iT_j(u_n)|^{p_i}dxdt.\end{aligned} \tag{33}$$

where $C_{j,\tau}^{p_i} = \max_{u_n \in [0,j]} u_n^{\tau p_i - 1}$. Moreover, since (21), there exist n_ε and δ_ε such that for every $E \subset \Omega$ with $meas(E) < \delta_\varepsilon$, we have

$$\int_0^T \int_{E \cap \{u_n \le j\}} |D_i T_j(u_n)|^{p_i} dxdt < \frac{\varepsilon}{2C_{j,\tau}^{p_i}}, \quad \forall n \ge n_\varepsilon.$$

Arguing this and from (32), (33) and taking $n \ge n_\varepsilon$, $j \ge j_0$ we see that

$$meas(E) < \delta_\varepsilon \Rightarrow \int_0^T \int_E u_n^{\tau p_i - 1} |D_i u_n|^{p_i} dxdt < \varepsilon.$$

So, this sequence is equi-integrable in Q_T. This proves (30). ☑

Lemma 3.6. *we have*

$$u_n \to u \quad \text{in} \quad C([0,T]; L^1(\Omega)). \tag{34}$$

Proof. We define the vector-valued function $\widehat{a}(t,x,s,\xi) : Q_t \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, where $\widehat{a}(t,x,s,\xi) = a_i(t,x,s)|\xi_i|^{p_i-2}\xi_i$. Taking $T_j(u_n - u_m)\mathbf{1}_{\{[0,t[\}}, (t \in [0,T])$ as test function in (P_n) for u_n and u_m, subtracting up both identities, we deduce that

$$\begin{aligned}\int_\Omega \Theta_j(u_n(t) - u_m(t))dx + \sum_{i=1}^N \int_0^t \int_\Omega [\widehat{a}(t,x,u_n,D_i u_n) - \widehat{a}(t,x,u_n,D_i u_m)] \\ D_i T_j(u_n - u_m) \\ + \sum_{i=1}^N \int_0^t \int_\Omega \left[u_n^{\tau p_i - 1}|D_i u_n|^{p_i} - u_m^{\tau p_i - 1}|D_i u_m|^{p_i}\right] T_j(u_n - u_m) dxdt \\ \le \int_0^t \int_\Omega |u_n^{r-1} - u_m^{r-1}||T_j(u_n - u_m)|dxdt + \int_\Omega |\Theta_j(u_n(0) - u_m(0))|dx.\end{aligned}$$

Note that I_n^m is well de?ned and $I_n^m \ge 0$, where

$$I_n^m = \int_0^t \int_\Omega [\widehat{a}(t,x,u_n,D_i u_n) - \widehat{a}(t,x,u_n,D_i u_m)] D_i(u_n - u_m) \ge 0.$$

Dropping non-negative terms, we get

$$\begin{aligned}\int_\Omega \Theta_j(u_n(t) - u_m(t))dx \le j \sum_{i=1}^N \int_0^t \int_\Omega \left|u_n^{\tau p_i - 1}|D_i u_n|^{p_i} - u_m^{\tau p_i - 1}|D_i u_m|^{p_i}\right| dxdt \\ + j \int_0^t \int_\Omega |u_n^{r-1} - u_m^{r-1}|dxdt + j \int_\Omega |(u_n(0) - u_m(0)|dx.\end{aligned}$$

Next, we divide this inequality by j and let $j \to 0$. Since (6), we obtain

$$\sup_{t\in[0,T]} \int_\Omega |u_n(t) - u_m(t)|dx \le \int_0^T \int_\Omega \left|u_n^{\tau p_i - 1}|D_i u_n|^{p_i} - u_m^{\tau p_i - 1}|D_i u_m|^{p_i}\right| dxdt + \int_0^T \int_\Omega |u_n^{r-1} - u_m^{r-1}|dxdt + \int_\Omega |(u_n(0) - u_m(0)|dx.$$

Taking into account (30), we deduce that (u_n) is a Cauchy sequence in $C([0,T]; L^1(\Omega))$. Consequently (34) holds true.

To finish the proof of the Theorem 2.2, by (2), (19), (20) and (29) together with Vitali?s Theorem, we arrive at

$$a_i(t,x,u_n)|D_i u_n|^{p_i-2} D_i u_n \to a_i(t,x,u)|D_i u|^{p_i-2} D_i u \quad \text{weakly in} \quad L^{p_i'}(Q_T).$$

Arguing this and the convergence (12),(30), we can pass to the limit for $n \longrightarrow +\infty$ in the weak formulation (17), we obtain that u is a weak solution for (P). ☑

4. Proof of Theorems 2.3

In Ω, let (u_{0n}), where $u_{0n} = T_n(u_0) \ge 0$, be a sequence of bounded functions. This sequence converges to u_0 in $L^1(\Omega)$. The existence of the approximate solution u_n of problems (P_n) is guaranteed by Theorem 2.2. Thus, $0 \le u_n \in L^{p^-}(0,T; W_0^{1,p_i}(\Omega)) \cap L^\infty(Q_T)$. Additionally, it satisfies the weak formulation (17). We will utilize the function defined by $G_j(r) = r - T_j(r)$ for every $r \in \mathbb{R}$.

Lemma 4.1. *Let u_0 be in $L^1(\Omega)$. Assume that* (1) *is true and let the sequence* (u_n) *be a solution that satisfies* (17). *Then we get*

- *the sequence* $\{u_n^{r-1}\}$ *is bounded in* $L^1(Q_T)$,
- *the sequence* $\{u_n\}$ *is bounded in* $L^\infty(0,T; L^1(\Omega))$.

***Proof*.** Take $\varphi = T_j(u_n)$ as test function in the weak formulation (17) and dropping the non negative term (see (13)), we have

$$\int_0^T \int_\Omega u_n^{\tau p^- -1}|D_i u_n|^{p^-} T_j(u_n)dxdt \le \sum_{i=1}^N \int_0^T \int_\Omega u_n^{\tau p_i -1}|D_i u_n|^{p_i} T_j(u_n)dxdt \le \int_0^T \int_\Omega u_n^{r-1} T_j(u_n)dxdt + \int_\Omega \Theta_j(u_n)(0)dx. \tag{35}$$

We can write the first term in the right hand of above inequality as

$$\int_0^T \int_\Omega u_n^{r-1} T_j(u_n)dxdt = \int_0^T \int_{\{u_n \le j\}} u_n^r dxdt + k \int_0^T \int_{\{u_n > j\}} u_n^{r-1} dxdt \le C_1 + C_2 \int_0^T \int_{\{u_n > j\}} (u_n - j)^{r-1} dxdt,$$

so, that

$$\int_0^T\int_{\{u_n>j\}}(G_j(u_n)+j)^{\tau p^- -1}|D_iu_n|^{p^-}dxdt \leq \int_0^T\int_\Omega u_n^{\tau p^- -1}|D_iu_n|^{p^-}T_j(u_n)dxdt$$
$$\leq C_3+C_4\int_0^T\int_{\{u_n>j\}}(G_j(u_n))^{r-1}dxdt+\|u_0\|_{L^1(\Omega)}.$$

where $G_j(u_n)=u_n-T_j(u_n)=u_n-j$ on the set $\{u_n>j\}$. Using that

$$(G_j(u_n)+j)^{\tau p^- -1}|D_iu_n|^{p^-}=\left(1+\frac{\tau p^- -1}{p^-}\right)^{-p^-}\left|D_i(G_j(u_n)+j)^{1+\frac{\tau p^- 1}{p^-}}\right|^{p^-},$$

which yields

$$\left(1+\frac{\tau p^- -1}{p^-}\right)^{-p^-}\int_0^T\int_{\{u_n>j\}}\left|D_i(G_j(u_n)+j)^{1+\frac{\tau p^- -1}{p^-}}\right|^{p^-}dxdt$$
$$\leq C_3+C_4\int_0^T\int_{\{u_n>j\}}(G_j(u_n))^{r-1}dxdt+C_0. \tag{36}$$

Now, Poincaré inequality implies

$$\int_0^T\int_{\{u_n>j\}}(G_j(u_n)+j)^{p^-(\tau+1)-1}dxdt\leq C_5+C_6\int_0^T\int_{\{u_n>j\}}(G_j(u_n))^{r-1}dxdt.$$

On the other hand, as we have $r\leq p^-\tau+1\leq p^-(\tau+1)$, we can use Young's inequality, then

$$\int_0^T\int_{\{u_n>j\}}|G_j(u_n)|^{p^-(\tau+1)-1}dxdt\leq C_7. \tag{37}$$

Therefore,

$$\int_0^T\int_{\{u_n>j\}}u_n^{r-1}dxdt=\int_0^T\int_{\{u_n>j\}}(G_j(u_n)+j)^{r-1}dxdt\leq C_8. \tag{38}$$

Hence, the sequence $\{u_n^{r-1}\}$ is bounded in $L^1(Q_T)$. By (6) and dropping non-negative terms in (13), we have

$$\sup_{t\in[0,T]}\int_\Omega u_n(t,x)dx\leq C. \tag{39}$$

This ends the proof of Lemma. ☑

Lemma 4.2. *The sequence $\{D_iu_n\}$ of solutions of problems (P_n) is bounded in $L^{q_i}(Q_T)$, where the exponent q_i defined as in (8).*

Proof. Similar to (13), by using (2), we obtain

$$\sum_{i=1}^{N}\int_0^T\int_\Omega \frac{|D_iT_j(u_n)|^{p_i}}{(1+u_n)^{\varrho^+}}dxdt \le j\int_0^T\int_{\{u_n>j\}} u_n^{r-1}dxdt + C_9.$$

From (38), we deduce that

$$\int_0^T\int_{\{u_n\le j\}}|D_iT_j(u_n)|^{p_i}dxdt = \int_0^T\int_{\{u_n\le j\}}\frac{|D_iT_j(u_n)|^{p_i}}{(1+u_n)^{\varrho^+}}.(1+u_n)^{\varrho^+}dxdt$$
$$\le (1+j)^{\varrho^+}\sum_{i=1}^{N}\int_0^T\int_\Omega\frac{|D_iT_j(u_n)|^{p_i}}{(1+u_n)^{\varrho^+}}dxdt \le (1+j)^{\varrho^+}\Big(jC_8+C_9\Big). \tag{40}$$

Let us write

$$\int_0^T\int_\Omega |D_iu_n|^{q_i}dxdt = \int_0^T\int_{\{u_n>j\}}|D_iu_n|^{q_i}dxdt + \int_0^T\int_{\{u_n\le j\}}|D_iu_n|^{q_i}dxdt \tag{41}$$

By (40) and since $q_i < p_i$, we have

$$\int_0^T\int_{\{u_n\le j\}}|D_iu_n|^{q_i}dxdt \le C_{10}. \tag{42}$$

For the first integral in the right hand of (41). We can assume that $\frac{q_i}{p_i} = \frac{\overline{q}}{\overline{p}}$. If not, we set $\gamma = \max\{q_i/p_i,\ i = 1,\dots,N\}$ and replace q_i by γp_i. Observe that, because $\gamma p_i \ge q_i$, the fact that (D_iu_n) is bounded in $L^{\gamma p_i}(Q_T)$ implies the result. From now on, we set $q_i = \gamma p_i$, $\gamma = \overline{q}/\overline{p}$ with $0<\gamma<1$. Applying Hölder's inequality for $\frac{p_i}{q_i} > 1$ and according (35) and (38), we obtain

$$\int_0^T\int_{\{u_n>j\}}u_n^{\tau q_i}|D_iu_n|^{q_i}dxdt = \int_0^T\int_{\{u_n>j\}}u_n^{\tau q_i}|D_iu_n|^{q_i}u_n^{\frac{-q_i}{p_i}}u_n^{\frac{q_i}{p_i}}dxdt$$
$$\le \left(\int_0^T\int_{\{u_n>j\}}u_n^{\tau p_i-1}|D_iu_n|^{p_i}dxdt\right)^{\gamma}\left(\int_0^T\int_{\{u_n>j\}}u_n^{\frac{\gamma}{1-\gamma}}dxdt\right)^{1-\gamma}$$
$$\le C_{11}\left(\int_0^T\int_{\{u_n>j\}}u_n^{d}dxdt\right)^{1-\gamma},\quad d=\overline{q}\left(\frac{N+1}{N}\right),$$

so, that

$$j^{\tau q_i}\int_0^T\int_{\{u_n>j\}}|D_iu_n|^{q_i}dxdt \le C_{12}\left(\int_0^T\int_{\{u_n>j\}}u_n^{d}dxdt\right)^{1-\gamma}.$$

witch implies that

$$\left(\int_0^T \int_{\{u_n>j\}} |D_i u_n|^{q_i} dxdt\right)^{1/q_i} \leq C_{13} \left(\int_0^T \int_{\{u_n>j\}} u_n^d dxdt\right)^{\frac{1-\gamma}{q_i}}. \tag{43}$$

Use the following interpolation argument and (39), we can write

$$\begin{aligned} \|u_n(\cdot,t)\|_{L^d(\{u_n>j\})} &\leq \|u_n(\cdot,t)\|^{1-\theta}_{L^1(\{u_n>j\})} \|u_n(\cdot,t)\|^{\theta}_{L^{\overline{q}^*}(\{u_n>j\})} \\ &\leq C_{14} \|u_n(\cdot,t)\|^{\theta}_{L^{\overline{q}^*}(\{u_n>j\})}. \end{aligned} \tag{44}$$

Integrating (44) on $[0,T]$, we obtain

$$\int_0^T \int_{\{u_n>j\}} u_n^d dxdt \leq C_{15} \int_0^T \|u_n(\cdot,t)\|^{d\theta}_{L^{\overline{q}^*}(\{u_n>j\})} dt = C_{13} \|u_n\|^{\overline{q}}_{L^{\overline{q}}(0,T;L^{\overline{q}^*}(\{u_n>j\}))}. \tag{45}$$

with $\theta = \frac{(1-d)\overline{q}^*}{(1-\overline{q}^*)d} = \frac{N}{N+1}$ and $\overline{q}^* = \frac{N\overline{q}}{N-\overline{q}}$ if $\overline{q} < N$ and $\overline{q}^* > 1$ satisfying $\theta.d = \overline{q}$ otherwise. Using Sobolev inequality (5), we get

$$\|u_n(t)\|_{L^{\overline{q}^*}(\{u_n>j\})} \leq C_{16} \prod_{i=1}^N \left(\|D_i u_n\|_{L^{q_i}(\{u_n>j\})}\right)^{\frac{1}{N}},$$

where

$$\frac{1}{\overline{q}} = \frac{1}{N} \sum_{i=1}^N \frac{1}{q_i}, \quad \left(\overline{q} < \overline{p} - \frac{N}{N+1}\right).$$

So that

$$\int_0^T \|u_n(t)\|^{\overline{q}}_{L^{\overline{q}^*}(\{u_n>j\})} dt \leq C_{17} \int_0^T \prod_{i=1}^N \left(\int_{\{u_n>j\}} |D_i u_n|^{q_i} dx\right)^{\frac{\overline{q}}{Nq_i}} dt.$$

The fact that $\sum_{i=1}^N \frac{\overline{q}}{Nq_i} = 1$, by the generalized Hölder inequality, lead to

$$\|u_n\|_{L^{\overline{q}}(0,T;L^{\overline{q}^*}(\{u_n>j\}))} \leq C_{18} \prod_{i=1}^N \left(\int_0^T \int_{\{u_n>j\}} |D_i u_n|^{q_i} dxdt\right)^{\frac{1}{Nq_i}}. \tag{46}$$

The combination of (46), (43) and (45), it gives that

$$\begin{aligned} \prod_{i=1}^N \left(\int_0^T \int_{\{u_n>j\}} |D_i u_n|^{q_i} dxdt\right)^{1/q_i} &\leq C_{19} \|u_n\|^{N\left(1-\frac{\overline{q}}{\overline{p}}\right)}_{L^{\overline{q}}(0,T;L^{\overline{q}^*}(\Omega))} \\ &\leq C_{20} \prod_{i=1}^N \left(\int_0^T \int_{\{u_n>j\}} |D_i u_n|^{q_i} dxdt\right)^{\frac{1}{q_i}\left(1-\frac{\overline{q}}{\overline{p}}\right)}. \end{aligned} \tag{47}$$

We get the desired results using (47) and the fact that $1 - \frac{\overline{q}}{\overline{p}} \in (0,1)$. ☑

4.1. Passage to the limit and end proof of Theorem 2.3

By Lemmas 4.1-4.2, we have the sequence $\{u_n\}$ is bounded in the space $L^\infty(0,T;L^1(\Omega)) \cap L^{q^-}(0,T;W_0^{1,q_i}(\Omega))$. Then, there exist a function $u \in L^\infty(0,T;L^1(\Omega)) \cap L^{q^-}(0,T;W_0^{1,q_i}(\Omega))$ and a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^{q^-}(0,T;W_0^{1,q_i}(\Omega)), \tag{48}$$

Moreover, we have

$$u_n \to u \quad \text{strongly in} \quad L^{q_i}(Q_T), \text{ and a.e. in } Q_T, \tag{49}$$

and

$$D_i u_n \longrightarrow D_i u \text{ a.e. in } Q_T. \tag{50}$$

By (49), we have the almost everywhere convergence:

$$u_n^{r-1} \to u^{r-1} \quad \text{a.e. in } Q_T.$$

In addition, we need to show that the sequence (u_n^{r-1}) is equi-integrable on Q_T. From (37)-(38), the condition $r-1 < p^-(\tau+1)-1$, and Hölder's inequality, we deduce the equi-integrability of (u_n^{r-1}).

Consequently, Vitali's convergence theorem yields the strong convergence:

$$u_n^{r-1} \longrightarrow u^{r-1} \quad \text{strongly in } L^1(Q_T). \tag{51}$$

By (2) and from the convergence (48),(50), we get

$$a_i(t,x,u_n)|D_iu_n|^{p_i-2}D_iu_n \to a_i(t,x,u)|D_iu|^{p_i-2}D_iu, \text{ a.e. in } Q_T,\ i=1,...,N. \tag{52}$$

and

$$u_n^{\tau p_i-1}|D_iu_n|^{p_i} \to u^{\tau p_i-1}|D_iu|^{p_i}, \text{ a.e. in } Q_T,\ i=1,...,N. \tag{53}$$

From (2), (52), Lemma 4.2 and the Vitali's Theorem, we derive for all $i = 1,...,N$

$$a_i(t,x,u_n)|D_iu_n|^{p_i-2}D_iu_n \to (a_i(t,x,u)|D_iu|^{p_i-2}D_iu, \text{ strongly in } L^{\sigma_i}(Q_T), \tag{54}$$

for all

$$1 \le \sigma_i < \frac{1}{p_i-1}\left(p_i - \frac{N}{N+1}\right).$$

Using (16),(53), we get for $i=1,...,N$

$$u_n^{\tau p_i-1}|D_iu_n|^{p_i} \to u^{\tau p_i-1}|D_iu|^{p_i}, \text{ weakly in } L^1(Q_T). \tag{55}$$

Finally, for $\varphi \in L^{p^-}(0,T;W_0^{1,p_i}(\Omega)) \cap L^\infty(Q_T)$, we have

$$\begin{aligned} -\int_0^T \int_\Omega u_n \partial_t \varphi dxdt - \int_\Omega \varphi(0,x) u_{0n}(x) dx \\ + \sum_{i=1}^N \int_0^T \int_\Omega a_i(t,x,u_n)|D_i u_n|^{p_i-2} D_i u_n D_i \varphi \\ + \sum_{i=1}^N \int_0^T \int_\Omega u_n^{\tau p_i - 1} |D_i u_n|^{p_i} \varphi dxdt = \int_0^T \int_\Omega u^{r-1} \varphi dxdt, \end{aligned} \tag{56}$$

Using (51),(54) and (55), we can easily pass to the limit in (56), to see that this last integral identity is true for u instead of u_n. This proves Theorem 2.3.

5. Concluding Remark and Outlook

This work establishes the existence and regularity of non-negative weak solutions for an anisotropic degenerate parabolic problem with a lower-order gradient term and a source. The main contributions are: the introduction of a truncation method to handle degenerate coercivity in the anisotropic setting, where the diffusion coefficients $a_i(t,x,u)$ may decay as $|u| \to \infty$, the derivation of a priori estimates and compactness results under minimal assumptions on the initial data $u_0 \in L^1(\Omega)$ or $u_0 \in L^\infty(\Omega)$, the proof of almost everywhere convergence of gradients using a novel time-regularization technique and careful passage to the limit, and the extension of previous results (e.g., from isotropic or non-degenerate cases) to the anisotropic degenerate setting with a nonlinear gradient term.

The results are valid under the structural conditions:

$$\tau \geq \frac{1}{p^-}, \quad 1 \leq r < \tau p^- + 1,$$

which ensure the balance between the source term and the gradient absorption. The proofs rely on anisotropic Sobolev embeddings, monotonicity arguments, and Vitali convergence theorems.

5.1. Outlook for Further Work

Several directions for future research arise naturally from this work:

Firstly, the question of uniqueness of weak solutions remains open, especially in the case of L^1 initial data. Stability with respect to initial conditions and parameters could also be investigated. Secondly, asymptotic properties such as convergence to steady states, decay estimates, or blow-up phenomena over large time scales are of great interest. Thirdly, extending the results to the case where $p_i = p_i(x)$ or $p_i = p_i(t,x)$ would further generalize the model and reflect more realistic applications. Fourthly, the case of non-zero Dirichlet or

Neumann boundary conditions could be studied, possibly with measure data. In addition, developing and analyzing numerical schemes for such degenerate anisotropic problems would be valuable for practical simulations. Finally, applying the results to specific models in physics, biology, or engineering such as fluid dynamics in anisotropic media, image processing, or population dynamics would demonstrate the utility of the theoretical framework.

This work lays a solid foundation for the analysis of degenerate anisotropic parabolic equations with gradient-dependent terms and sources. The techniques developed here may also be adapted to other classes of nonlinear partial differential equations with similar structural features.

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