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Renormalized Solutions for Nonlinear and Non-Coercive Elliptic Problems Involving Hardy Potentials and L^1-Data

Soluciones renormalizadas para problemas elípticos no lineales y no coercitivos que involucran potenciales Hardy y funciones de tipo L1

Bouchaib Ferrahi✉, Hassane Hjiaj, Rajae Zerouali

Abdelmalek Essaadi University, Tetouan, Morocco

Abstract. This work is centered around the study of the following non-coercive elliptic problem :

$$\begin{cases} Au + g(x,u,\nabla u) = f(x) + \frac{|u|^{p_0-2}u}{|x|^{p_0}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

In the anisotropic Sobolev space, where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ that includes the origin, with $g(x,s,\xi)$ subject to certain growth conditions and $f \in L^1(\Omega)$. We prove the existence of renormalized solutions for the strongly nonlinear and non-coercive elliptic Dirichlet problem. Furthermore, we establish several regularity results.

Key words and phrases. Anisotropic Sobolev spaces, strongly nonlinear elliptic equation, non-coercive problems, renormalized solutions.

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Resumen. Este trabajo se centra en el estudio del siguiente problema coercitivo elíptico:

$$\begin{cases} Au + g(x,u,\nabla u) = f(x) + \frac{|u|^{p_0-2}u}{|x|^{p_0}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

en un espacio anisotrópico de Sobolev, donde Ω es un subconjunto abierto acotado de $\mathbb{R}^N$ $(N \geq 2)$ que incluye el origen, donde $g(x,s,\xi)$ está sujeto a restricciones relacionadas a su crecimiento y $f \in L^1(\Omega)$. Probamos la existencia de soluciones renormalizadas para el problema elíptico de Dirichlet. Más aún, probamos varios resultados acerca de la regularidad de las soluciones.

Palabras y frases clave. Espacios anisotrópicos de Sobolev, ecuaciones elípticas fuertemente no lineales, problemas no coercitivos, soluciones renormalizadas.

1. Introduction

In [1], B. Abdellaoui and al. studied the following nonlinear elliptic problem:

$$\begin{cases} -\Delta u \pm |\nabla u|^2 = \lambda.\frac{u}{|x|^2} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where λ is a positive real number. They have proved the existence of positives solutions for the problem (1) in the case $(+|\nabla u|^2)$, with $f \in L^1(\Omega)$ that is the absorption case. Note that for the reaction case $(-|\nabla u|^2)$ the non-existence of solutions is proved even in a very weak sense.

In [15], M. M. Porzio investigated the existence of weak solutions for the following quasilinear elliptic problem:

$$\begin{cases} -div(M(x,u)\nabla u) + \gamma|u|^{p-1}u = a.\dfrac{u}{|x|^2} + f(x) - divF & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

here p is greater than $\frac{N}{N-2}$, the element a is a positive constant and the Caratheodory function $M(x,s)$ satisfies the growth and coercivity conditions. For further details, we refer the reader to [4].

Recently, there has been growing interest among scientists and mathematicians in the anisotropic Sobolev spaces $W^{1,\vec{p}}(\Omega)$. This interest stems from their relevance in studying nonhomogeneous materials, which exhibit different behaviors across various spatial directions. These spaces are particularly applicable in fields such as electrophysiology and thermoelectric fluid dynamics. For more information, we refer to [3] and [13].

Many authors considered special cases of such problems, for example, in [9], R. Di Nardo and F. Feo examined the following quasilinear elliptic problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial_i a_i(x,u,\nabla u) + \sum_{i=1}^{N} H_i(x,\nabla u) = f - \sum_{i=1}^{N} \partial_i g_i & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

They proved both the existence and uniqueness of weak solutions for this anisotropic elliptic Dirichlet problem, assuming that the data resides in the dual space.

E. Azroul et al. studied the following anisotropic quasilinear elliptic Dirichlet problem, as detailed in [5].

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f + \lambda.\frac{|u|^{p_0(x)-2}u}{|x|^{p_0(x)}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

Where $Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u)$ is a Leray-Lions operator satisfying the standard growth, monotonicity, and coercivity conditions, and $f \in L^1(\Omega)$ with λ being a positive real number. The authors proved the existence of an entropy solution for the quasilinear elliptic problem (4).

This paper focuses on the study of a nonlinear and non-coercive elliptic Dirichlet problem, formulated as follows:

$$\begin{cases} -\sum_{i=1}^{N} D^i(a_i(x,u,\nabla u)) + g(x,u,\nabla u) = f(x) + \dfrac{|u|^{p_0-2}u}{|x|^{p_0}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where the strongly nonlinear term $g(x,s,\xi)$ is a Caratheodory function satisfying certain growth conditions, and f belongs to $L^1(\Omega)$.

The aim of this paper is to prove the existence of renormalized solutions for certain nonlinear and non-coercive elliptic problems of type (5) in anisotropic Sobolev spaces by using an approximation procedure and a priori estimates.

This paper is organized as follows: First, in Section 2, we recall some definitions and basic properties concerning anisotropic Sobolev spaces. In Section 3, we present some non-standard assumptions on the Carathéodory functions $a_i(x,s,\xi)$ and $g(x,s,\xi)$ for which our nonlinear elliptic problem has at least one renormalized solution. In Section 4, we state the main results and prove the existence of renormalized solutions for our nonlinear elliptic problem. Additionally, some regularity results are concluded.

2. Preliminaries

Let Ω be a bounded open subset in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$.

Let $p_1, \ldots, p_N$ be N be real constants numbers, which satisfy $1 < p_i < \infty$ for $i = 1, \ldots, N$.

We denote

$$\vec{p} = (1, p_1, \ldots, p_N), \qquad D^0 u = u \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for} \qquad i = 1, \ldots, N,$$

and we define

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad \underline{p}^+ = \max\{p_1, p_2, \ldots, p_N\}.$$

We recall the définition of the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$:

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \;\; \text{for} \;\; i = 1, 2, \ldots, N \right\},$$

equipped with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{6}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|u\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (see [14]). Recall that $W_0^{1,\vec{p}}(\Omega)$ is defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ with respect to the norm (6).

We now revisit the Poincaré and Sobolev type inequalities in the anisotropic Sobolev spaces.

Proposition 2.1. *Let $u \in W_0^{1,\vec{p}}(\Omega)$, we have*

(i) *Poincaré inequality : there exists a real constant number $C_p > 0$, such that the following inequality holds*

$$\|u\|_{L^{p_i}(\Omega)} \leq C_p \, \|D^i u\|_{L^{p_i}(\Omega)} \qquad \text{for any} \quad i = 1, \ldots, N.$$

(ii) *Sobolev inequality : there exists a second real constant number $C_s > 0$, such that the following inequality holds*

$$\|u\|_{L^q(\Omega)} \leq \frac{C_s}{N} \sum_{i=1}^{N} \Big\| \frac{\partial u}{\partial x_i} \Big\|_{L^{p_i}(\Omega)},$$

with

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \qquad \text{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N - \overline{p}} & \text{if} \quad \overline{p} < N \\ q \in [1, +\infty[& \text{if} \quad \overline{p} \geq N \end{cases}$$

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we define*

$$s = \max(q, \max_{1 \leq i \leq N} p_i),$$

then, the following embedding hold :

- *if $\overline{p} < N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, s[$,*
- *if $\overline{p} = N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ is compact for any $r \in [1, +\infty[$,*
- *if $\overline{p} > N$ then the embedding $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ is compact.*

The proof of this lemma follows from the Proposition 2.1.

Proposition 2.3. *The space $W^{-1,\vec{p'}}(\Omega)$ denotes the dual of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$, here $\vec{p'} = (s', p'_1, \ldots, p'_N)$ and $s = \max(q, \underline{p}^+)$.*

For every $F \in W^{-1,\vec{p'}}(\Omega)$, there exist $F_0 \in L^{s'}(\Omega)$ and $F_i \in L^{p'_i}(\Omega)$ for $i = 1, \ldots, N$, satisfying the following equality : $F = F_0 - \sum_{i=1}^{N} D^i F_i$. Furthermore, *for every* $u \in W_0^{1,\vec{p}}(\Omega)$ *we have*

$$\langle F, u\rangle = \sum_{i=0}^{N} \int_{\Omega} F_i \, D^i u \, dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p'}} = \inf\Big\{ \textstyle\sum_{i=0}^{N} \|F_i\|_{p'_i} \quad / \quad F = F_0 - \sum_{i=1}^{N} D^i F_i \quad \text{with} \quad F_0 \in L^{s'}(\Omega) \quad \text{and} \quad F_i \in L^{p'_i}(\Omega)\Big\}.$$

Definition 2.4. Let k be a positive real number. We define the truncation function $T_k(.) : I\!R \longmapsto I\!R$, by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \le k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we set

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \{u : \Omega \mapsto \mathbb{R} \text{ measurable } \; / \; T_k(u) \in W_0^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.5. *Let* u *be an element of the space* $\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. *For any* $i \in \{1, \ldots, N\}$, *there exists an unique measurable function* $v_i : \Omega \mapsto \mathbb{R}$ *which is measurable and satisfies :*

$$\forall k > 0 \qquad D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \qquad a.e. \quad x \in \Omega,$$

For a measurable subset A, *let* χ_A *denote its characteristic function. The functions* v_i *are referred to as the weak partial derivatives of* u *and are denoted* $D^i u$. *Moreover, if* u *is in* $W_0^{1,1}(\Omega)$, *then* v_i *matched with the standard distributional derivative of* u, *in other world,* $v_i = D^i u$.

The proof of Proposition 2.5 uses the standard methods outlined in [8] for the case of Sobolev spaces. For further information on anisotropic Sobolev spaces, we refer the reader to [2, 7, 9, 10].

Lemma 2.6. (see [11], Theorem 13.47) *Let* $(u_n)_n$ *be a sequence in* $L^1(\Omega)$ *and* $u \in L^1(\Omega)$ *satisfying the following items.*

(i) $u_n \to u$ *a.e. in* Ω,

(ii) $u_n \ge 0$ *and* $u \ge 0$ *a.e. in* Ω,

(iii) $\displaystyle\int_{\Omega} u_n \, dx \to \int_{\Omega} u \, dx$,

then $u_n \to u$ *in* $L^1(\Omega)$.

3. Essential Assumptions and main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, and let $1 < p_i < N$ for $i = 1, \dots, N$, we set

$$\underline{p} = \min\{p_1, p_2, \dots, p_N\} \qquad \text{and} \qquad \underline{p}^+ = \max\{p_1, p_2, \dots, p_N\}.$$

Let f be a function in $L^1(\Omega)$. We examine the strongly nonlinear anisotropic elliptic problem given by:

$$\begin{cases} Au + g(x, u, \nabla u) = f(x) + \dfrac{|u|^{p_0-2}u}{|x|^{p_0}} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $1 \leq p_0 < \dfrac{N(\underline{p}^+ - \lambda)}{N + \underline{p}^+ - \lambda - 1}$. Here A is the Leray-Lions operator mapping from $W_0^{1,\vec{p}}(\Omega)$ into its dual $W^{-1,\vec{p}'}(\Omega)$ and defined by:

$$Au = -\sum_{i=1}^{N} D^i a_i(x, u, \nabla u) \tag{8}$$

where $a_i(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions (i.e. measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) and satisfy the following conditions:

$$(a_i(x, s, \xi) - a_i(x, s, \eta))(\xi_i - \eta_i) > 0 \qquad \text{for any} \quad \xi_i \neq \eta_i, \tag{9}$$

$$|a_i(x, s, \xi)| \leq \beta \left(K_i(x) + |s|^{p_i-1} + |\xi_i|^{p_i-1} \right), \tag{10}$$

for any given positive function $K_i(x) \in L^{p_i'}(\Omega)$ for $i = 1, \dots, N$, and $\beta > 0$.

We further assume the existence of a positive, decreasing function $b(\cdot) : [0, \infty[\to]0, \infty[$, and a constant $b_0 > 0$ such that

$$a_i(x, s, \xi)\xi_i \geq b(|s|)|\xi_i|^{p_i} \qquad \text{with} \quad b(|s|) \geq \frac{b_0}{(1 + |s|)^\lambda}, \tag{11}$$

a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $0 \leq \lambda < \min(1, p - 1)$.

As a consequence of (9) and the continuity of $a_i(x, s, \cdot)$ with respect to ξ, we have

$$a_i(x, s, 0) = 0 \qquad \text{for} \quad i = 1, \dots, N.$$

The lower order term $g(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ is a Carathéodory function which satisfy the growth condition given by:

$$|g(x, s, \xi)| \leq g_0(x) + \sum_{i=1}^{N} d(|s|)|\xi_i|^{p_i}, \tag{12}$$

where $g_0(x)$ is assumed to be a positive measurable function in $L^1(\Omega)$, and the continuous decreasing function $d(\cdot) : \mathbb{R} \mapsto \mathbb{R}^+$ is assumed to satisfy $\dfrac{d(|s|)}{b(|s|)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

We now recall some important lemmas that will be useful in proving our main result.

Lemma 3.1. (see [6]) *Under assumptions* $(9)-(12)$*, let* $(u_n)_{n\in\mathbb{N}}$ *be a sequence in* $W_0^{1,\vec{p}}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,\vec{p}}(\Omega)$ *and*

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,u_n,\nabla u_n)-a_i(x,u_n,\nabla u)\Big)(D^iu_n-D^iu)\,dx\\&+\sum_{i=1}^{N}\int_\Omega \Big(|u_n|^{\underline{p}-2}u_n-|u|^{\underline{p}-2}u\Big)(u_n-u)\,dx\longrightarrow 0 \qquad \text{as}\quad n\to\infty,\end{aligned} \tag{13}$$

then $u_n \to u$*weakly in*$W_0^{1,\vec{p}}(\Omega)$ *for a subsequence.*

We recall the definition of renormalized solutions for the strongly nonlinear elliptic equation (7).

Definition 3.2. A measurable function u is called a renormalized solution for the strongly nonlinear elliptic problem (7), if $u \in \mathcal{T}_0^{1,\vec{p}}(\Omega)$, $g(x,u,\nabla u) \in L^1(\Omega)$,

$$\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u|\le h\}} a_i(x,u,\nabla u)D^iu\,dx = 0, \tag{14}$$

and u satisfies the following equality

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,u,\nabla u)\cdot\big(S'(u)\varphi D^iu+S(u)D^i\varphi\big)\,dx+\int_\Omega g(x,u,\nabla u)S(u)\varphi\,dx\\&=\int_\Omega fS(u)\varphi\,dx+\int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}}S(u)\varphi\,dx,\end{aligned} \tag{15}$$

for every $\varphi \in W_0^{1,\vec{p}}(\Omega)\cap L^\infty(\Omega)$ and for any smooth function $S(\cdot) \in W^{1,\infty}(\mathbb{R})$ with a compact support.

The aim of this paper is to prove the following existence result:

Theorem 3.3. *Let* $f \in L^1(\Omega)$*, and assume that conditions* $(9)-(12)$ *are satisfied. Then, there exists at least one renormalized solution u for the strongly nonlinear and non-coercive elliptic problem* (7)*. Moreover, we have* $u \in L^r(\Omega)$ *for any* $0 < r < \underline{p}^+ - \lambda - 1$.

4. Proof of Theorem 3.3

The proof is structured in several steps.

Step 1: Approximate problems.

Let $(f_n)_n$ be a sequence of measurable functions in $W^{-1,\vec{p}'}(\Omega) \cap L^1(\Omega)$ that satisfy: $fn \to f$ strongly in $L^1(\Omega)$ and $|f_n| \le |f|$ (e.g., $f_n = T_n(f)$). We consider the approximate problem:

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x, T_n(u_n), \nabla u_n) + g_n(x, u_n, \nabla u_n) = f_n(x) + \dfrac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0} + \frac{1}{n}} \text{ in } \Omega, \\ u_n = \ 0 \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{16}$$

where $g_n(x,s,\xi) = T_n(g(x,s,\xi))$. Let's consider the operators A_n and G_n from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ satisfying:

$$\langle A_n u, v\rangle = \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v \, dx \qquad \forall u, v \in W_0^{1,\vec{p}}(\Omega),$$

and

$$\langle G_n u, v\rangle = \int_\Omega g_n(x, u, \nabla u) v \, dx - \int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0} + \frac{1}{n}} v \, dx \quad \forall u, v \in W_0^{1,\vec{p}}(\Omega).$$

Lemma 4.1. *The bounded operator $B_n = A_n + G_n$, which acts from $W_0^{1,\vec{p}}(\Omega)$ into $W^{-1,\vec{p}'}(\Omega)$ is pseudo-monotone. Furthermore, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}}} \longrightarrow \infty \qquad \text{as} \quad \|v\|_{1,\vec{p}} \to \infty \quad \text{for any} \quad v \in W_0^{1,\vec{p}}(\Omega).$$

For the proof of Lemma 4.1, see Appendix. In view of Lemma 4.1 (see [12], Theorem 8.2), there exists at least one weak solution $u_n \in W_0^{1,\vec{p}}(\Omega)$ for problem (16), that is:

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i v \, dx + \int_\Omega g_n(x, u_n, \nabla u_n) \ v \, dx \\ &\quad = \int_\Omega f_n \ v \, dx + \int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \ v \, dx \qquad \text{for any} \quad v \in W_0^{1,\vec{p}}(\Omega). \end{aligned} \tag{17}$$

Step 2 : Some regularity results

Lemma 4.2. *Assume that the conditions* $(9)-(12)$ *are satisfied, then there exists a constant* $C>0$*, independent of* k *and* n*, such that the following estimates hold:*

$$\sum_{i=1}^{N}\int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\lambda+\theta}}\,dx \le C(k^\lambda + k^{\underline{p}^+-\theta} + k^{p_0-1}) \qquad \text{for any} \quad 1<\theta<\underline{p}, \tag{18}$$

$$\sum_{i=1}^{N}\int_\Omega |D^i T_k(u_n)|^{p_i}\,dx \le C(k^{\lambda+\theta} + k^{\underline{p}^+} + k^{p_0-1+\theta}) \qquad \text{for any} \quad k>0. \tag{19}$$

$$\int_\Omega |u_n|^s\,dx \le C \qquad \text{for any} \quad 0<s<\underline{p}^+-\lambda-1. \tag{20}$$

Let $k\ge 1$ and $1<\theta<\underline{p}^+ - \dfrac{N(1-p_0)}{N-p_0}$, we set

$$\varphi(s)=\left(1-\frac{1}{(1+|s|)^{\theta-1}}\right)\operatorname{sign}(s) \qquad \text{and} \qquad B(|s|)=\int_0^s \frac{2d(|\tau|)}{b(|\tau|)}\,d\tau.$$

By taking $v=\varphi(u_n)(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)} \in W_0^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (16), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i(\varphi(u_n)(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)})\,dx\\
&\quad+\int_\Omega g_n(x,u_n,\nabla u_n)\varphi(u_n)(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&=\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\varphi(u_n)(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&\quad+\int_\Omega f_n\varphi(u_n)(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx.
\end{aligned} \tag{21}$$

It follows that

$$
\begin{aligned}
&(\theta-1)\sum_{i=1}^{N}\int_\Omega \frac{a_i(x,T_n(u_n),\nabla u_n)D^iu_n}{(1+|u_n|)^\theta}(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&+\lambda\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u_n)(1+|T_k(u_n)|)^{\lambda-1}|\varphi(u_n)|e^{B(|u_n|)}\,dx\\
&+2\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&\le \int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&+\int_\Omega |f_n(x)|\;|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&+\int_\Omega |g_n(x,u_n,\nabla u_n)|\;|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx.
\end{aligned}
\tag{22}
$$

Thanks to (11) and (12) we obtain

$$
\begin{aligned}
&(\theta-1)\sum_{i=1}^{N}\int_\Omega \frac{b(|u_n|)|D^iu_n|^{p_i}}{(1+|u_n|)^\theta}(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&\quad+\lambda\sum_{i=1}^{N}\int_\Omega b(|u_n|)|D^iT_k(u_n)|^{p_i}(1+|T_k(u_n)|)^{\lambda-1}|\varphi(u_n)|e^{B(|u_n|)}\,dx\\
&\quad+2\sum_{i=1}^{N}\int_\Omega b(|u_n|)|D^iu_n|^{p_i}\frac{d(|u_n|)}{b(|u_n|)}\;|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&\le \int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&\quad+\int_\Omega (|f_n(x)|+g_0(x))|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx\\
&\quad+\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda e^{B(|u_n|)}\,dx.
\end{aligned}
\tag{23}
$$

Since $|\varphi(u_n)|\le 1$, we conclude that

$$
\begin{aligned}
&b_0(\theta-1)\sum_{i=1}^{N}\int_\Omega \frac{|D^iT_k(u_n)|^{p_i}}{(1+|u_n|)^\theta}\,dx+b_0(\theta-1)(1+k)^\lambda\sum_{i=1}^{N}\int_{\{|u_n|>k\}}\frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx\\
&+b_0\lambda\sum_{i=1}^{N}\int_\Omega \frac{|D^iT_k(u_n)|^{p_i}}{1+|T_k(u_n)|}|\varphi(u_n)|\,dx+\sum_{i=1}^{N}\int_\Omega d(u_n)|D^iu_n|^{p_i}|\varphi(u_n)|(1+|T_k(u_n)|)^\lambda\,dx\\
&\le e^{B(\infty)}\int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}(1+|T_k(u_n)|)^\lambda\,dx+e^{B(\infty)}(1+k)^\lambda\int_\Omega (|f(x)|+g_0(x))\,dx.
\end{aligned}
\tag{24}
$$

For the first term on the left-hand side of (24), applying the Poincaré inequality, we obtain:

$$
\begin{aligned}
& b_0(\theta-1)\sum_{i=1}^{N}\int_\Omega \frac{|D^iT_k(u_n)|^{p_i}}{(1+|u_n|)^\theta}\,dx \\
&= b_0(\theta-1)\sum_{i=1}^{N}\int_\Omega \Big|\frac{D^iT_k(u_n)}{(1+|u_n|)^{\frac{\theta}{p_i}}}\Big|^{p_i}\,dx \\
&= b_0(\theta-1)\sum_{i=1}^{N}\int_\Omega \Big|D^i\int_0^{|T_k(u_n)|}\frac{d\tau}{(1+|\tau|)^{\frac{\theta}{p_i}}}\Big|^{p_i}\,dx \\
&\geq \sum_{i=1}^{N}\frac{b_0(\theta-1)}{C_p^{p_i}}\int_\Omega \Big|\int_0^{|T_k(u_n)|}\frac{d\tau}{(1+|\tau|)^{\frac{\theta}{p_i}}}\Big|^{p_i}\,dx \qquad (25)\\
&\geq \sum_{i=1}^{N}\frac{b_0(\theta-1)}{C_p^{p_i}}\int_\Omega \frac{|T_k(u_n)|^{p_i}}{(1+|T_k(u_n)|)^\theta}\,dx \\
&\geq \sum_{i=1}^{N}\frac{b_0(\theta-1)}{2C_p^{p_i}}\int_\Omega |T_k(u_n)|^{p_i-\theta}\,dx - C_0 \\
&\geq C_1\int_\Omega |T_k(u_n)|^{\underline{p}^+-\theta}\,dx - C_2.
\end{aligned}
$$

with $C_1 = \dfrac{b_0(\theta-1)}{2C_p^{\underline{p}^+}}$. Concerning the second term on the left-hand side of (24), we have

$$
\begin{aligned}
& b_0(\theta-1)\sum_{i=1}^{N}\int_{\{|u_n|>k\}} \frac{|D^iu_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx \\
&= b_0(\theta-1)\sum_{i=1}^{N}\int_\Omega \Big|D^i\int_{|T_k(u_n)|}^{|u_n|}\frac{d\tau}{(1+|\tau|)^{\frac{\theta+\lambda}{p_i}}}\Big|^{p_i}\,dx \\
&\geq \sum_{i=1}^{N}\frac{b_0(\theta-1)}{C_p^{p_i}}\int_\Omega \Big|\int_{|T_k(u_n)|}^{|u_n|}\frac{d\tau}{(1+|\tau|)^{\frac{\theta+\lambda}{p_i}}}\Big|^{p_i}\,dx \\
&\geq \sum_{i=1}^{N}\frac{b_0(\theta-1)}{C_p^{p_i}}\int_\Omega \frac{(|u_n|-|T_k(u_n)|)^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx \\
&\geq \sum_{i=1}^{N}\frac{b_0(\theta-1)}{2^{p_i-1}C_p^{p_i}}\int_\Omega \frac{|u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx - \sum_{i=1}^{N}\frac{b_0(\theta-1)}{C_p^{p_i}}\int_\Omega \frac{|T_k(u_n)|^{p_i}}{(1+|u_n|)^{\theta+\lambda}}\,dx \\
&\geq \sum_{i=1}^{N}\frac{b_0(\theta-1)}{2^{p_i+\theta+\lambda-1}C_p^{p_i}}\int_\Omega |u_n|^{p_i-\theta-\lambda}\,dx - \sum_{i=1}^{N}\frac{b_0(\theta-1)}{C_p^{p_i}}\int_\Omega k^{p_i-\theta-\lambda}\,dx - C_3 \\
&\geq C_4\int_\Omega |u_n|^{\underline{p}^+-\theta-\lambda}\,dx - C_5k^{\underline{p}^+-\theta-\lambda} - C_6.
\end{aligned}
\tag{26}
$$

On the other hand, applying Young's inequality, we get:

$$
\begin{aligned}
& e^{B(\infty)} \int_\Omega \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}(1+|T_k(u_n)|)^\lambda \, dx \\
& \leq e^{B(\infty)} \int_{\{|u_n|\leq k\}} \frac{|T_k(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}(1+|T_k(u_n)|)^\lambda \, dx + e^{B(\infty)}(1+k)^\lambda \int_{\{|u_n|>k\}} \frac{|u_n|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx \\
& \leq \frac{C_1}{2} \int_\Omega |T_k(u_n)|^{\underline{p}^+-\theta} \, dx + C_6 \int_\Omega \frac{1}{|x|^{\frac{p_0(\underline{p}^+-\theta)}{\underline{p}^+-\theta-p_0+1-\lambda}}} \, dx + e^{B(\infty)} \int_{\{|u_n|\leq k\}} \frac{|T_k(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}} \, dx \\
& \quad + \frac{C_4}{2}(1+k)^\lambda \int_\Omega |u_n|^{\underline{p}^+-\theta-\lambda} \, dx + C_7(1+k)^\lambda \int_\Omega \frac{1}{|x|^{\frac{p_0(\underline{p}^+-\theta-\lambda)}{\underline{p}^+-\theta-\lambda-p_0+1}}} \, dx \\
& \leq \frac{C_1}{4} \int_\Omega |T_k(u_n)|^{\underline{p}^+-\theta} \, dx + \frac{C_4}{2}(1+k)^\lambda \int_\Omega |u_n|^{\underline{p}^+-\theta-\lambda} \, dx + C_8(1+k)^\lambda + C_9 k^{p_0-1}.
\end{aligned}
\tag{27}
$$

since $\dfrac{p_0(\underline{p}^+-\theta-\lambda)}{\underline{p}^+-\theta-\lambda-p_0+1} < \dfrac{p_0(\underline{p}^+-\theta)}{\underline{p}^+-\theta-p_0+1-\lambda} < N$ then $\displaystyle\int_\Omega \frac{1}{|x|^{\frac{p_0(\underline{p}^+-\theta)}{\underline{p}^+-\theta-p_0-\lambda+1}}} dx \in L^1(\Omega)$.

By combining (24) and (25) – (27), it follows that

$$
\begin{aligned}
& \frac{C_1}{4} \int_\Omega |T_k(u_n)|^{\underline{p}^+-\theta} \, dx + \frac{C_4}{4}(1+k)^\lambda \int_\Omega |u_n|^{\underline{p}^+-\theta-\lambda} \, dx \\
& + \frac{b_0(\theta-1)}{2} \sum_{i=1}^N \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{(1+|u_n|)^\theta} \, dx \\
& + \frac{b_0(\theta-1)(1+k)^\lambda}{2} \sum_{i=1}^N \int_{\{|u_n|>k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} \, dx \\
& + b_0 \lambda \sum_{i=1}^N \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{1+|T_k(u_n)|} |\varphi(u_n)| \, dx \\
& + \sum_{i=1}^N \int_\Omega d(u_n) |D^i u_n|^{p_i} \varphi(u_n)(1+|T_k(u_n)|)^\lambda \, dx \\
& \leq C_{10}(k^\lambda + k^{\underline{p}^+-\theta} + k^{p_0-1}) \qquad \text{for any} \quad k \geq 1,
\end{aligned}
\tag{28}
$$

where C_9 is a constant independent of n and k. Therefore, we have:

$$
\begin{aligned}
& \sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} \, dx \\
& = \sum_{i=1}^N \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} \, dx + \sum_{i=1}^N \int_{\{|u_n|>k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} \, dx \\
& \leq \sum_{i=1}^N \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{(1+|u_n|)^\theta} \, dx + (1+k)^\lambda \sum_{i=1}^N \int_{\{|u_n|>k\}} \frac{|D^i u_n|^{p_i}}{(1+|u_n|)^{\theta+\lambda}} \, dx \\
& \leq C_{11}(k^\lambda + k^{\underline{p}^+-\theta} + k^{p_0-1}) \qquad \text{for any} \quad k \geq 1,
\end{aligned}
\tag{29}
$$

and

$$\frac{1}{(1+k)^\theta}\sum_{i=1}^{N}\int_\Omega |D^iT_k(u_n)|^{p_i}\,dx \leq \sum_{i=1}^{N}\int_\Omega \frac{|D^iT_k(u_n)|^{p_i}}{(1+|u_n|)^\theta}dx \leq C_{12}(k^\lambda + k^{\underline{p}^+-\theta} + k^{p_0-1}), \tag{30}$$

Thus the estimates given by $(18)-(19)$ hold. Moreover, thanks to (29) we have

$$\int_\Omega |u_n|^{\underline{p}^+-\theta-\lambda}dx \leq \int_\Omega |u_n|^{\underline{p}^+-\theta-\lambda}(1+|T_k(u_n)|)^\lambda\,dx \leq C_{13}(k^\lambda + k^{\underline{p}^+-\theta} + k^{p_0-1}). \tag{31}$$

for any $k \geq 1$ and $\theta > 1$, which completes the proof of (20).

Step 3 : Weak convergence of truncations

In view of (19), the sequence $(T_k(u_n))_n$ remains bounded in $W_0^{1,\vec{p}}(\Omega)$, and there exists a subsequence, still denoted by $(T_k(u_n))_n$, and a measurable function $\psi_k \in W_0^{1,\vec{p}}(\Omega)$ such that:

$$\begin{cases} T_k(u_n) \rightharpoonup \psi_k & \text{in} \quad W_0^{1,\vec{p}}(\Omega), \\ T_k(u_n) \to \psi_k & \text{in} \quad L^1(\Omega) \quad \text{and a.e. in} \quad \Omega. \end{cases} \tag{32}$$

Furthermore, due to (20), for any $0 < r < \underline{p}^+ - 1 - \lambda$, the sequence $(|u_n|^r)_n$ is bounded in $L^1(\Omega)$. As a result, we have:

$$\begin{aligned} k^r \text{ meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|^r\,dx \\ &\leq \int_\Omega |u_n|^r\,dx \\ &\leq C \qquad\qquad \text{for any} \quad k \geq 1, \end{aligned} \tag{33}$$

It is therefore necessary that

$$\limsup_{n\to\infty} \text{ meas}\{|u_n| > k\} \leq \frac{C}{k^r} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{34}$$

We will now prove that $(u_n)_n$ is a Cauchy sequence in measure. In fact, for every $\delta > 0$, we have:

$$\begin{aligned} \text{meas}\{|u_n - u_m| > \delta\} &\leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ &\quad + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned}$$

Given $\varepsilon > 0$, according to (34), we can select $k = k(\varepsilon)$ sufficiently large such that:

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{35}$$

Furthermore, by (32), we have $T_k(u_n) \to \psi_k$ strongly in $L^1(\Omega)$ and almost everywhere in Ω. Therefore, the sequence $T_k(u_n)$ is a Cauchy sequence in measure. For any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that:

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3} \qquad \text{for all } m, n \geq n_0(h, \delta, \varepsilon). \tag{36}$$

By combining (35) and (36), we conclude that for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon \qquad \text{for any} \quad n, m \geq n_0.$$

It follows that $(u_n)_n$ is a Cauchy sequence in measure, then after going to a subsequence, it converges almost everywhere, to some measurable function u. Thanks to (32) we have

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } \quad W_0^{1,\vec{p}}(\Omega). \tag{37}$$

Using Lebesgue dominated convergence theorem, we obtain:

$$T_k(u_n) \to T_k(u) \text{ strongly in } \quad L^{p_i}(\Omega) \quad \text{for} \quad i = 1, \ldots, N. \tag{38}$$

Step 4: Equi-integrability of the term $\left(\dfrac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}}\right)_n$

Now, we will show the strong convergence of $\left(\dfrac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}}\right)_n$ in $L^1(\Omega)$.
We have

$$\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \to \frac{|u|^{p_0-2}u}{|x|^{p_0}} \qquad \text{a.e. in} \quad \Omega. \tag{39}$$

Using Vitali's theorem, it is sufficient to prove that $\left(\dfrac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}}\right)_n$ is uniformly equi-integrable.
Let $r, \varepsilon > 0$ such that $p_0 - 1 < r < r + \varepsilon < \underline{p}^+ - 1 - \lambda$, and let E be a measurable function in Ω. Thanks to Young's inequality we have

$$\begin{aligned}\int_E \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0} + \frac{1}{n}}\, dx \quad &\leq \int_E \frac{1}{|x|^{\frac{p_0 r}{r - p_0 + 1}}}\, dx + \int_E |T_h(u_n)|^r\, dx \\ &\quad + \int_{\{|u_n| > h\}} |u_n|^r\, dx.\end{aligned} \tag{40}$$

Thanks to (20), there exists $h(\eta) > 0$ such that

$$\int_{\{|u_n|>h\}} |u_n|^r\, dx \leq \frac{1}{h^\varepsilon} \int_{\{|u_n|>h\}} |u_n|^{r+\varepsilon}\, dx \leq \frac{C}{h^\varepsilon} \longrightarrow 0 \qquad \text{as} \quad h \to \infty. \tag{41}$$

Thus, for any $\eta > 0$ there exists $h(\eta) > 0$ such that

$$\int_{\{|u_n|>h(\eta)\}} |u_n|^r \, dx \leq \frac{\eta}{2}. \tag{42}$$

On the other hand, there exists $\mu(\eta) > 0$, such that for all $E \subset \Omega$, we have

$$\int_E \frac{1}{|x|^{\frac{p_0 r}{r-p_0+1}}} \, dx + \int_E |T_h(u_n)|^r \, dx \leq \frac{\eta}{2} \qquad \text{for} \quad meas(E) \leq \mu(\eta). \tag{43}$$

We conclude that, for any measurable subset E of Ω that

$$\int_E \frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0} + \frac{1}{n}} \, dx \leq \eta \qquad \text{for} \quad meas(E) \leq \mu(\eta). \tag{44}$$

Thus, the sequence $\Big(\dfrac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}}\Big)_n$ is equi-integrability, we deduce that

$$\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \longrightarrow \frac{|u|^{p_0-2}u}{|x|^{p_0}} \qquad \text{strongly in} \quad L^1(\Omega). \tag{45}$$

Step 5 : Some regularity results

The goal of this step is to show that

$$\lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u_n|\leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = 0.$$

Let $h > k \geq 1$, by using $\dfrac{T_h(u_n)}{h} e^{B(|u_n|)} \in W_0^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (16), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i \Big(\frac{T_h(u_n)}{h} e^{B(|u_n|)}\Big) \, dx \\ &\quad + \int_\Omega g_n(x, u_n, \nabla u_n) \frac{T_h(u_n)}{h} e^{B(|u_n|)} \, dx \\ &= \int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} \frac{T_h(u_n)}{h} e^{B(|u_n|)} \, dx + \int_\Omega f_n \frac{T_h(u_n)}{h} e^{B(|u_n|)} \, dx. \end{aligned} \tag{46}$$

It follows that

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_\Omega a_i(x,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+\frac{2}{h}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}|T_h(u_n)|e^{B(|u_n|)}\,dx\\
&\le\frac{1}{h}\int_\Omega\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|T_h(u_n)|e^{B(|u_n|)}dx+\frac{1}{h}\int_\Omega|f_n(x)||T_h(u_n)|e^{B(|u_n|)}dx\\
&\quad+\frac{1}{h}\int_\Omega|g_n(x,u_n,\nabla u_n)||T_h(u_n)|e^{B(|u_n|)}\,dx.
\end{aligned}\tag{47}$$

In view of (11) and (12) we conclude that

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\\
&\quad+\frac{2}{h}\sum_{i=1}^{N}\int_\Omega b(|u_n|)|D^iu_n|^{p_i}\frac{d(|u_n|)}{b(|u_n|)}|T_h(u_n)|e^{B(|u_n|)}\,dx\\
&\le\frac{e^{B(\infty)}}{h}\int_\Omega\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\;|T_h(u_n)|\,dx+\frac{e^{B(\infty)}}{h}\int_\Omega(|f_n(x)|+g_0(x))\;|T_h(u_n)|\,dx\\
&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_h(u_n)|e^{B(|u_n|)}\,dx.
\end{aligned}\tag{48}$$

It follows that

$$\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx\\
&\quad+\frac{1}{h}\sum_{i=1}^{N}\int_\Omega d(|u_n|)|D^iu_n|^{p_i}|T_h(u_n)|e^{B(|u_n|)}\,dx\\
&\le e^{B(\infty)}\int_\Omega\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\;\frac{|T_h(u_n)|}{h}\,dx+e^{B(\infty)}\int_\Omega(|f(x)|+g_0(x))\;\frac{|T_h(u_n)|}{h}\,dx.
\end{aligned}\tag{49}$$

For the two terms on the right-hand side of (49), we have meas $\{|u_n|>h\}\to 0$ as h tends to infinity, then $\dfrac{|T_h(u_n)|}{h}\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$. Thanks to (45), we have $\dfrac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\to\dfrac{|u|^{p_0-2}u}{|x|^{p_0}}$ strongly in $L^1(\Omega)$, it follows that

$$\varepsilon_1(h)=\int_\Omega\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}\;\frac{|T_h(u_n)|}{h}\,dx\longrightarrow 0\qquad\text{as}\quad h\to\infty.\tag{50}$$

Moreover, since $f(x)$ and $g_0(x)$ belong to $L^1(\Omega)$ then

$$\varepsilon_2(h)=\int_\Omega(|f(x)|+g_0(x))\;\frac{|T_h(u_n)|}{h}\,dx\longrightarrow 0\qquad\text{as}\quad h\to\infty.\tag{51}$$

By combining (49) and (50) − (51), we deduce that

$$\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n dx+\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega} d(|u_n|)|D^i u_n|^{p_i}|T_h(u_n)|dx \leq \varepsilon_3(h). \tag{52}$$

By letting h tends to infinity in (52) we conclude that

$$\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\leq h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\, dx = 0. \tag{53}$$

Moreover, we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i}\, dx = 0. \tag{54}$$

Step 6 : Strong convergence of truncations

Let $h > k \geq 1$ and

$$\psi_h(u_n) = 1 - \frac{|T_h(u_n - T_h(u_n))|}{h} \qquad \phi(s) = s.\exp(\frac{\gamma^2 s^2}{2}),$$

where $\gamma = 3\left\|\frac{d(|\cdot|)}{b(|\cdot|)}\right\|_{L^\infty(\mathbb{R})}$. Note that $\phi'(s) - \gamma|\phi(s)| \geq \frac{1}{2} \qquad \forall s \in \mathbb{R}$.

By using $v = \phi(T_k(u_n)-T_k(u))\psi_h(u_n)e^{B(|u_n|)} \in W_0^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (16), we obtain :

$$\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n) - D^iT_k(u))\phi'(T_k(u_n)-T_k(u))\psi_h(u_n)e^{B(|u_n|)}\, dx\\
&-\frac{1}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n|\phi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\, dx\\
&+2\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_n),\nabla u_n)D^i u_n\frac{d(|u_n|)}{b(|u_n|)}\text{sign}(u_n)\phi(T_k(u_n)-T_k(u))\psi_h(u_n)e^{B(|u_n|)}\, dx\\
&+\int_{\Omega} g_n(x,u_n,\nabla u_n)\phi(T_k(u_n)-T_k(u))\psi_h(u_n)e^{B(|u_n|)}\, dx\\
&=\int_{\Omega}\frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\phi(T_k(u_n)-T_k(u))\psi_h(u_n)e^{B(|u_n|)}\, dx\\
&+\int_{\Omega} f_n(x)\phi(T_k(u_n)-T_k(u))\psi_h(u_n)e^{B(|u_n|)}\, dx.
\end{aligned} \tag{55}$$

Since $\phi(T_k(u_n)-T_k(u))$ have the same sign as u_n on the set $\{|u_n|>k\}$, and $\psi_h(u_n)=1$ on the set $\{|u_n|\le k\}$. In view of (11) and (12) we obtain

$$\begin{aligned}
&\sum_{i=1}^N\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\phi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&-\sum_{i=1}^N\int_{\{k<|u_n|\le 2h\}} |a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\;|D^iT_k(u)|\phi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&-2\sum_{i=1}^N\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&+2\sum_{i=1}^N\int_{\{k<|u_n|\le 2h\}} b(|u_n|)|D^iu_n|^{p_i}\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n)-T_k(u))||\psi_h(u_n)|e^{B(|u_n|)}\,dx\\
&\le\int_\Omega\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac1n}\;|\phi(T_k(u_n)-T_k(u))|\psi_h(u_n)|\;e^{B(|u_n|)}\,dx\\
&+\int_\Omega(|f_n(x)|+g_0(x))\;|\phi(T_k(u_n)-T_k(u))|\;e^{B(|u_n|)}\,dx\\
&+\int_\Omega d(|u_n|)|D^iu_n|^{p_i}\;|\phi(T_k(u_n)-T_k(u))||\psi_h(u_n)|\;e^{B(|u_n|)}\,dx\\
&+\frac1h\sum_{i=1}^N\int_{\{h<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n|\phi(T_k(u_n)-T_k(u))|\;e^{B(|u_n|)}\,dx.
\end{aligned}\tag{56}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\phi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\
&-3\sum_{i=1}^N\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&\le e^{B(\infty)}\int_\Omega\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac1n}\;|\phi(T_k(u_n)-T_k(u))|\,dx\\
&+e^{B(\infty)}\int_\Omega(|f(x)|+g_0(x))\;|\phi(T_k(u_n)-T_k(u))|\,dx\\
&+\frac{\phi(2k)e^{B(\infty)}}{h}\sum_{i=1}^N\int_{\{h<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_{2h}(u_n)\,dx\\
&+\phi'(2k)\sum_{i=1}^N\int_{\{k<|u_n|\le 2h\}} |a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\;|D^iT_k(u)|e^{B(|u_n|)}\,dx.
\end{aligned}\tag{57}$$

For the two first terms on the right-hand side of (57), we have $T_k(u_n)\rightharpoonup T_k(u)$ weak$-*$ in $L^\infty(\Omega)$ and thanks to (45) we conclude that :

$$\varepsilon_1(n)=\int_\Omega\frac{|T_n(u_n)|^{p_0-1}}{|x|^{p_0}+\frac1n}\;|\phi(T_k(u_n)-T_k(u))|\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty,\tag{58}$$

Similarly, we have

$$\varepsilon_2(n)=\int_\Omega(|f(x)|+g_0(x))\;|\phi(T_k(u_n)-T_k(u))|\,dx\longrightarrow 0\quad\text{as}\quad n\to\infty.\tag{59}$$

Moreover, in view of (53) we obtain :

$$\varepsilon_3(h) = \frac{\phi(2k)e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))D^iT_{2h}(u_n)\,dx \longrightarrow 0 \quad \text{as} \quad h\to\infty. \tag{60}$$

Concerning the last term on the right-hand side of (57), we have $(|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|)_n$ is bounded in $L^{p'_i}(\Omega)$, then there exists $\eta_i\in L^{p'_i}(\Omega)$ such that $|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\rightharpoonup \eta_i$ weakly in $L^{p'_i}(\Omega)$ for any $i=1,\dots,N$, therefore,

$$\begin{aligned}\varepsilon_4(n) &= \sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} |a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\;|D^iT_k(u)|\;e^{B(|u_n|)}\,dx\\ &\le e^{B(\infty)}\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} |a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\;|D^iT_k(u)|\,dx \qquad (61)\\ &\longrightarrow e^{B(\infty)}\sum_{i=1}^{N}\int_{\{k<|u|\le 2h\}} \eta_i\;|D^iT_k(u)|\,dx = 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

By combining (57) and (58) − (61), we conclude that :

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\phi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\,dx\\ &\qquad -3\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u_n)\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\ &\le \varepsilon_5(n,h).\end{aligned} \tag{62}$$

It follows that

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega (a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u)))(D^iT_k(u_n)-D^iT_k(u))\\ &\qquad\qquad \times\Big(\phi'(T_k(u_n)-T_k(u))-3\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n)-T_k(u))|\Big)e^{B(|u_n|)}\,dx\\ &\le -\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\\ &\qquad\qquad \times\Big(\phi'(T_k(u_n)-T_k(u))-3\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n)-T_k(u))|\Big)e^{B(|u_n|)}\,dx\\ &\qquad +3\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u_n))D^iT_k(u)\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\ &\qquad\qquad +\varepsilon_5(n,h).\end{aligned} \tag{63}$$

For the first term on the right-hand side of (63), we have $T_k(u_n) \to T_k(u)$ strongly in $L^{p_i}(\Omega)$, then, $a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u))$ strongly in $L^{p_i'}(\Omega)$, and since $D^i T_k(u_n)$ converges to $D^i T_k(u)$ weakly in $L^{p_i}(\Omega)$ for $i = 1, \dots, N$, we obtain

$$\begin{aligned}
|\varepsilon_6(n)| \leq & \Big| \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u))(D^i T_k(u_n) - D^i T_k(u)) \\
& \quad \times (\phi'(T_k(u_n) - T_k(u)) - 3\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n) - T_k(u))|) e^{B(|u_n|)}\, dx \Big| \\
\leq & \Big(\phi'(2k) + 3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\mathbb{R})}\phi(2k)\Big) e^{B(\infty)} \\
& \quad \times \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u))|\; |D^i T_k(u_n) - D^i T_k(u)|\, dx \to 0 \text{ as } n \to \infty,
\end{aligned} \tag{64}$$

Concerning the second term on the right-hand side of (63), we have that $(|a_i(x, T_k(u_n), \nabla T_k(u_n))|)_n$ is uniformly bounded in $L^{p_i'}(\Omega)$, then there exists $\nu_i \in L^{p_i'}(\Omega)$ such that $|a_i(x, T_k(u_n), \nabla T_k(u_n))| \rightharpoonup \nu_i$ weakly in $L^{p_i'}(\Omega)$ for any $i = 1, \dots, N$, and we have $T_k(u_n) \to T_k(u)$ in $L^{p_i}(\Omega)$ therefore,

$$\begin{aligned}
|\varepsilon_7(n)| & \leq \Big| \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u) \frac{d(|u_n|)}{b(|u_n|)} |\phi(T_k(u_n) - T_k(u))| e^{B(|u_n|)}\, dx \Big| \\
& \leq \Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_\infty \phi(2k) e^{B(\infty)} \sum_{i=1}^{N} \int_\Omega |a_i(x, T_k(u_n), \nabla T_k(u_n))|\; |D^i T_k(u)|\, dx \\
& \longrightarrow \Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_\infty \phi(2k) e^{B(\infty)} \sum_{i=1}^{N} \int_\Omega \nu_i\; |D^i T_k(u)|\, dx \to 0 \quad \text{as} \quad n \to \infty,
\end{aligned} \tag{65}$$

By combining (63) and (64) – (65), we conclude that

$$\begin{aligned}
& \frac{1}{2} \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u))\, dx \\
& \leq \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)))(D^i T_k(u_n) - D^i T_k(u)) \\
& \qquad \times \Big(\phi'(T_k(u_n) - T_k(u)) - 3\frac{d(|u_n|)}{b(|u_n|)}|\phi(T_k(u_n) - T_k(u))|\Big) e^{B(|u_n|)}\, dx \\
& \leq \varepsilon_5(n, h).
\end{aligned} \tag{66}$$

Having in mind that $T_k(u_n) \to T_k(u)$ strongly in $L^{\underline{p}}(\Omega)$, we obtain by letting n then h tend to infinity

$$\sum_{i=1}^{N} \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)(D^i T_k(u_n) - D^i T_k(u))\, dx + \int_\Omega (|T_k(u_n)|^{\underline{p}-2} T_k(u_n) - |T_k(u)|^{\underline{p}-2} T_k(u))(T_k(u_n) - T_k(u))\, dx \longrightarrow 0. \tag{67}$$

In view of Lemma 3.1, we conclude that

$$\begin{cases} T_k(u_n) \to T_k(u) & \text{strongly in} \quad W_0^{1,\vec{p}}(\Omega), \\ D^i u_n \to D^i u & \text{a.e. in} \quad \Omega \quad \text{for} \quad i = 1, \dots, N. \end{cases} \tag{68}$$

Moreover, we have $a_i(x, T_n(u_n), \nabla u_n) D^i u_n$ tends to $a_i(x, u, \nabla u) D^i u$ almost everywhere in Ω, and in view of Fatou's lemma and (53), we conclude that

$$\begin{aligned} &\lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{h<|u|\le 2h\}} a_i(x, u, \nabla u) D^i u\, dx \\ &\le \lim_{h\to\infty} \liminf_{n\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{h<|u_n|\le 2h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, dx \\ &\le \lim_{h\to\infty} \limsup_{n\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{h<|u_n|\le 2h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n\, dx = 0. \end{aligned} \tag{69}$$

Moreover, thanks to (54) and since $d(|u_n|)|D^i u_n|^{p_i}$ tends to $d(|u|)|D^i u|^{p_i}$ almost everywhere in Ω, it follows that

$$\begin{aligned} &\lim_{h\to\infty} \sum_{i=1}^{N} \int_{\{|u|>h\}} d(|u|)|D^i u|^{p_i}\, dx \\ &\le \lim_{h\to\infty} \liminf_{n\to\infty} \sum_{i=1}^{N} \int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i}\, dx \\ &\le \lim_{h\to\infty} \limsup_{n\to\infty} \sum_{i=1}^{N} \int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i}\, dx = 0. \end{aligned} \tag{70}$$

Step 7 : The equi-integrability of the sequence $(g_n(x, u_n, \nabla u_n))_n$

Now, we will show that

$$g_n(x, u_n, \nabla u_n) \longrightarrow g(x, u, \nabla u) \qquad \text{strongly in} \quad L^1(\Omega),$$

using Vitali's theorem, it is sufficient to show that the sequence $(g_n(x, u_n, \nabla u_n))_n$ is uniformly equi-integrable.

Indeed, thanks to (54) we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^p\,dx = 0. \tag{71}$$

and since $g_0 \in L^1(\Omega)$ we obtain

$$\begin{aligned}&\int_{\{|u_n|>h\}} |g(x,u_n,\nabla u_n)|\,dx\\ &\le \int_{\{|u_n|>h\}} |g_0(x)|\,dx + \sum_{i=1}^{N}\int_{\{|u_n|>h\}} d(|u_n|)|D^i u_n|^{p_i}\,dx \to 0 \text{ as } h\to\infty.\end{aligned}$$

We conclude that for any $\varepsilon > 0$ there exists $h_0(\varepsilon) > 0$ such that

$$\int_{\{|u_n|>h\}} |g(x,u_n,\nabla u_n)|\,dx \le \frac{\varepsilon}{2} \quad \text{for any} \quad h \ge h_0(\varepsilon). \tag{72}$$

On the other hand, it's clear that for any measurable subset $E \subset \Omega$ we have

$$\begin{aligned}&\int_E |g_n(x,u_n,\nabla u_n)|\,dx\\ &\le \int_E |g_n(x,T_h(u_n),\nabla T_h(u_n))|\,dx + \int_{\{|u_n|>h\}} |g(x,u_n,\nabla u_n)|\,dx.\end{aligned} \tag{73}$$

Thanks to (68), there exists $\beta(\eta) > 0$ small enough such that

$$\begin{aligned}&\int_E |g_n(x,T_h(u_n),\nabla T_h(u_n))|\,dx\\ &\le \int_E |g_0(x)|\,dx + \sum_{i=1}^{N}\int_E d(|T_h(u_n)|)|D^i T_h(u_n)|^{p_i}\,dx \le \frac{\varepsilon}{2}.\end{aligned} \tag{74}$$

By combining (72), (73) and (74) we deduce that

$$\int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \varepsilon, \text{with } E\subseteq\Omega \text{ such that } \operatorname{meas}(E) \le \beta(\varepsilon). \tag{75}$$

It follows that the sequence $(g_n(x,u_n,\nabla u_n))_n$ is uniformly equi-integrable, and thanks to (68) we have

$$g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \qquad \text{a.e. in } \Omega. \tag{76}$$

Thus, in view of Vitali's Theorem we conclude that

$$g_n(x,u_n,\nabla u_n) \longrightarrow g(x,u,\nabla u) \qquad \text{strongly in} \quad L^1(\Omega). \tag{77}$$

Step 8 : Passage to the limit.

Let $\varphi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and let $S(\cdot)$ be a smooth function in $C_0^1(\mathbb{R})$ such that supp $(S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$.
By choosing $S(u_n)\varphi \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function in the approximate problem (16), we obtain

$$\begin{aligned}\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \left(D^i u_n S'(u_n)\varphi + S(u_n) D^i \varphi\right) \, dx \\ + \int_\Omega g_n(x, u_n, \nabla u_n) S(u_n) \varphi \, dx \\ = \int_\Omega \frac{|T_n(u_n)|^{p_0-2} T_n(u_n)}{|x|^{p_0} + \frac{1}{n}} S(u_n) \varphi \, dx + \int_\Omega f_n S(u_n) \varphi \, dx.\end{aligned} \tag{78}$$

To begin, we examine the first term on the left-hand side of (78), and it follows that

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \left(D^i u_n S'(u_n)\varphi + S(u_n) D^i \varphi\right) \, dx \\ &= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n))(S'(u_n)\varphi D^i T_M(u_n) + S(u_n) D^i \varphi) \, dx,\end{aligned}$$

in view of (10), we have that $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p_i'}(\Omega)$, and since $a_i(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a_i(x, T_M(u), \nabla T_M(u))$ almost everywhere in Ω, it follows that

$$a_i(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a_i(x, T_M(u), \nabla T_M(u)) \qquad \text{weakly in} \qquad L^{p_i'}(\Omega),$$

and since $S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi$ tends strongly to $S'(u)\varphi \, D^i T_M(u) + S(T_M(u)) D^i \varphi$ in $L^{p_i}(\Omega)$, we deduce that

$$\begin{aligned}&\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \left(D^i u_n S'(u_n)\varphi + S(u_n) D^i \varphi\right) \, dx \\ &= \lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \left(D^i T_M(u_n) S'(T_M(u_n))\varphi + S(T_M(u_n)) D^i \varphi\right) \, dx \\ &= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u), \nabla T_M(u)) \left(D^i T_M(u) S'(T_M(u))\varphi + S(T_M(u)) D^i \varphi\right) \, dx \\ &= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \left(D^i u S'(u)\varphi + S(u) D^i \varphi\right) \, dx.\end{aligned} \tag{79}$$

Regarding the second term on the right-hand side of (78), it is evident that $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\Omega)$. Given (77) we observe that

$g_n(x,u_n,\nabla u_n)\to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$, hence we obtain

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega g_n(x,u_n,\nabla u_n)S(T_M(u_n))\varphi\,dx &= \int_\Omega g(x,u,\nabla u)S(T_M(u))\varphi\,dx\\ &= \int_\Omega g(x,u,\nabla u)S(u)\varphi\,dx.\end{aligned} \tag{80}$$

For the terms on the right-hand side of (78), we have $f_n\to f$ strongly in $L^1(\Omega)$, and using (45) we find $\dfrac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}\to\dfrac{|u|^{p_0-2}u}{|x|^{p_0}}$ strongly in $L^1(\Omega)$, Thus, we obtain

$$\lim_{n\to\infty}\int_\Omega f_nS(T_M(u_n))\varphi\,dx=\int_\Omega fS(T_M(u))\varphi\,dx=\int_\Omega fS(u)\varphi\,dx. \tag{81}$$

and

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega \frac{|T_n(u_n)|^{p_0-2}T_n(u_n)}{|x|^{p_0}+\frac{1}{n}}S(T_M(u_n))\varphi dx &= \int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}}S(T_M(u))\varphi dx\\ &= \int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}}S(u)\varphi\,dx.\end{aligned} \tag{82}$$

Combining (78) – (82), we obtain that

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,u,\nabla u)\left(D^iuS'(u)\varphi+S(u)D^i\varphi\right)\,dx+\int_\Omega g(x,u,\nabla u)S(u)\varphi\,dx\\ &=\int_\Omega \frac{|u|^{p_0-2}u}{|x|^{p_0}}S(u)\varphi\,dx+\int_\Omega fS(u)\varphi\,dx,\end{aligned} \tag{83}$$

which concludes the proof of Theorem 3.3.

5. Appendix

Proof of Lemma 4.1

Using Hölder's inequality and the growth condition (10), we can prove that the operator A_n is bounded, and since

$$\begin{aligned}|\langle G_nu,v\rangle| &\le \int_\Omega |g_n(x,u,\nabla u)|\,|v|\,dx+\int_\Omega \Big|\frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}}\Big|\,|v|\,dx\\ &\le n\int_\Omega |v|\,dx+n^{p_0}\int_\Omega |v|\,dx\\ &\le (n+n^{p_0})\|v\|_{1,\vec{p}} \qquad \text{for any}\quad u,v\in W_0^{1,\vec{p}}(\Omega),\end{aligned}$$

it follows that B_n is bounded.

For the coercivity, we have for any $v \in W_0^{1,\vec{p}}(\Omega)$

$$\begin{aligned}\langle B_n v, v\rangle &= \langle A_n v, v\rangle + \langle G_n v, v\rangle \\ &= \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(v), \nabla v) D^i v\, dx + \int_\Omega g_n(x, v, \nabla v) v\, dx \\ &\quad - \int_\Omega \frac{|T_n(v)|^{p_0-1}}{|x|^{p_0} + \frac{1}{n}} v\, dx \\ &\geq \sum_{i=1}^{N} \int_\Omega b(|T_n(v)|)|D^i v|^{p_i}\, dx - n\, \|v\|_1 - n^{p_0}\|v\|_1 \\ &\geq \frac{b_0}{(1+n)^\lambda} \sum_{i=1}^{N} \int_\Omega |D^i v|^{p_i}\, dx - C(n + n^{p_0})\|v\|_{1,\vec{p}}.\end{aligned}$$

We conclude that

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}}} \geq \frac{b_0}{(1+n)^\lambda \|v\|_{1,\vec{p}}} \sum_{i=1}^{N} \int_\Omega |D^i v|^{p_i}\, dx - \frac{C(n + n^{p_0})\|v\|_{1,\vec{p}}}{\|v\|_{1,\vec{p}}} \longrightarrow \infty. \tag{84}$$

We still need to prove that B_n is pseudo-monotone.

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{weakly in } W_0^{1,\vec{p}}(\Omega), \\ B_n u_k \rightharpoonup \chi_n & \text{weakly in } W^{-1,\vec{p}'}(\Omega), \\ \limsup_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{cases} \tag{85}$$

we will show that

$$\chi_n = B_n u \quad \text{and} \quad \langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \qquad \text{as } k \to +\infty.$$

Thanks to Lemma 2.1, we have $W_0^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$ with $\underline{p} > 1$, then $u_k \to u$ in $L^{\underline{p}}(\Omega)$ and a.e. in Ω, for a subsequence denoted again by $(u_k)_{k\in\mathbb{N}}$.

As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}}(\Omega)$, and by the growth condition, the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'}(\Omega)$. Therefore, there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \quad \text{weakly in} \quad L^{p_i'}(\Omega) \quad \text{as } k \to \infty, \tag{86}$$

Also, we have $(g_n(x, u_k, \nabla u_k))_{k\in\mathbb{N}}$ is uniformly bounded in $L^{\underline{p}'}(\Omega)$, then there exists a measurable function $\psi_n \in L^{\underline{p}'}(\Omega)$ such that

$$g_n(x, u_k, \nabla u_k) \rightharpoonup \psi_n \quad \text{weakly in} \quad L^{\underline{p}'}(\Omega) \quad \text{as } k \to \infty. \tag{87}$$

Moreover, we have $\Big(\frac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0} + \frac{1}{n}}\Big)_{k\in\mathbb{N}}$ is bounded in $L^{\underline{p}'}(\Omega)$, and

$\dfrac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0}+\frac{1}{n}} \to \dfrac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}}$ almost everywhere in Ω. By Lebesgue dominated convergence theorem, we deduce that

$$\frac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0}+\frac{1}{n}} \longrightarrow \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}} \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega) \tag{88}$$

Firstly, for any $v \in W_0^{1,\vec{p}}(\Omega)$ we have

$$\begin{aligned}\langle \chi_n, v\rangle &= \lim_{k\to\infty}\langle B_n u_k, v\rangle \\ &= \lim_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v dx \\ &\quad + \lim_{k\to\infty}\int_\Omega g_n(x,u_k,\nabla u_k)v dx - \lim_{k\to\infty}\int_\Omega \frac{|T_n(u_k)|^{p_0-2}T_n(u_k)}{|x|^{p_0}+\frac{1}{n}} v\,dx \\ &= \sum_{i=1}^{N}\int_\Omega \varphi_i\ D^i v\,dx + \int_\Omega \psi_n\ v\,dx - \int_\Omega \frac{|T_n(u)|^{p_0-2}T_n(u)}{|x|^{p_0}+\frac{1}{n}} v\,dx.\end{aligned} \tag{89}$$

In view of (85) and (89), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle &= \limsup_{k\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx \\ &\quad + \int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx - \int_\Omega \frac{|T_n(u_k)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|u_k|\,dx\Big) \\ &\le \sum_{i=1}^{N}\int_\Omega \varphi_i\ D^i u\,dx + \int_\Omega \psi_n\ u\,dx - \int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|u|\,dx.\end{aligned} \tag{90}$$

Due to $(87)-(88)$ and the fact that u_k converges strongly to u in $L^{\underline{p}}(\Omega)$, we have

$$\int_\Omega g_n(x,u_k,\nabla u_k)u_k\,dx \longrightarrow \int_\Omega \psi_n\ u\,dx \quad \text{as} \quad k\to\infty, \tag{91}$$

and

$$\int_\Omega \frac{|T_n(u_k)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|u_k|\,dx \longrightarrow \int_\Omega \frac{|T_n(u)|^{p_0-1}}{|x|^{p_0}+\frac{1}{n}}|u|\,dx \quad \text{as} \quad k\to\infty. \tag{92}$$

Consequently, we have

$$\limsup_{k\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx \le \sum_{i=1}^{N}\int_\Omega \varphi_i\ D^i u\,dx. \tag{93}$$

Moreover, thanks to (9), we get

$$\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^i u_k - D^i u)\,dx \ge 0, \tag{94}$$

then

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx\\ &\geq \sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u\,dx\\ &\quad+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u)(D^i u_k - D^i u)\,dx.\end{aligned}$$

By applying Lebesgue dominated convergence theorem, we have $T_n(u_k) \to T_n(u)$ strongly in $L^{p_i}(\Omega)$ then $a_i(x,T_n(u_k),\nabla u) \to a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'}(\Omega)$, and using (86) we get

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx \geq \sum_{i=1}^{N}\int_{\Omega}\varphi_i\ D^i u\,dx.$$

Taking into account (93), we conclude that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx = \sum_{i=1}^{N}\int_{\Omega}\varphi_i\ D^i u\,dx. \tag{95}$$

According to (89), (91) – (92) and (95), we obtain

$$\langle B_n u_k, u_k\rangle \longrightarrow \langle \chi_n, u\rangle \quad\text{as}\quad k\to+\infty. \tag{96}$$

Now, using (95) we can show that

$$\lim_{k\to+\infty}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u))(D^i u_k - D^i u)\,dx = 0.$$

We have $u_n \to u$ strongly in $L^{\underline{p}}(\Omega)$ and by using Lemma 3.1, we get

$$u_k \longrightarrow u \ \text{ strongly in } W_0^{1,\vec{p}}(\Omega) \quad\text{and}\quad D^i u_k \longrightarrow D^i u \text{ a.e. in } \Omega,$$

it follows that $a_i(x,T_n(u_k),\nabla u_k) \to a_i(x,T_n(u),\nabla u)$ and $g_n(x,u_k,\nabla u_k) \to g_n(x,u,\nabla u)$ almost everywhere in Ω, we conclude that

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup a_i(x,T_n(u),\nabla u) \qquad \text{weakly in} \quad L^{p_i'}(\Omega) \quad\text{for}\quad i=1,\ldots,N,$$

and

$$g_n(x,u_k,\nabla u_k) \rightharpoonup g_n(x,u,\nabla u) \qquad \text{weakly in} \quad L^{\underline{p}'}(\Omega) \quad\text{for}\quad i=1,\ldots,N.$$

Thanks to (88) we conclude that $\chi_n = B_n u$, which ends the proof of Lemma 4.1.

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Department of Mathematics
LaR2A, Faculty of Sciences,
Abdelmalek Essaadi University,
Tetouan - Morocco
e-mail: bferrahi@uae.ac.ma, rajae.zerouali@etu.uae.ac.ma

Department of Mathematics
AFNLA, Faculty of Sciences,
Abdelmalek Essaadi University,
Tetouan - Morocco
e-mail: hhjiaj@uae.ac.ma