

# Solving Strongly Sublinear $p(x)$ -Laplacian Dirichlet Problems: Existence via Fixed-Point Theorems

Resolución de problemas de Dirichlet para  $p(x)$ -Laplacianos fuertemente sublineales: existencia utilizando teoremas de punto fijo

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**ABSTRACT.** We study the existence of weak solutions to the nonlinear elliptic problem  $-\Delta_{p(x)}u = \lambda|u|^{s(x)-2}u + f(x, u, \nabla u)$  in a bounded domain  $\Omega \subset \mathbb{R}^N$  with smooth boundary  $\partial\Omega$ , under homogeneous Dirichlet conditions. The equation features a variable exponent  $p(x)$  in the  $p(x)$ -Laplacian operator, an eigenvalue term  $\lambda|u|^{s(x)-2}u$ , and a Carathéodory perturbation  $f$  with sublinear growth, depending on both  $u$  and  $\nabla u$ . Using a topological approach based on fixed-point theorems, we establish the existence of weak solutions to the problem.

*Key words and phrases.* Strongly nonlinear elliptic problem,  $p(x)$ -Laplacian, generalized Lebesgue-Sobolev spaces, fixed point.

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**RESUMEN.** Estudiamos la existencia de soluciones débiles para el problema elíptico no lineal  $-\Delta_{p(x)}u = \lambda|u|^{s(x)-2}u + f(x, u, \nabla u)$  en un dominio acotado  $\Omega \subset \mathbb{R}^N$  con frontera suave  $\partial\Omega$ , bajo condiciones de Dirichlet homogéneas. La ecuación involucra una variable exponencial  $p(x)$  en el operador Laplaciano, un término  $\lambda|u|^{s(x)-2}u$ , y una perturbación de Carathéodory  $f$  con crecimiento sublineal, que depende tanto de  $u$  como de  $\nabla u$ . Usando un método topológico de puntos fijos, probamos la existencia de soluciones débiles al problema.

*Palabras y frases clave.* Problemas elípticos fuertemente no lineales,  $p(x)$ -Laplaciano, espacios generalizados de Lebesgue-Sobolev, teoremas de punto fijo.

## 1. Introduction

Nonlinear partial differential equations (PDEs) with variable exponents have become a focal point in modern mathematical analysis, offering a powerful framework for modeling complex physical phenomena with spatially adaptive behavior. At the heart of this field lies the  $p(x)$ -Laplacian operator, a generalization of the classical Laplacian that adjusts its nonlinearity across a domain through a variable exponent  $p(x)$ . This adaptability makes the  $p(x)$ -Laplacian an indispensable tool in applications ranging from nonlinear elasticity where it models stretch and strain in materials to the dynamics of electrorheological fluids, which change viscosity under electric fields [15], and advanced image restoration algorithms [4, 14]. The mathematical foundation of this operator rests on Lebesgue and Sobolev spaces with variable exponents [5, 6, 7, 10, 12, 16], providing a flexible framework that mirrors the adaptability of the phenomena it describes. This rich interplay of theory and application has sparked significant interest in boundary value problems involving variable exponents, nonlinearities, and boundary conditions, presenting both profound challenges and exciting opportunities.

In this paper, we explore a boundary value problem involving the  $p(x)$ -Laplacian, enriched by an eigenvalue term and gradient-dependent nonlinearity. Specifically, we consider the following problem:

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\Delta_{p(x)} u = \lambda |u|^{s(x)-2} u + f(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) with a smooth boundary  $\partial\Omega$ , and  $\lambda > 0$  is a real parameter.

- The variable exponent  $p(x)$  is log-Hölder continuous on  $\overline{\Omega}$  *i.e.* there exists a constant  $C$  such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x - y|}, \quad \forall x, y \in \overline{\Omega} \text{ with } |x - y| < \frac{1}{2},$$

and satisfies:

$$1 < p^- := \inf_{\overline{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < N.$$

- The eigenvalue term  $\lambda |u|^{s(x)-2} u$  introduces a new layer of complexity, with the exponent  $s(x)$  also log-Hölder continuous and constrained by:

$$1 < s^- < s^- := \inf_{\overline{\Omega}} s(x) \leq s(x) \leq s^+ := \sup_{\overline{\Omega}} s(x) < p^-.$$

- The right-hand side term  $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a Carathéodory function, measurable in  $x$  for fixed  $(\eta, \zeta)$  and continuous in  $(\eta, \zeta)$  for almost every  $x \in \Omega$ , satisfying the growth condition:

$$(\mathcal{H}_f) \quad |f(x, \eta, \zeta)| \leq A(x) \left( |\eta|^{q(x)-1} + |\zeta|^{q(x)-1} \right) + B(x),$$

where  $1 < q^- - 1 \leq q(x) \leq q^+ < p^-$ , and the weight functions  $A \in L^{\alpha(x)}(\Omega)$  and  $B \in L^{\beta(x)}(\Omega)$  have exponents:

$$\alpha(x) = \frac{p^*(x)}{p^*(x) - q^+}, \quad \beta(x) = \frac{p^*(x)}{p^*(x) - 1}.$$

Our goal in this paper is to establish the existence of weak solutions to  $(\mathcal{P}_\lambda)$ . To achieve this, we employ a topological approach rooted in fixed-point theorems [17, 19], building on and generalizing prior works in the field. Below, we summarize key contributions closely related to our problem  $(\mathcal{P}_\lambda)$ , highlighting the methodologies and results that inform our approach:

- **Problems without eigenvalue terms or gradient dependence:** For the case where  $\lambda = 0$  and the nonlinearity  $f$  does not depend on  $\nabla u$ , Fan and Zhang [8] established sufficient conditions for the existence of solutions to problems similar to  $(\mathcal{P}_\lambda)$ . Their analysis leverages the theory of variable exponent Lebesgue spaces  $L^{p(x)}(\Omega)$  and Sobolev spaces  $W_0^{1,p(x)}(\Omega)$ , laying a foundational framework for studying nonlinear PDEs with variable exponents. In a complementary study, Iliáš [13] employed variational and topological methods to derive sufficient conditions for the existence of weak solutions, offering alternative perspectives on this class of problems.
- **Eigenvalue problems:** When the nonlinearity  $f \equiv 0$  and the exponents  $p(x) = q(x)$ , Fan et al. [9] studied the eigenvalue problem associated with  $(\mathcal{P}_\lambda)$ . They provided sufficient conditions under which the infimum of the set of eigenvalues  $\Lambda$  satisfies either  $\inf \Lambda = 0$  or  $\inf \Lambda > 0$ , offering valuable insights into the spectral properties of the  $p(x)$ -Laplacian operator in variable exponent spaces.
- **Problems with competing nonlinearities:** Alsaedi [3] considered a problem closely related to  $(\mathcal{P}_\lambda)$ , but with two competing nonlinear terms:

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2} u + \mu |u|^{q(x)-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $\mu$  is an additional parameter. Using a combination of the Ekeland variational principle, the mountain pass theorem, and energy arguments,

Alsaedi established sufficient conditions for the existence of non-trivial weak solutions, demonstrating the effectiveness of variational techniques in handling multiple nonlinearities.

- **Problems with gradient-dependent nonlinearities:** The presence of gradient-dependent terms, as in  $(\mathcal{P}_\lambda)$ , introduces additional complexity, inspiring creative approaches to prove the existence of solutions. For instance, Ait Hammou et al. [1, 2] harnessed topological degree theory to establish existence results for strongly nonlinear  $p(x)$ -elliptic problems, including those featuring gradient-dependent nonlinearities. Similarly, Wang [18] employed methods based on pseudomonotone operators, which provide a powerful framework for proving both existence and uniqueness of solutions under certain conditions. These studies highlight the versatility of topological and operator-theoretic methods in addressing the challenges posed by gradient-dependent terms, offering valuable insights that inform our analysis of  $(\mathcal{P}_\lambda)$ .

**Outline of the Paper.** Section 2 introduces key definitions and properties of variable exponent Lebesgue and Sobolev spaces, including embedding theorems essential for the  $p(x)$ -Laplacian problem. Section 3 presents our main results, proving the existence of weak solutions to problem  $(\mathcal{P}_\lambda)$  using a topological approach based on fixed-point theorems.

## 2. Preliminaries on Variable Exponent Spaces

It is well known that generalized Lebesgue-Sobolev spaces, such as  $L^{p(x)}(\Omega)$  and  $W^{k,p(x)}(\Omega)$ , share many essential properties with classical Sobolev spaces, rendering them invaluable for studying nonlinear partial differential equations with variable exponents. Nevertheless, certain properties, including convolution and translation invariance, fail to hold in these spaces, as shown in [5, 6, 7, 10, 11]. This necessitates the development of tailored techniques for problems in variable exponent settings.

We begin by defining the variable exponent Lebesgue and Sobolev spaces, which form the functional framework for our analysis. Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with a smooth boundary  $\partial\Omega$ . Define the set of continuous functions on  $\bar{\Omega}$  that are strictly greater than 1 as:

$$C_+(\bar{\Omega}) = \{h \in C(\bar{\Omega}) : h(x) > 1 \text{ for all } x \in \bar{\Omega}\}.$$

For  $p \in C_+(\bar{\Omega})$ , the variable exponent Lebesgue space  $L^{p(x)}(\Omega)$  consists of all measurable functions  $u : \Omega \rightarrow \mathbb{R}$  satisfying:

$$\varrho_p(u) = \int_{\Omega} |u(x)|^{p(x)} dx < +\infty,$$

equipped with the Luxemburg norm:

$$\|u\|_{p(x)} = \inf \left\{ \lambda > 0 : \varrho_p \left( \frac{u}{\lambda} \right) \leq 1 \right\}.$$

The dual space of  $L^{p(x)}(\Omega)$  is  $L^{p'(x)}(\Omega)$ , where the conjugate exponent is defined by  $p'(x) = \frac{p(x)}{p(x)-1}$ .

The variable exponent Sobolev space  $W^{1,p(x)}(\Omega)$  is defined as:

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \nabla u \in L^{p(x)}(\Omega)^N \right\},$$

endowed with the norm:

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space  $W_0^{1,p(x)}(\Omega)$  is the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p(x)}(\Omega)$  with respect to the norm  $\|\cdot\|_{1,p(x)}$ . The spaces  $L^{p(x)}(\Omega)$ ,  $W^{1,p(x)}(\Omega)$ , and  $W_0^{1,p(x)}(\Omega)$  are separable and reflexive Banach spaces [5, 10].

Below, we list key properties of these spaces, which are essential for our analysis.

**Proposition 2.1** (Hölder’s Inequality [5, 10]). *For  $u \in L^{p(x)}(\Omega)$  and  $v \in L^{p'(x)}(\Omega)$ , the following inequality holds:*

$$\int_{\Omega} |uv| \, dx \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(x)} \|v\|_{p'(x)} \leq 2 \|u\|_{p(x)} \|v\|_{p'(x)},$$

where  $p'^- = \inf_{\bar{\Omega}} p'(x)$ .

**Proposition 2.2** (Norm-Modular Relationship [5, 10]). *For  $u \in L^{p(x)}(\Omega)$ , the following bounds hold:*

$$\min \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\} \leq \varrho_p(u) \leq \max \left\{ \|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+} \right\}.$$

Moreover,

(i)  $\|u\|_{p(x)} < 1$  (resp.  $= 1, > 1$ ) if and only if  $\varrho_p(u) < 1$  (resp.  $= 1, > 1$ ).

(ii) If  $\|u\|_{p(x)} < 1$ , then  $\|u\|_{p(x)}^{p^+} \leq \varrho_p(u) \leq \|u\|_{p(x)}^{p^-}$ .

(iii) If  $\|u\|_{p(x)} > 1$ , then  $\|u\|_{p(x)}^{p^-} \leq \varrho_p(u) \leq \|u\|_{p(x)}^{p^+}$ .

(iv)  $\varrho_p \left( \frac{u}{\|u\|_{p(x)}} \right) = 1$ .

**Proposition 2.3** (Nested Norms [5, 6]). *Let  $p, q \in L^\infty(\Omega)$  be log-Hölder continuous functions such that  $1 \leq p(x)q(x) \leq \infty$  almost everywhere in  $\Omega$ . For  $u \in L^{q(x)}(\Omega)$ ,  $u \neq 0$ , the following hold:*

- (i) *If  $\|u\|_{p(x)q(x)} \leq 1$ , then  $\|u\|_{p(x)q(x)}^{p^+} \leq \| |u|^{p(\cdot)} \|_{q(x)} \leq \|u\|_{p(x)q(x)}^{p^-}$ .*
- (ii) *If  $\|u\|_{p(x)q(x)} \geq 1$ , then  $\|u\|_{p(x)q(x)}^{p^-} \leq \| |u|^{p(\cdot)} \|_{q(x)} \leq \|u\|_{p(x)q(x)}^{p^+}$ .*

*In particular, if  $p(x) \equiv p$  is constant, then  $\| |u|^p \|_{q(x)} = \|u\|_{pq(x)}^p$ .*

**Proposition 2.4** (Convergence in  $L^{p(x)}(\Omega)$  [5, 10]). *For  $u, u_n \in L^{p(x)}(\Omega)$ ,  $n = 1, 2, \dots$ , the following statements are equivalent:*

- (i)  $\lim_{n \rightarrow \infty} \|u_n - u\|_{p(x)} = 0$ ,
- (ii)  $\lim_{n \rightarrow \infty} \varrho_p(u_n - u) = 0$ ,
- (iii)  $u_n \rightarrow u$  in measure in  $\Omega$  and  $\lim_{n \rightarrow \infty} \varrho_p(u_n) = \varrho_p(u)$ .

**Proposition 2.5** (Poincaré Inequality [5, 10]). *Let  $p \in L^{p(x)}(\Omega)$  be log-Hölder continuous function. Then, there exists a constant  $C > 0$  such that:*

$$\|u\|_{p(x)} \leq C \|\nabla u\|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

*Consequently, the norms  $\|\nabla u\|_{p(x)}$  and  $\|u\|_{1,p(x)}$  are equivalent on  $W_0^{1,p(x)}(\Omega)$ .*

**Proposition 2.6** (Critical Embedding [5, 11]). *If  $p : \bar{\Omega} \rightarrow \mathbb{R}$  is Lipschitz continuous, there exists a constant  $c > 0$  such that:*

$$\|u\|_{p^*(x)} \leq c \|u\|_{1,p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega),$$

*where  $p^*(x) = \frac{Np(x)}{N-p(x)}$  is the critical Sobolev exponent.*

**Proposition 2.7** (General Embedding [5, 10]). *Let  $p, q \in C(\bar{\Omega})$  be log-Hölder continuous functions satisfying  $1 \leq q(x) \leq p^*(x)$  for all  $x \in \bar{\Omega}$ . Then the embedding  $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$  is continuous. Moreover, if  $\inf_{\Omega} (p^* - q) > 0$ , the embedding is compact.*

To unify the notation throughout this paper, we define the Banach space  $X = W_0^{1,p(x)}(\Omega)$ , a separable and reflexive Sobolev space with variable exponent, equipped with the norm  $\|u\|_X = \|\nabla u\|_{p(x)}$ . The dual space of  $X$  is  $X^* = W^{-1,p'(x)}(\Omega)$ , where  $p'(x) = \frac{p(x)}{p(x)-1}$  is the conjugate exponent. We denote weak convergence by “ $\rightharpoonup$ ” and strong convergence by “ $\rightarrow$ ”. Throughout the paper,  $C_i, C'_i$  ( $i = 1, 2, \dots$ ) represent positive constants, which may vary from line to line.

### 3. Main Result

The goal of this section is to prove the existence of weak solutions to problem  $(\mathcal{P}_\lambda)$ . A function  $u \in X$  is a weak solution to  $(\mathcal{P}_\lambda)$  if it satisfies the variational formulation:

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx = \int_{\Omega} \left( \lambda |u|^{s(x)-2} uv + f(x, u, \nabla u)v \right) \, dx, \quad \forall v \in X.$$

To this end, we define the operators  $T, N : X \rightarrow X^*$  by:

$$\begin{aligned} \langle Tu, v \rangle &= \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx, \\ \langle Nu, v \rangle &= \int_{\Omega} \left( \lambda |u|^{s(x)-2} uv + f(x, u, \nabla u)v \right) \, dx, \end{aligned}$$

for all  $u, v \in X$ . Here,  $T$  represents the weak form of the  $p(x)$ -Laplacian operator, while  $N$  encapsulates the nonlinear terms, including the eigenvalue contribution and the gradient-dependent nonlinearity, both mapping from  $X$  to its dual  $X^*$ .

Our approach relies on the following Bohnenblust-Karlin fixed-point theorem, applied to a suitably defined set-valued operator  $S : B \rightarrow 2^B$ , where  $B \subset X$  is a closed ball.

**Theorem 3.1** (Bohnenblust-Karlin, [17]). *Let  $Z$  be a Banach space,  $B \subset Z$  a nonempty, closed, convex set, and  $S : B \rightarrow 2^B$  a set-valued mapping satisfying:*

- (a) *For each  $u \in B$ , the set  $S(u)$  is nonempty, closed, and convex;*
- (b)  *$S$  is closed;*
- (c) *The set  $S(B) = \bigcup_{u \in B} S(u)$  is relatively compact.*

*Then,  $S$  has a fixed point in  $B$ .*

We now state our main result.

**Theorem 3.2.** *If hypothesis  $(\mathcal{H}_f)$  holds, then the equation  $T(u) = N(u)$  has a weak solution  $u \in X$ .*

To apply the Bohnenblust-Karlin fixed-point theorem and establish the existence of a weak solution to problem  $(\mathcal{P}_\lambda)$ , we need to verify several properties of the operators  $T$  and  $N$ . The lemmas below establish the necessary conditions, including the continuity and monotonicity of  $T$ , the compactness of  $N$ , and the existence of a suitable closed ball  $B \subset X$ .

**Lemma 3.3.** ([8]) *The operator  $T : X \rightarrow X^*$ , defined by  $\langle Tu, v \rangle = \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \, dx$  for all  $u, v \in X$ , has the following properties:*

- (i)  $T$  is strictly monotone, i.e.,  $\langle Tu - Tv, u - v \rangle > 0$  for all  $u, v \in X$ ,  $u \neq v$ ;
- (ii)  $T$  is hemicontinuous, i.e., for any  $u, v, w \in X$ , the mapping  $t \mapsto \langle T(u + tv), w \rangle$  is continuous on  $\mathbb{R}$ ;
- (iii)  $T$  is bounded, i.e., it maps bounded sets in  $X$  to bounded sets in  $X^*$ ;
- (iv)  $T$  is coercive, i.e.,  $\frac{\langle Tu, u \rangle}{\|u\|_X} \rightarrow \infty$  as  $\|u\|_X \rightarrow \infty$ ;
- (v)  $T$  satisfies the  $(S_+)$  condition, i.e., if  $u_n \rightharpoonup u$  in  $X$  and  $\limsup_{n \rightarrow \infty} \langle Tu_n, u_n - u \rangle \leq 0$ , then  $u_n \rightarrow u$  in  $X$ .

**Lemma 3.4.** *The operator  $T : X \rightarrow X'$  satisfies the following property*

$$(S_2) : [u_n \rightharpoonup u, Tu_n \rightarrow Tu] \Rightarrow u_n \rightarrow u \text{ in } X$$

The proof of the preceding lemma follows readily from [19], which establishes the relationship between the  $(S_+)$  and  $(S_2)$  conditions, specifically showing that  $(S_+) \implies (S_2)$ .

**Lemma 3.5.** *Assume hypothesis  $(\mathcal{H}_f)$ . The operator  $N : X \rightarrow X^*$ , defined by*

$$\langle Nu, v \rangle = - \int_{\Omega} \left( \lambda |u|^{s(x)-2} uv + f(x, u, \nabla u) v \right) dx, \quad \forall u, v \in X,$$

*is compact.*

**Proof.** To prove that  $N$  is compact, we introduce the auxiliary operator  $\phi : X \rightarrow L^{p'(x)}(\Omega)$ , defined by:

$$\begin{aligned} \phi u &= -\lambda |u|^{s(x)-2} u - f(x, u, \nabla u), \\ &= \phi_1 u + \phi_2 u. \end{aligned}$$

where  $\phi_1 u(x) = -\lambda |u(x)|^{s(x)-2} u(x)$  and  $\phi_2 u(x) = -f(x, u(x), \nabla u(x))$ .

**Step 1: Boundedness of  $\phi$ .**

Indeed, denotes:  $\Omega_1 = \{x \in \Omega, |\theta(x)| \geq 1\}$ ,  $\Omega_2 = \{x \in \Omega, |\theta(x)| < 1\}$ .

Clearly, we have

$$|\langle \phi u(x), v \rangle| \leq |\langle \phi_1 u(x), v \rangle| + |\langle \phi_2 u(x), v \rangle|$$

Using Hölder's inequality, we get

$$\begin{aligned} |\langle \phi_1 u(x), v \rangle| &= \left| \int_{\Omega} \lambda |u|^{s(x)-2} uv dx \right|, \\ &\leq \lambda C_1 \| |u|^{s(\cdot)-1} \|_{p'(x)} \|v\|_{p(x)}, \\ &\leq 2\lambda C_2 \max\{ \|u\|_{p(x)}^{s^+-1}, \|u\|_{p(x)}^{s^- -1} \} \|v\|_{p(x)} < \infty. \end{aligned}$$

Since  $q^- - 1 > 1$  we get  $p^*(x)/(q^- - 1) < p^*(x)$ , and  $p^*(x)/(q^+ - 1) < p^*(x)$ , then the following embeddings hold true:

$$W^{1,p(x)} \hookrightarrow L^{(p^*(x)/(q^+-1))p(x)}(\Omega), W^{1,p(x)} \hookrightarrow L^{(p^*(x)/(q^--1))p(x)}(\Omega).$$

Due of Proposition 3, we have

$$\|u\|_{p^*(x)/(q^+-1)}^{q^+-1} \leq C \|u\|_{p^*(x)}^{q^+-1} \quad \text{and} \quad \|u\|_{p^*(x)/(q^--1)}^{q^--1} \leq C \|u\|_{p^*(x)}^{q^--1}.$$

In view of assumption  $(\mathcal{H}_f)$ , we apply Hölder's inequality and Sobolov embeddings, we obtain

$$\begin{aligned} \left| \int_{\Omega_1} f(x, u, \nabla u) v dx \right| &\leq \left| \int_{\Omega_1} \left( A(x) (|u|^{q(x)-1} + |\nabla u|^{q(x)-1}) + B(x) \right) v dx \right|, \\ &\leq C_3 \left[ |A|_\alpha ( \|u\|_{p^*(x)/(q^+-1)}^{q^+-1} + \|\nabla u\|_{p^*(x)/(q^+-1)}^{q^+-1} ) + \|B\|_\beta \right] \|v\|_{p^*(x)}, \\ &\leq C_5 \left[ \|A\|_\alpha ( \|u\|_{p^*(x)}^{q^+-1} + \|\nabla u\|_{p^*(x)}^{q^+-1} ) + \|B\|_\beta \right] \|v\|_{p^*(x)}, \\ &\leq C_6 ( \|A\|_\alpha \|u\|_{p(x)}^{q^+-1} + \|B\|_\beta ) \|v\|_{p(x)} < \infty. \end{aligned}$$

Similarly, we get

$$\left| \int_{\Omega_2} f(x, u, \nabla u) v dx \right| \leq C'_6 ( \|A\|_\alpha \|u\|_{p(x)}^{q^--1} + \|B\|_\beta ) \|v\|_{p(x)} < \infty.$$

This implies that  $\phi_2$  is bounded. Thus,  $\phi$  is bounded on  $X$ .

**Step 2: Continuity of  $\phi$ .** Let  $u_n \rightarrow u$  in  $X$  then  $u_n \rightarrow u$  in  $L^{p(x)}(\Omega)$  and  $\nabla u_n \rightarrow \nabla u$  in  $(L^{p(x)}(\Omega))^N$ , and there exists a subsequence  $(u_k)$  of  $(u_n)$  and measurable functions  $h \in L^{p(x)}(\Omega)$  and  $g \in (L^{p(x)}(\Omega))^N$  such that

$$\begin{aligned} u_k &\rightarrow u & \text{and} & \quad \nabla u_k \rightarrow \nabla u, \\ |u_k(x)| &\leq h(x) & \text{and} & \quad |\nabla u_k| \leq |g(x)|, \end{aligned}$$

for a.e  $x \in \Omega$  and for all  $k \in \mathbb{N}$ . Since  $f$  satisfies Carathéodory condition, we obtain

$$f(x, u_k(x), \nabla u_k(x)) \rightarrow f(x, u(x), \nabla u(x)) \text{ a.e } x \in \Omega,$$

it follows from  $(\mathcal{H}_f)$  that

$$f(x, u_k, \nabla u_k) \leq A(x) \left( |h(x)|^{q(x)-1} + |g(x)|^{q(x)-1} \right) + B(x),$$

for a.e  $x \in \Omega$  and for all  $k \in \mathbb{N}$ . Since  $\lambda|h|^{s(x)-1} + A \left( |h|^{q(x)-1} + |g|^{q(x)-1} \right) + B \in L^{p'(x)}(\Omega)$ , and taking into account that

$$\begin{aligned} \rho_{p'(x)}(\phi u_k - \phi u) &= \int_{\Omega} \left| \left( \lambda |u_k|^{s(x)-1} + f(x, u_k(x), \nabla u_k(x)) \right) \right. \\ &\quad \left. - \left( \lambda |u|^{s(x)-1} + f(x, u(x), \nabla u(x)) \right) \right|^{p'(x)} dx, \end{aligned}$$

hence, by the dominated convergence theorem and Proposition 4, we get

$$\phi u_k \rightarrow \phi u \text{ in } L^{p'(x)}(\Omega).$$

However, by uniqueness of the limit, it extends to the whole sequence.

Note that  $N$  can be expressed as  $N = I^* \circ \phi_1 + I^* \circ \phi_2$ , where  $I^* : L^{p'(x)}(\Omega) \rightarrow X^*$  is the adjoint of the embedding  $I : X \hookrightarrow L^{p(x)}(\Omega)$ . Since  $I$  is compact and  $I^*$  is also compact, therefore the compositions  $I^* \circ \phi_1$  and  $I^* \circ \phi_2$  and that  $N$  is compact. Completing the proof.  $\square$

**Lemma 3.6.** *Suppose hypothesis  $(\mathcal{H}_f)$  holds. Then, there exists a constant  $k > 0$  such that if  $u, \sigma \in X$  satisfy  $T(u) = N(\sigma)$  and  $\|\sigma\|_X \leq k$ , it follows that  $\|u\|_X \leq k$ .*

**Proof.** Let  $u, \sigma \in X$  be such  $T(u) = N(\sigma)$  then  $\langle T(u), v \rangle = \langle N(\sigma), v \rangle \forall v \in X$ . In particular if we have  $x \in \Omega_1$  and  $u = v$ , then  $\langle T(u), u \rangle = \langle N(\sigma), u \rangle$  i.e.

$$\begin{aligned} \|u\|_{p(x)}^{p^-} &\leq \|u\|_{p(x)}^{p(x)}, \\ &= \langle T(u), u \rangle, \\ &= \rho_p(\nabla u), \\ &= \int_{\Omega_1} \lambda |\sigma|^{s(x)-1} u + f(x, \sigma, \nabla \sigma) u dx, \\ &\leq \int_{\Omega_1} \left( \lambda |\sigma|^{s(x)-1} + A(x)(|\sigma|^{q(x)-1} + |\nabla \sigma|^{q(x)-1}) + B(x) \right) u dx, \\ &\leq C_7 (\lambda \|\sigma\|_{p(x)}^{s^+-1} + \|A\|_{\alpha} \|\sigma\|_{p(x)}^{q^+-1} + \|B\|_{\beta}) \|u\|_{p(x)} < \infty. \\ &\leq C_8 \left( \lambda \|\sigma\|_{p(x)}^{s^+-1} + \|\sigma\|_{p(x)}^{q^+-1} + 1 \right) \|u\|_{p(x)}, \forall u \in X. \end{aligned}$$

Since  $1 < q^+, s^+ < p^-$  there is  $k > 0$  such that  $(k/2)^{p^- - 1} - \lambda C_8 k^{s^+ - 1} - C_8 k^{q^+ - 1} - C_8 \geq 0$ , for sufficiently small constant  $\lambda$ . So if  $\|\sigma\|_{p(x)} \leq k$  we get  $\|u\|_{p(x)} \leq k$ .

Similar calculation gives us the following results: on  $\Omega_2$  and since  $1 < q^-, s^- < p^+$  there is  $k > 0$  such that  $(k/2)^{p^+ - 1} - \lambda C'_8 k^{s^- - 1} - C'_8 k^{q^- - 1} - C'_8 \geq 0$ , for sufficiently small constant  $\lambda$ . So if  $\|\sigma\|_{p(x)} \leq k$  we get  $\|u\|_{p(x)} \leq k$ . The proof finished.  $\square$

*Proof of Theorem 9.* We establish the existence of a solution by applying the Bohnenblust-Karlin fixed-point theorem. To this end, we construct a closed,

set-valued operator  $S$  mapping a closed ball  $B$  into its power set  $2^B$ , show that  $S(B)$  is relatively compact, and demonstrate that  $S$  admits a fixed point. The proof proceeds in five steps.

**Step 1: Define the Setting and Operator  $S$**  Let  $B \subset X$  denote the closed ball of radius  $k$  centered at the origin in the Banach space  $X$ , as provided by Lemma 13. Define the operator  $S : B \rightarrow 2^B$  by

$$S(\sigma) = \{u \in B : T(u) = N(\sigma)\},$$

**Step 2: Establish Properties of  $S$**  The operator  $S$  inherits key properties from  $T$  and  $N$ :

- (1) **Nonemptiness and Convexity:** For each  $\sigma \in B$ , the set  $S(\sigma)$  is nonempty, strictly convex, closed, and bounded in  $B$ . This follows from Browder's theorem (see [19, p. 557]) due to the strict monotonicity, hemicontinuity, and boundedness of  $T$ .
- (2) **Closedness of  $S$ :** The operator  $S$  is closed. To see this, consider sequences  $\sigma_n \rightarrow \sigma$  in  $B$  and  $u_n \in S(\sigma_n)$  with  $u_n \rightarrow u$  in  $B$ . By the continuity of  $N$ ,  $N(\sigma_n) \rightarrow N(\sigma)$  weakly in  $X^*$ . The demicontinuity of  $T$  ensures  $T(u_n) \rightarrow T(u)$  weakly, so  $T(u) = N(\sigma)$ . Thus,  $u \in S(\sigma)$ , proving  $S$  is closed.

**Step 3: Relative Compactness of  $S(B)$**  We show that  $S(B) = \bigcup_{\sigma \in B} S(\sigma)$  is relatively compact. Let  $(u_n) \subset S(B)$  be a sequence, and let  $(\sigma_n) \subset B$  be a corresponding sequence such that  $T(u_n) = N(\sigma_n)$ . Since  $N$  is compact,  $N(B)$  is relatively compact in  $X^*$ . Thus, there exists  $H \in X^*$  such that, up to a subsequence,  $N(\sigma_n) \rightarrow H$  in  $X^*$ .

By the coercivity of  $T$ , the sequence  $(u_n)$  is bounded in  $X$ . The reflexivity of  $X$  ensures that, up to a further subsequence,  $u_n \rightarrow u_0$  weakly in  $X$ . Since  $B$  is closed and convex, it is weakly closed, so  $u_0 \in B$ .

**Step 4: Strong Convergence of  $(u_n)$  to  $u_0$ .** We now prove that  $(u_n)$  converges strongly to  $u_0$ . By the monotonicity of  $T$ , for all  $u \in X$  with  $u \neq u_n$ ,

$$\langle T(u) - T(u_n), u - u_n \rangle \geq 0.$$

Taking the limit as  $n \rightarrow \infty$ , and noting that  $T(u_n) = N(\sigma_n) \rightarrow H$  in  $X^*$  and  $u_n \rightarrow u_0$  weakly in  $X$ , we obtain

$$\langle T(u) - H, u - u_0 \rangle \geq 0 \quad \text{for all } u \in X.$$

By the hemicontinuity of  $T$  we have  $T(u_0) = H$ . Taking the condition  $(S_2)$  into account, we obtain that the sequence  $(u_n)$  converges strongly to  $u_0$  in  $X$ . This establishes the relative compactness of  $S(B)$ , as every sequence in  $S(B)$  has a strongly convergent subsequence.

**Step 5: Application of the Fixed-Point Theorem** The operator  $S : B \rightarrow 2^B$  is closed, and  $S(B)$  is relatively compact in  $B$ . By the Bohnenblust-Karlin fixed-point theorem,  $S$  admits a fixed point, *i.e.*, there exists  $\sigma_0 \in B$  such that  $\sigma_0 \in S(\sigma_0)$ . This implies  $T(\sigma_0) = N(\sigma_0)$ , completing the proof.  $\square$

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