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Existence results of renormalized solutions to nonlinear parabolic equations in Musielak-Orlicz spaces

Resultados de existencia de soluciones renormalizadas para ecuaciones parabólicas no lineales en espacios de Musielak-Orlicz

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Abstract. In this paper, we prove the existence of renormalized solutions for a nonlinear parabolic problem of the type $\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(u)\big) + g(u)\varphi(x,|\nabla u|) = f$ in Q (a bounded subset of $\mathbb{R}^N$), in the setting of Musielak-Orlicz, where $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator, $\Phi \in C^0\left(\mathbb{R},\mathbb{R}^N\right)$. The term $g(u)\varphi(x,|\nabla u|)$ represents a nonlinear lower-order term with natural growth with respect to $|\nabla u|$ and satisfies a sign condition. Note that the Δ_2-condition is not assumed on the Musielak function and the source term f belongs to $L^1(Q)$.

Key words and phrases. Inhomogeneous Musielak-Orlicz-Sobolev spaces, Parabolic problems, Musielak-Orlicz function, Renormalized solutions.

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Resumen. En este artículo, probamos la existencia de soluciones renormalizadas para el problema no lineal parabólico del tipo $\frac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(u)\big) + g(u)\varphi(x,|\nabla u|) = f$ en Q (un subconjunto acotado de $\mathbb{R}^N$), en el espacio de Musielak-Orlicz, donde $-\operatorname{div}(a(x,t,u,\nabla u))$ es un operador de tipo Leray-Lions, $\Phi \in C^0\left(\mathbb{R},\mathbb{R}^N\right)$. El término $g(u)\varphi(x,|\nabla u|)$ representa un término no lineal de menor orden con crecimiento natural con respecto a $|\nabla u|$ y satisface una condición de signo. Note que no se asume la condición Δ_2 en la función de Musielak function y la función fuente f pertenece a $L^1(Q)$.

Palabras y frases clave. Espacios inhomogéneos de Musielak-Orlicz-Sobolev.

1. Introduction

In the present paper, we prove the existence of renormalized solutions to the following nonlinear parabolic problem with homogeneous Dirichlet boundary value conditions

$$\begin{cases} \frac{\partial u}{\partial t} + A(u) - \operatorname{div}\big(\Phi(u)\big) + g(u)\varphi(x,|\nabla u|) = f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \partial Q, \\ u(x,0) = u_0 & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where Ω be a bounded subset of $\mathbb{R}^N$, T is a positive real number, $Q = \Omega\times(0,T)$. The operator $A(u) = -\operatorname{div}\big(a(x,t,u,\nabla u)\big)$ is a Leray-Lions operator defined on a subset of $W_0^{1,x}L_\varphi(Q)$ where φ is a Musielak-Orlicz function, the right-hand side $f \in L^1(Q)$, we assume that g is an integrable function in $\mathbb{R}$ and satisfying the sign condition, while the function Φ is a continuous function on $\mathbb{R}$.

Solving the problem (1) presents some difficulties due to the fact that the data f only belong to L^1 and that the function Φ is not restricted by any growth condition at infinity so that proving the existence of a weak solution (i.e. in the distribution meaning) seems to be an arduous task. Loosely speaking it would require an $L^1_{loc}(Q)$ a priori estimate on $\Phi(u)$ in order to be able to define the nonlinear term $\operatorname{div}(\Phi(u))$ as a distribution on Q. In order to define the solution of (1) we will then use the notion of renormalized solutions introduced by R.-J. DiPerna and P.-L. Lions [11] for the study of the Boltzmann equation. It was then adapted to the study of some nonlinear elliptic or parabolic problems and evolution problems in fluid mechanics, see for example [8, 10]. Let us mention that in a joint work with F. Murat (see D. Blanchard, F. Murat [8]) the first author has obtained an existence and uniqueness result for Problem (1) in the case $\Phi = 0$. The existence and uniqueness of renormalized solution of (1) have been proved in [21] in the case where $g = 0$. Where $g = \Phi = 0$ and f is replaced by $f + \operatorname{div}(F)$, the existence and uniqueness of renormalized solution have been proved in [20].

The functional setting in these works is the usual Sobolev space $W^{1,p}$. Accordingly the function $a(.)$ is supposed to satisfy polynomial growth conditions with respect to u and its derivatives ∇u. When trying to perform an analysis for the function $a(.)$ with more general growth conditions, one is led to replace $W^{1,p}$ by more general Sobolev spaces. In this direction, generalizations to the Orlicz-Sobolev spaces W^1L_M and to the variable exponent Sobolev spaces $W^{1,p(x)}$ have been the subject of a large number of research papers.

On Orlicz spaces, Elmahi and Meskine [13] have proved existence of solutions for (1) where $\Phi \equiv g \equiv 0$ and where $g(u)\varphi(x,|\nabla u|) \equiv g(x,t,u,\nabla u)$ in [14], without assuming any restriction on the N-function M.

In the framework of variable exponent Sobolev spaces, in [4] Azroul, Benboubker, Redwane, and Yazough have shown the existence of renormalized solutions for the problem (1) without sign condition involving nonstandard growth in the case where $u \equiv b(u)$ and $g(u)\varphi(x, |\nabla u|) \equiv g(x, t, u, \nabla u)$ and in the elliptic case (see [5]).

In the setting of Musielak-Orlicz spaces, M. S. B. Elemine Vall, A. Ahmed, A. Touzani, A. Benkirane [12] proved the existence of solutions for the problem (1) where $\Phi \equiv \Phi(x, t, u)$ and $g \equiv 0$. The problem (1) has recently been solved by A.Talha, A. Benkirane, and M.S.B. Elemine Vall in [23] when the right hand side function f is measure data, $\Phi \equiv 0$ and $g(u)\varphi(x, |\nabla u|) \equiv g(x, t, u, \nabla u)$.

The study of the nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field. The generalized Orlicz (Musielak-Orlicz) spaces are of interest not only as the natural generalization of these important examples, but also in their own right. They have appeared in many problems in PDEs and the calculus of variations [3] and have applications to image processing [9, 15] and fluid dynamics [17].

It is our purpose in this paper, to prove the existence of renormalized solutions to a strongly nonlinear parabolic equation with minimal restrictions for the Musielak Orlicz functions and $\Phi(u) \neq 0$, while the right hand side function f in (1) is in $L^1(Q)$. This result can be applied, for example, for finding a renormalized solution for the following equation:

$$\frac{\partial u}{\partial t} - \operatorname{div}\big(\frac{m(x, |\nabla u|)}{|\nabla u|}\nabla u + u|u|^{\sigma}\big) + \frac{sign(u)}{1+u^2}\varphi(x, |\nabla u|) = f \in L^1(Q),$$

where φ is a Musielak function and m is the derivative of φ with respect to t.

The paper is organized as follows: We introduce some basic definitions and properties in inhomogeneous Musielak-Orlicz-Sobolev spaces as well as an abstract theorem, in section 2. In Section 3, we prepare some auxiliary results to prove our theorem. In Section 4, we give Basic assumptions on a, Φ, g, f and the definition of a renormalized solution of (1).

Finally in Section 5, we state the main result and proofs.

2. Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard references are [7, 18].

2.1. Musielak–Orlicz function:

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a) $\varphi(x,\cdot)$ is an N-function for all $x \in \Omega$ (i.e. convex, strictly increasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$, for all $t>0$, $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=0$ and $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=\infty$),

(b) $\varphi(\cdot,t)$ is a measurable function.

The function φ is called a Musielak–Orlicz function.

For a Musielak-Orlicz function φ we put $\varphi_x(t)=\varphi(x,t)$ and we associate its non-negative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t)\le k\,\varphi(x,t)+h(x) \text{ for all } x\in\Omega \text{ and } t\ge 0. \tag{2}$$

When (2) holds only for $t\ge t_0>0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma\prec\varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$

$$\gamma(x,t)\le\varphi(x,ct) \text{ for all } t\ge t_0, \quad (\text{ resp. for all } t\ge 0 \text{ i.e. } t_0=0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma\prec\prec\varphi$ if for every positive constant c we have

$$\lim_{t\longrightarrow 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0, \quad (\text{resp. } \lim_{t\longrightarrow\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0).$$

Definition 2.1. A Musielak function φ is called locally integrable on Ω if

$$\int_E\varphi(x,t)dx=\int_\Omega\varphi\left(x,t\chi_E(x)\right)\mathrm{d}x<+\infty,$$

for all $t\ge 0$ and all measurable set $E\subset\Omega$ with $\operatorname{mes}(E)<+\infty$.

***Remark* 2.2.** If $\gamma\prec\prec\varphi$ and γ is locally integrable on Ω, then $\forall c>0$ there exists a nonnegative integrable function h such that

$$\gamma(x,t)\le\varphi(x,ct)+h(x), \text{ for all } t\ge 0 \text{ and for a.e. } x\in\Omega. \tag{3}$$

Definition 2.3. A Musielak function φ satisfies the log-Hölder continuity condition on Ω if there exists a constant $A>0$ such that

$$\frac{\varphi(x,t)}{\varphi(y,t)}\le t^{\left(\frac{A}{\log(\frac{1}{|x-y|})}\right)}$$

for all $t\ge 1$ and for all $x,y\in\Omega$ with $|x-y|\le\frac{1}{2}$.

Lemma 2.4. [2] *Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the log-Hölder Continuity, then there exists an N-function M such that*

$$\varphi(x,t) \leq M(t), \text{ for all } t \geq 1 \text{ and for all } x \in \Omega.$$

***Remark* 2.5.** The latter Lemma proves that the log-Hölder Continuity condition implies the local integrability.

2.2. Musielak-Orlicz space:

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \Big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} \mid \rho_{\varphi,\Omega}(u) < \infty\Big\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} \mid \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < \infty, \text{ for some } \lambda > 0\big\}.$$

For a Musielak-Orlicz function φ we put: $\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}$, ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0 \,\Big|\, \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)\, dx \leq 1\right\},$$

which is called the Luxemburg norm and the so–called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|\, dx,$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent [18].

We will also use the space $E_\varphi(\Omega)$ defined by

$$E_\varphi(\Omega) = \big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} \mid \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < \infty, \text{ for all } \lambda > 0\big\}.$$

***Remark* 2.6.** [2] The set E_φ is a closed subset of L_φ.

Theorem 2.7. [2] *Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the log-Hölder Continuity condition. Then $(E_\varphi(\Omega))'$ is isomorphic to $L_\psi(\Omega)$.*

We say that the sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed non-negative integer m we define

$$W^m L_\varphi(\Omega) = \Big\{ u \in L_\varphi(\Omega) : \forall |\alpha| \leq m,\ D^\alpha u \in L_\varphi(\Omega) \Big\}.$$

and

$$W^m E_\varphi(\Omega) = \Big\{ u \in E_\varphi(\Omega) : \forall |\alpha| \leq m,\ D^\alpha u \in E_\varphi(\Omega) \Big\}.$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative integers α_i, $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}$$

for $u \in W^m L_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L_\varphi(\Omega), \|\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition [18]:

$$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c_0. \tag{4}$$

The space $W^m L_\varphi(\Omega)$ will always be identified as a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W^m_0 L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W^m_0 E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \Big\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \Big\}$$

and

$$W^{-m}E_\psi(\Omega)=\left\{f\in D'(\Omega);\ f=\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha f_\alpha \text{ with } f_\alpha\in E_\psi(\Omega)\right\}.$$

We say that a sequence of functions $u_n\in W^mL_\varphi(\Omega)$ is modular convergent to $u\in W^mL_\varphi(\Omega)$ if there exists a constant $k>0$ such that

$$\lim_{n\to\infty}\overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n-u}{k}\Big)=0.$$

For φ and its complementary function ψ, the following inequality is called the Young's inequality [18]:

$$ts\le\varphi(x,t)+\psi(x,s),\quad \forall t,s\ge 0, x\in\Omega. \tag{5}$$

This inequality implies that

$$|||u|||_{\varphi,\Omega}\le\rho_{\varphi,\Omega}(u)+1. \tag{6}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega}\le\rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega}>1, \tag{7}$$

$$\|u\|_{\varphi,\Omega}\ge\rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega}\le 1. \tag{8}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u\in L_\varphi(\Omega)$ and $v\in L_\psi(\Omega)$, then we have the Hölder inequality [18]

$$\left|\int_\Omega u(x)v(x)\,dx\right|\le\|u\|_{\varphi,\Omega}|||v|||_{\psi,\Omega}. \tag{9}$$

2.3. Inhomogeneous Musielak–Orlicz-Sobolev spaces

Let Ω be a bounded domain of $\mathbb{R}^N$ and $\mathrm{Q}=\Omega\times[0,T], T>0$ and let φ be a Musielak function. For each $\alpha\in\mathbb{N}^N$, denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x\in\mathbb{R}^N$.

The inhomogeneous Musielak-Sobolev spaces of order 1 are defined as follows:

$$W^{1,x}L_\varphi(\mathrm{Q})=\left\{u\in L_\varphi(\mathrm{Q}): D_x^\alpha u\in L_\varphi(\mathrm{Q})\forall|\alpha|\le 1\right\},$$
$$W^{1,x}E_\varphi(\mathrm{Q})=\left\{u\in E_\varphi(\mathrm{Q}): D_x^\alpha u\in E_\varphi(\mathrm{Q})\forall|\alpha|\le 1\right\}.$$

The latter space is a subspace of the former, and both are Banach spaces with respect to the norm defined by norm $\|u\|=\sum_{|\alpha|\le 1}\|D_x^\alpha u\|_{\varphi,\mathrm{Q}}$.

These spaces are regarded as subspaces of the product space $\Pi L_\varphi(Q)$, which consists of $(N+1)$ copies. We will also examine the weak topologies $\sigma(\Pi L_\varphi,\Pi E_\psi)$ and $\sigma(\Pi L_\varphi,\Pi L_\psi)$. If $u\in W^{1,x}L_\varphi(Q)$, then the function $t\mapsto$

$u(t)=u(t,\cdot)$ is defined on $(0,T)$ and takes values in $W^1L_\varphi(Q)$. Furthermore, if $u\in W^{1,x}E_\varphi(Q)$, this function is also $W^{1,x}E_\varphi(Q)$ valued and is strongly measurable.

Additionally, the following embedding holds :

$$W^{1,x}E_\varphi(Q)\subset L^1\left(0,T,W^1E_\varphi(\Omega)\right)$$

It is worth noting that the space $W^{1,x}L_\varphi(Q)$ is not generally separable. The space $W^{1,x}E_\varphi(Q)$ is characterized as the norm closure of $\mathcal{D}(Q)$ within $W^{1,x}$ $L_\varphi(Q)$. We can show as in [6] that when Ω is a Lipschitz domain, then each element u of the closure of $\mathcal{D}(Q)$ with respect to the weak$*$ topology $\sigma\left(\Pi L_\varphi,\Pi E_\psi\right)$ is the limit, in $W^{1,x}E_\varphi(Q)$, of some subsequence $(u_n)\subset\mathcal{D}(Q)$ for modular convergence, i.e. there exists $\lambda>0$ such that for all $|\alpha|\leq 1$, we have

$$\int_Q\varphi\left(x,\frac{D_x^a u_n-D_x^\alpha u}{\lambda}\right)dxdt\to 0 \text{ as } n\to\infty.$$

This implies that (u_n) converges to u in $W^{1,x}(Q)$ for the weak topology $\sigma\left(\Pi L_\varphi,\Pi E_\psi\right)$. Consequently $\overline{(\mathcal{D}(Q))}^{\sigma(\Pi L_\varphi,\Pi E_\psi)}=\overline{(\mathcal{D}(Q))}^{\sigma(\Pi L_\varphi,\Pi L_\psi)}$. This space will be denoted by $W_0^{1,x}L_\psi(Q)$. Furthermore, $W_0^{1,x}E_\varphi(Q)=W_0^{1,x}L_\varphi(Q)\cap\Pi E_\varphi$. We have then the following complementary system

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F\\ W_0^{1,x}E_\varphi(Q) & F_0\end{pmatrix}$$

where F is the dual space of $W_0^{1,x}L_\varphi(Q)$. Additionally, except for an isomorphism, F corresponds to the quotient of ΠL_ψ by the polar set $\left(W_0^{1,x}E_\varphi(Q)\right)^\perp$. We denote this space as $F=W^{-1,x}L_\psi(Q)$, and it can be shown that

$$W^{-1,x}L_\psi(Q)=\left\{f=\sum_{|\alpha|<1}D_r^\alpha f_\alpha : f_\alpha\in L_\varphi(Q)\right\}.$$

This space is equipped with the usual quotient norm

$$\|f\|=\inf\sum_{|\alpha|\leq 1}\|f_\alpha\|_{\psi,Q}$$

where the infimum is taken over all possible decompositions

$$f=\sum_{|\alpha|<1}D_x^\alpha f_\alpha,\quad f_\alpha\in L_\psi(Q).$$

The space F_0 is then given by

$$F_0=\left\{f=\sum_{|\alpha|\leq 1}D_x^\alpha f_\alpha : f_\alpha\in E_\psi(Q)\right\}$$

and is denoted by $W^{-1,x}E_\psi(Q)$.

3. Auxiliary results

In this section we cite some preliminaries lemmas from the literature that will come handy in the next sections.

Lemma 3.1. [2] *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$ and let φ be a Musielak function satisfying the log-Hölder continuity such that*

$$\psi(x,1) \leq c_1 \text{ a.e in } \Omega \text{ for some } c_1 > 0. \tag{10}$$

Then $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1 L_\varphi(\Omega)$ for modular convergence.

***Remark* 3.2.** Note that if $\lim_{t\to\infty} \inf_{x\in\Omega} \dfrac{\varphi(x,t)}{t} = \infty$, then (10) holds.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1 L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

The following lemma gives the modular Poincaré's inequality in Musielak-Orlicz spaces.

Lemma 3.3. [2] *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$ $(N \geq 2)$ and let φ be a Musielak function satisfying the conditions of lemma (3.1). Then there exist positive constants a, b and λ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x, |u(x)|)dx \leq a + b \int_\Omega \varphi(x, \lambda|\nabla u(x)|)dx \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{11}$$

Corollary 3.4. [2] (Poincaré Inequality) *Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N(N \geq 2)$ and let φ be a Musielak function satisfying the same conditions of Lemma* 3.3 *Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \forall v \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.5. [7] *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak–Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} \text{ a.e in } \{x \in \Omega : u(x) \in D\} \\ \quad 0 \quad \text{ a.e in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 3.6. [24] *Let $u_n, u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

Definition 3.7. Let Ω be an open subset of $\mathbb{R}^N$. We say that Ω has the segment property if there exist a locally finite open covering $\{O_i\}$ of the boundary $\partial\Omega$ of Ω and corresponding vectors $\{y_i\}$ such that if $x \in \Omega \cap O_i$ for some i, then $x + ty_i \in \Omega$ for $0 < t < 1$.

Lemma 3.8. [7]. *Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $||u_n||_\infty \leq (N+1)||u||_\infty$.

Lemma 3.9. [1] *Let Ω be an open bounded subset of $\mathbb{R}^N$ satisfying the segment property. If $u \in (W_0^1 L_\varphi(\Omega))^N$ then*

$$\int_\Omega div\, u\, dx = 0.$$

Lemma 3.10. (The Nemytskii Operator) *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak-Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{12}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$.

Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x, u(x))$ is continuous from

$$\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2})\right)^p = \prod \left\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\right\}.$$

into $(L_\psi(\Omega))^q$ for the modular convergence.

Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\left(\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2})\right)^p$ to $(E_\gamma(\Omega))^q$.

Lemma 3.11. [1] *Let Ω be a bounded open Lipschitz domain of $\mathbb{R}^N$, then*

$$\left\{w \in W_0^{1,x} L_{\bar\varphi}(Q) : \frac{\partial w}{\partial t} \in W^{-1,x} L_{\bar\varphi}(\Omega) + L^1(Q)\right\} \subset \mathcal{C}\left([0,T], L^1(\Omega)\right).$$

Proposition 3.12. [23] *Let φ be a Musielak function and let (u_n) be a sequence of $W^{1,x} L_\varphi(Q)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x} L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi L_\psi),$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q),$$

with (h_n) bounded in $W^{-1,x} L_\psi(Q)$ and (k_n) bounded in the space $\mathcal{M}(Q)$ of measures on Q. Then $u_n \to u$ strongly in $L^1_{loc}(Q)$. If further $(u_n) \subset W_0^{1,x} L_\varphi(Q)$ then $u_n \to u$ strongly in $L^1(Q)$.

4. Basic assumptions and definition of a renormalized solution

Let Q be the cylinder $\Omega \times (0,T)$, $0 < T < +\infty$, Ω is a bounded domain of $\mathbb{R}^N$ with the segment property and let φ and ψ two complementary Musielak–Orlicz functions.

Let $A : D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ be a mapping given by

$$A(u) = -\operatorname{div}(a(x,t,u,\nabla u)),$$

where $a(x,t,s,\xi) : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function. There exist two Musielak–Orlicz functions φ and γ such that $\gamma \prec\prec \varphi$, a positive function $c(x,t) \in E_\psi(Q)$ and two positive constants ν, β such that for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$, ξ, $\xi' \in \mathbb{R}^N$, $\xi \neq \xi'$

$$|a(x,t,s,\xi)| \leq \beta\Big(c(x,t) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|\xi|)\Big), \tag{13}$$

$$\Big(a(x,t,s,\xi) - a(x,t,s,\xi')\Big)(\xi - \xi') > 0, \tag{14}$$

$$a(x,t,s,\xi)\xi \geq \alpha\varphi(x,|\xi|), \tag{15}$$

$$\Phi : \mathbb{R} \longrightarrow \mathbb{R}^N \text{ is a continuous function,} \tag{16}$$

$$g : \mathbb{R} \longrightarrow \mathbb{R} \text{ is an integrable function on } \mathbb{R} \text{ and } g(u)u \geq 0, \tag{17}$$

$$f \in L^1(Q), \tag{18}$$

$$u_0 \text{ is an element of } L^1(\Omega). \tag{19}$$

***Remark* 4.1.** As already mentioned in the introduction, problem (1) does not admit a weak solution under assumptions (13)–(19) since the growths of $a(x,t,u,\nabla u)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs to $W^{1,x}L_\varphi(Q)$.

Throughout this paper $\langle\cdot,\cdot\rangle$ means either the pairing between $W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ and $W^{-1,x}L_\psi(Q) + L^1(Q)$ or between $W^{1,x}L_\varphi(Q)$ and $W^{-1,x}L_\psi(Q)$. We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases}$$

We shall use the following definition of renormalized solutions for problem (1) in the following sense:

Definition 4.2. Let $f \in L^1(Q)$. A renormalized solution of problem (1) is a function u defined on Q, satisfying the following conditions:

$$u \in L^\infty(0,T;L^1(\Omega)) \text{ and } T_k(u) \in W_0^{1,x}L_\varphi(Q) \text{ for all } k \geq 0, \tag{20}$$

$$\int_{\{(x,t)\in Q; m\le |u(x,t)|\le m+1\}} a(x,t,u,\nabla u)\nabla u \; dx\, dt \longrightarrow 0 \text{ as } m \longrightarrow \infty, \tag{21}$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$ and such that S has a compact support, we have

$$\frac{\partial S(u)}{\partial t} - \operatorname{div}(S'(u)a(x,t,u,\nabla u)) + S''(u)a(x,t,u,\nabla u)\nabla u - \operatorname{div}(S'(u)\Phi(u)) \\ + S''(u)\Phi(u)\nabla u + S'(u)g(u)\varphi(x,|\nabla u|) = fS'(u) \text{ in } \mathcal{D}'(Q), \tag{22}$$

$$S(u)(t=0) = S(u_0) \text{ in } \Omega. \tag{23}$$

The following remarks are concerned with a few comments on Definition 4.2.

***Remark* 4.3.** Equation (22) is formally obtained through pointwise multiplication of (1) by $S'(u)$. However, while $a(x,t,u,\nabla u), \Phi(u)$ and $g(u)\varphi(x,|\nabla u|)$ does not in general make sense in $\mathcal{D}'(Q)$, all the terms in (22) have a meaning in $\mathcal{D}'(Q)$. Indeed, if k is such that supp $S' \subset [-k,k]$, the following identifications are made in (22):

- $S'(u)a(x,t,u,\nabla u)$ identifies with $S'(u)a(x,t,T_k(u),\nabla T_k(u))$ a.e. in Q. Since $|T_k(u)| \le k$ a.e. in Q and $S'(u) \in L^\infty(Q)$, we obtain from (13) that
$$S'(u)a(x,t,T_k(u),\nabla T_k(u)) \in (L_\psi(Q))^N.$$
- $S''(u)a(x,t,u,\nabla u)\nabla u$ identifies with $S''(u)a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u)$ and
$$S''(u)a(x,t,T_k(u),\nabla T_k(u))\nabla T_k(u) \in L^1(Q).$$
- $S'(u)\Big(g(u)\varphi(x,|\nabla u|)\Big)$ identifies with $S'(u)\Big(g(T_k(u))\varphi(x,|\nabla T_k(u)|)\Big)$ a.e in Q. Since $|T_k(u)| \le k$ a.e. in Q and $S'(u) \in L^\infty(Q)$, we obtain from (13), (17) that
$$S'(u)\Big(g(T_k(u))\varphi(x,|\nabla T_k(u)|)\Big) \in L^1(Q).$$
- $S'(u)\Phi(u)$ and $S''(u)\Phi(u)\nabla u$ respectively identify with $S'(u)\Phi(T_K(u))$ and $S''(u)\Phi(T_K(u))\nabla T_K(u)$. Due to the properties of S and Φ beeing a continuous function, the functions S', S'' and $\Phi \circ T_K$ are bounded on $\mathbb{R}$ so that (20) implies that $S'(u)\Phi(T_K(u)) \in (L^\infty(Q))^N$, and $S''(u)\Phi(T_K(u))\nabla T_K(u) \in (L_\varphi(Q))^N$.
- $S'(u)f$ belongs to $L^1(Q)$ by (18).

5. Statement of main results

Our main results are collected in the following theorem.

Theorem 5.1. *Assume that* (13)–(19) *hold true. Then, there exists a renormalized solution* u *of problem* (1) *in the sense of Definition 4.2.*

Proof. The proof of this theorem is done in six steps. ☑

Step 1: Approximate problem

Let us introduce the following regularization of the data,

$$a_n(x,t,r,\xi) = a(x,t,T_n(r),\xi) \quad a.e.\ (x,t) \in Q,\ \forall r \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N, \tag{24}$$

$$\Phi_n \text{ is a Lipschitz continuous bounded function from } \mathbb{R} \text{ into } \mathbb{R}^N, \tag{25}$$

such that Φ_n uniformly converges to Φ on any compact subset of $\mathbb{R}$ as $n \to \infty$ n,

$$f_n \in C_0^\infty(Q) \quad : ||f_n||_{L^1} \leq ||f||_{L^1} \text{ and } f_n \longrightarrow f \text{ in } L^1(Q) \text{ as n tends to } +\infty, \tag{26}$$

$$u_{0n} \in C_0^\infty(\Omega) : ||u_{0n}||_{L^1} \leq ||u_0||_{L^1} \text{ and } u_{0n} \longrightarrow u_0 \text{ in } L^1(\Omega) \text{ as n tends to } +\infty. \tag{27}$$

Let consider the following approximate problem:

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}\big(a(x,t,u_n,\nabla u_n) + \Phi_n(u_n)\big) + g(u_n)\varphi(x, \mid \nabla u_n \mid) = f_n \text{ in } Q, \\ u_n(x,0) = u_{0n}(x) \text{ in } \Omega. \end{cases} \tag{28}$$

For fixed $n > 0$, since Φ is continuous, it is obvious that $|\Phi_n(t)| \leq \max_{|t|\leq n} |\Phi(t)| = C_n$. Moreover, let $\{f_n\} \subset \mathcal{D}(Q)$ be such that $f_n \to f$ strongly in $L^1(Q)$, and let $\{u_{0n}\} \subset \mathcal{D}(\Omega)$ be such that $u_{0n} \to u_0$ strongly in $L^1(\Omega)$. Then, the existence of a weak solution $u_n \in W_0^{1,x}L_\varphi(Q)$ to problem (28) is an easy task (see, e.g., [16]).

Step 2: A priori estimates

The estimates derived in this step rely on usual techniques for problems of the type (28).

Proposition 5.2. *Assume that* (13)–(19) *are satisfied and let* u_n *be a solution of the approximate problem* (28)*. Then for all* $\ell, n > 0$*, we have*

i) $\|T_\ell(u_n)\|_{W_0^{1,x}L_\varphi(Q)} \leq C\ell$,

ii) $\lim\limits_{\ell\longrightarrow\infty} meas\{(x,t) \in Q : |u_n| > \ell\} = 0$ *uniformly with respect to* n, *iii)* $\displaystyle\int_Q g(u_n)\varphi(x,|\nabla u_n|)\,dx\,dt \le C_g$, *where* C_g *is a positive constant not depending on* n.

Proof. We take $T_\ell(u_n)\chi_{(0,\tau)}$ as a test function in (28), we get for every $\tau \in (0,T)$

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t}, T_\ell(u_n)\chi_{(0,\tau)}\rangle + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)dx\,dt\\ &+\int_{Q_\tau}\Phi_n(u_n)\nabla T_\ell(u_n)dx\,dt + \int_{Q_\tau} g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)dx\,dt\\ &=\int_{Q_\tau} f_nT_\ell(u_n)dx\,dt\end{aligned}\tag{29}$$

which implies that

$$\begin{aligned}&\int_\Omega S_\ell(u_n(\tau))\;dx + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,dx\,dt\\ &+\int_{Q_\tau}\Phi_n(u_n)\nabla T_\ell(u_n)dx\,dt\\ &=\int_{Q_\tau} f_nT_\ell(u_n)\,dx\,dt - \int_{Q_\tau} g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)\,dx\,dt + \int_\Omega S_\ell(u_{0n})\;dx\end{aligned}$$

where

$$S_\ell(r) = \int_0^r T_\ell(\sigma)d\sigma = \begin{cases}\frac{r^2}{2} & \text{if } |r|\le\ell,\\ \ell|r| - \frac{r^2}{2} & \text{if } |r|>\ell.\end{cases}\tag{30}$$

The Lipschitz character of Φ_n, Stokes formula together with the boundary condition $u_n = 0$ on $(0,T)\times\partial\Omega$, make it possible to obtain

$$\int_{Q_\tau}\Phi_n(u_n)\nabla T_\ell(u_n)\,dx\,dt = 0.\tag{31}$$

Thanks to the definition of S_ℓ and (27) we have

$$0 \le \int_\Omega S_\ell(u_{0n})\;dx \le \ell\int_\Omega |u_{0n}|\;dx \le \ell||u_0||_{L^1(\Omega)}.\tag{32}$$

Consider now for $\theta,\epsilon>0$ a function $\varrho_\theta^\epsilon \in C^1(\mathbb{R})$ such that

$$\varrho_\theta^\epsilon(s) = \begin{cases} 0 & \text{if } |s|\le\theta,\\ \operatorname{sign}(s) & \text{if } |s|>\theta+\epsilon.\end{cases}\tag{33}$$

and

$$(\varrho_\theta^\epsilon)'(s)\ge 0 \qquad \forall s\in\mathbb{R}.$$

Then, by using $\varrho_\theta^\epsilon(u_n)$ as a test function in (28) and following [19], we can see that

$$\int_{\{|u_n|>\theta\}} |g(u_n)\varphi(x,|\nabla u_n|)|\,dx\,dt \le \int_{\{|u_n|>\theta\}} |f_n|\,dx\,dt + \int_{\{|u_n|>\theta\}} |u_{0n}|\,dx\,dt \tag{34}$$

and so by letting $\theta \to 0$ and using Fatou's lemma, we deduce that $g(u_n)$ $\varphi(x,|\nabla u_n|)$ is a bounded sequence in $L^1(Q_\tau)$, then we obtain iii). By using (31), (32), iii) and (17), it yields

$$\begin{aligned}
&\int_\Omega S_\ell(u_n(\tau))\,dx + \int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,dx\,dt \\
&= \int_{Q_\tau} f_n T_\ell(u_n)\,dx\,dt - \int_{Q_\tau} g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)\,dx\,dt + \int_\Omega S_\ell(u_{0n})\,dx \\
&= \int_{Q_\tau} f_n T_\ell(u_n)\,dx\,dt - \int_{Q_\tau} |g(u_n)\varphi(x,|\nabla u_n|)T_\ell(u_n)|\,dx\,dt + \int_\Omega S_\ell(u_{0n})\,dx \\
&\le \ell||f_n||_{L^1(Q_\tau)} + \ell C_g + \ell||u_0||_{L^1(\Omega)} \\
&\le \ell(||f_n||_{L^1(Q_\tau)} + \ell C_g + ||u_0||_{L^1(\Omega)}) \\
&\le \ell C_0,
\end{aligned} \tag{35}$$

where here and below C_i denote positive constants not depending on n and ℓ. By using (35) and the fact that $S_\ell(u_n) \ge 0$, we can deduce that

$$\int_{Q_\tau} a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,dx\,dt \le \ell C_0, \tag{36}$$

which implies by virtue of (15) that

$$\int_{Q_\tau} \varphi(x,|\nabla T_\ell(u_n)|)\,dx\,dt \le \ell C_1. \tag{37}$$

We deduce from that above inequality (35) that

$$\int_\Omega S_\ell(u_n(\tau))\,dx \le \ell C_0, \text{ for any } \tau \text{ in } [0,T]. \tag{38}$$

On the other hand, thanks to Lemma 3.3, there exist positive constants a, b and c depending only on Ω and φ such that

$$\int_{Q_\tau} \varphi(x,|v(x)|)\,dx\,dt \le a + b\int_{Q_\tau} \varphi(x,\lambda|\nabla v(x)|)\,dx\,dt \quad \forall v \in W_0^{1,x}L_\varphi(Q_\tau). \tag{39}$$

Taking $v = \frac{T_\ell(u_n)}{\lambda}$ in (39) and using (37), one has

$$\int_{Q_\tau} \varphi(x,\frac{|T_\ell(u_n)|}{\lambda})\,dx\,dt \le \ell C_1. \tag{40}$$

On the other hand, one has

$$\begin{aligned}\operatorname{meas}\{|u_n|>\ell\} &\le \frac{1}{\underset{x\in\Omega}{\operatorname{ess\,inf}}\varphi(x,\frac{\ell}{\lambda})}\int_{\{|u_n|>\ell\}}\varphi(x,\frac{\ell}{\lambda})\,dx\,dt\\ &\le \frac{1}{\underset{x\in\Omega}{\operatorname{ess\,inf}}\varphi(x,\frac{\ell}{\lambda})}\int_{\Omega}\varphi(x,\frac{1}{\lambda}|T_\ell(u_n)|)\,dx\,dt\\ &\le \frac{C_1\ell}{\underset{x\in\Omega}{\operatorname{ess\,inf}}\varphi(x,\frac{\ell}{\lambda})},\quad \forall n,\quad \forall \ell\ge 0.\end{aligned} \tag{41}$$

For any $\beta>0$, we have

$$\begin{aligned}\operatorname{meas}\{|u_n-u_m|>\beta\} \le& \operatorname{meas}\{|u_n|>\ell\}+\operatorname{meas}\{|u_m|>\ell\}\\ &+\operatorname{meas}\{|T_\ell(u_n)-T_\ell(u_m)|>\beta\}\end{aligned}$$

and so that

$$\operatorname{meas}\{|u_n-u_m|>\beta\} \le \frac{2C_1\ell}{\underset{x\in\Omega}{\operatorname{ess\,inf}}\varphi(x,\frac{\ell}{\lambda})}+\operatorname{meas}\{|T_\ell(u_n)-T_\ell(u_m)|>\beta\}. \tag{42}$$

By using (40) and Poincaré's inequality in Musielak–Orlicz spaces, we deduce that $(T_\ell(u_n))_n$ is bounded in $W_0^{1,x}L_\varphi(Q)$, and then there exists $\omega_\ell \in W_0^{1,x}L_\varphi(Q)$ such that $T_\ell(u_n) \rightharpoonup \omega_\ell$ weakly in $W_0^{1,x}L_\varphi(Q)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$, strongly in $L^1(Q)$ and a.e. in Q.

Consequently, we can assume that $(T_\ell(u_n))_n$ is a Cauchy sequence in measure in Q.

Let $\varepsilon>0$, then by (42) and the fact that $\frac{2C_1\ell}{\underset{x\in\Omega}{\operatorname{ess\,inf}}\varphi(x,\frac{\ell}{\lambda})}\to 0$ as $\ell\to+\infty$ there exists some $\ell=\ell(\varepsilon)>0$ such that

$$\operatorname{meas}\{|u_n-u_m|>\lambda\}<\varepsilon,\qquad \text{for all } n,m\ge h_0(\ell(\varepsilon),\lambda).$$

This proves that (u_n) is a Cauchy sequence in measure, thus, (u_n) converges almost everywhere to some measurable function u. ☑

Which completes the proof. We now prove the following proposition:

Proposition 5.3. *Let u_n be a solution to the approximate problem* (28)*, then we have the following properties:*

$$u_n \longrightarrow u \text{ a.e. } Q. \tag{43}$$

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) \rightharpoonup \phi_\ell \text{ weakly in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\varphi,\Pi E_\psi), \tag{44}$$

for some $\phi_\ell\in(L_\psi(Q))^N$.

Proof. We have from (37) that $(T_\ell(u_n))$ is bounded in $W_0^{1,x}L_\varphi(Q)$ for every $\ell > 0$. Consider now a $C^2(\mathbb{R})$ nondecreasing function $\zeta_\ell(s) = s$ for $|s| \leq \frac{\ell}{2}$ and $\zeta_\ell(s) = \ell$ sign (s) for $|s| \geq 0$. Multiplying the approximating equation by $\zeta'_\ell(u_n)$, we obtain

$$\begin{aligned}\frac{\partial(\zeta_\ell(u_n))}{\partial t} &= \operatorname{div}\big(a(x,t,u_n,\nabla u_n)\zeta'_\ell(u_n)\big) - a(x,t,u_n,\nabla u_n)\zeta''_\ell(u_n)\nabla u_n \\ &+ \operatorname{div}(\Phi_n(u_n)\zeta'_\ell(u_n)\big) - \Phi_n(u_n)\zeta''_\ell(u_n)\nabla u_n - g(u_n)\varphi(x,|\nabla u_n|)\zeta'_\ell(u_n) + f_n\zeta'_\ell(u_n),\end{aligned} \tag{45}$$

in the sense of distributions. This implies, thanks to (40) and the fact that ζ'_ℓ has compact support, that $\zeta'_\ell(u_n)$ is bounded in $W_0^{1,x}L_\varphi(Q)$ while its time derivative $\frac{\partial(\zeta_\ell(u_n))}{\partial t}$ is bounded in $W_0^{-1,x}L_\varphi(Q) + L^1(Q)$, hence Proposition 3.12 allows us to conclude that $\zeta_\ell(u_n)$ is compact in $L^1(Q)$. Due to the choice of ζ_ℓ, we conclude that for each ℓ, the sequence $(T_\ell(u_n))$ converges almost everywhere in Q, which implies that (u_n) converges almost everywhere to some measurable function u in Q. Therefore, following [8], we can see that there exists a measurable function $u \in L^\infty(0,T;L^1(\Omega))$ such that for every $\ell > 0$ and a subsequence, not relabeled,

$$u_n \to u \text{ a. e. in } Q,$$

and

$$\begin{aligned} T_\ell(u_n) \rightharpoonup T_\ell(u) \text{ weakly in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi),\\ \text{strongly in } L^1(Q) \text{ and a. e. in } Q.\end{aligned} \tag{46}$$

We prove that $a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))$ is bounded sequence in $(L_\psi(Q))^N$.

Let $w \in (E_\varphi(Q)^N$ with $\|w\|_{\varphi,Q} \leq 1$, By using (14), we have

$$\Big(a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\frac{w}{\nu})\Big)(\nabla T_\ell(u_n) - \frac{w}{\nu}) > 0,$$

hence

$$\begin{aligned}\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\frac{w}{\nu}\,dx\,dt &\leq \int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,dx\,dt \\ &- \int_Q a(x,t,T_\ell(u_n),\frac{w}{\nu})(\nabla T_\ell(u_n) - \frac{w}{\nu})\,dx\,dt.\end{aligned} \tag{47}$$

Thanks to (36), we have

$$\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(u_n)\,dx\,dt \leq C_2. \tag{48}$$

where C_2 is a positive constant which is independent of n.

On the other hand, for λ large enough ($\lambda > \beta$), we have by using (13).

$$\begin{aligned}
&\int_Q \psi_x\Big(\frac{a(x,t,T_\ell(u_n),\frac{w}{\nu})}{3\lambda}\Big)\,dx\,dt\\
&\leq \int_Q \psi_x\Big(\frac{\beta\Big(c(x,t)+\psi_x^{-1}(\gamma(x,|T_\ell(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))\Big)}{3\lambda}\Big)\,dx\,dt\\
&\leq \frac{\beta}{\lambda}\int_Q \psi_x\Big(\frac{c(x,t)+\psi_x^{-1}(\gamma(x,|T_\ell(u_n)|))+\psi_x^{-1}(\varphi(x,|w|))}{3}\Big)\,dx\,dt\\
&\leq \frac{\beta}{3\lambda}\bigg(\int_Q \psi_x(c(x,t))\,dx\,dt+\int_Q \gamma(x,|T_\ell(u_n)|)\,dx\,dt+\int_Q \varphi(x,|w|)\,dx\,dt\bigg)\\
&\leq \frac{\beta}{3\lambda}\bigg(\int_Q \psi_x(c(x,t))\,dx\,dt+\int_Q \gamma(x,|T_\ell(u_n)|)\,dx\,dt+\int_Q \varphi(x,|w|)\,dx\,dt\bigg)\\
&\leq C_3,
\end{aligned}$$

Now, since γ grows essentially less rapidly than φ near infinity and by using the Remark 2.2, there exists $h \in L^1(Q)$ such that $\forall c > 0$ we have $\gamma(x,|T_\ell(u_n)|) \leq \varphi(x,c|T_\ell(u_n)|)+h(x,t)$ and so we also have

$$\begin{aligned}
\int_Q \psi_x\Big(\frac{a(x,t,T_\ell(u_n),\frac{w}{\nu})}{3\lambda}\Big)\,dx\,dt \leq \frac{\beta}{3\lambda}\bigg(&\int_Q \psi_x(c(x,t))\,dx\,dt+\int_Q \varphi(x,c|T_\ell(u_n)|)\,dx\,dt\\
&+\int_Q h(x,t)\,dx\,dt+\int_Q \varphi(x,|w|)\,dx\,dt\bigg).
\end{aligned}$$

hence $a(x,t,T_\ell(u_n),\frac{w}{\nu})$ is bounded in $(L_\psi(Q))^N$. Which implies that the second term of the right hand side of (5) is bounded, consequently we obtain

$$\int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))w\,dx\,dt \leq C_3, \quad \text{for all } w \in (E_\varphi(Q)^N \text{ with } \|w\|_{\varphi,Q} \leq 1,$$

where C_3 is a positive constant which is independent of n.

Hence, thanks to the Banach-Steinhaus Theorem, the sequence $(a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)))_n$ is a bounded sequence in $(L_\psi(Q))^N$, thus up to a subsequence

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) \rightharpoonup \phi_k \text{ weakly star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi,\Pi E_\varphi) \tag{49}$$

for some $\phi_k \in (L_\psi(Q))^N$. ☑

Let use prove the following lemma which will be needed later:

Lemma 5.4. *Let u_n be a solution to the approximate problem* (28)*. Then*

$$\lim_{m\longrightarrow\infty}\lim_{n\longrightarrow\infty}\int_{\left\{m\leq|u_n|\leq m+1\right\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt=0. \tag{50}$$

Proof. Considering the following function $v=T_1(u_n-T_m(u_n))$ as test function in (28) we obtain,

$$\begin{aligned}&\langle\frac{\partial u_n}{\partial t},T_1(u_n-T_m(u_n))\rangle+\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\&+\int_Q \operatorname{div}\big[\int_0^{u_n}\Phi(r)T_1'(u_n-T_m(u_n))dr\big]\,dx\,dt=\int_Q f_nT_1(u_n-T_m(u_n))\,dx\,dt\\&-\int_Q g(u_n)\varphi(x,|\nabla u_n|)T_1(u_n-T_m(u_n))\,dx\,dt.\end{aligned} \tag{51}$$

Using the fact that $\int_0^{u_n}\Phi(r)T_1'(u_n-T_m(u_n))dr\in W_0^{1,x}L_\varphi(Q)$ and Stokes formula, we get

$$\begin{aligned}&\int_\Omega U_n^m(u_n(T))dx+\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\&\leq\int_Q|f_nT_1(u_n-T_m(u_n))|\,dx\,dt+\int_Q|g(u_n)\varphi(x,|\nabla u_n|)T_1(u_n-T_m(u_n))|\,dx\,dt\\&+\int_\Omega U_n^m(x,u_{0n})dx\leq\int_Q(|f_n+g(u_n)\varphi(x,|\nabla u_n|)|)|T_1(u_n-T_m(u_n))|\,dx\,dt\\&+\int_\Omega U_n^m(x,u_{0n})dx,\end{aligned} \tag{52}$$

where $U_n^m(r)=\int_0^n\frac{\partial u_n}{\partial t}T_1(s-T_m(s))ds$. In order to pass to the limit as n tends to $+\infty$ in (52), we use the fact that $U_n^m(u_n(T))\geq 0$ and iii) in Proposition 5.2, we obtain that,

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\&\qquad\leq\int_{\{|u_n|>m\}}(|f|+C_g)\,dx\,dt+\int_{\{|u_0|>m\}}|u_0|dx.\end{aligned} \tag{53}$$

Finally by (53) we obtain (50). ☑

Step 3: Almost everywhere convergence of the gradients

Let ρ_m be a truncation defined by

$$\rho_m(s) = \begin{cases} 1 & \text{if} \quad |s| \leq m, \\ m+1-|s| & \text{if} \quad |s| \leq m+1, \\ 0 & \text{if} \quad |s| \geq m+1, \end{cases} \tag{54}$$

where $m > \ell$.

we set $T_\ell^*(s) = \Big(\int_0^{T_\ell(s)} \exp\Big(\int_0^t g(s)ds \Big) dt \Big) \Big(\exp\Big(- \int_0^\infty g(s)ds \Big) \Big)$,

$R_m(s) = \Big(\int_0^s \rho_m(t) \exp\Big(\int_0^t g(s)ds \Big) dt$,

$\omega_{\mu,j}^i = T_\ell(v_j)_\mu + \exp(-\mu t) T_\ell(w_i)$.

Let $(v_j) \in D(Q)$ be a sequence such that

$$v_j \to T_\ell^*(u) \text{ in } W_0^{1,x} L_\varphi(Q) \text{ for the modular convergence,} \tag{55}$$

and let $(\omega_j) \subset \mathcal{D}(Q)$ be a sequence such that $v_j \geq T_\ell^*(\omega_j)$ and ω_j converges strongly to $T_\ell^*(u_0)$ in $L^2(\Omega)$.

Also $T_\ell(v_j)_\mu$ is the mollification with respect to time of $T_\ell(v_j)$, see [7]. Note that $\omega_{\mu,j}^i$ is a smooth function having the following properties:

$$\frac{\partial}{\partial t}(\omega_{\mu,j}^i) = \mu(T_\ell(v_j) - \omega_{\mu,j}^i), \omega_{\mu,j}^i(0) = T_\ell(\omega_i), |\omega_{\mu,j}^i| \leq \ell, \tag{56}$$

$$\omega_{\mu,j}^i \to T_\ell^*(u)_\mu + \exp(-\mu t) T_\ell(w_i) \text{ in } W_0^{1,x} L_\varphi(Q), \tag{57}$$

for the modular convergence as $j \to \infty$,

$$T_\ell^*(u)_\mu + \exp(-\mu t) T_\ell(w_i) \to T_\ell^*(u) \text{ in } W_0^{1,x} L_\varphi(Q), \tag{58}$$

for the modular convergence as $\mu \to \infty$.

Using the admissible test function $Z_{i,j,n}^{\mu,m} = (T_\ell^*(u_n) - \omega_{\mu,j}^i)\rho_m(u_n) \exp\Big(\int_0^{u_n} g(s)ds \Big)$ as a test function in (28) leads to

$$\Big\langle \frac{\partial u_n}{\partial t}, Z_{i,j,n}^{\mu,m} \Big\rangle + \int_Q a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla \omega_{\mu,j}^i)\rho_m(u_n) \exp\Big(\int_0^{u_n} g(s)ds \Big)\, dx\, dt \quad (1)$$

$$+ \int_{\{m \leq u_n \leq m+1\}} a(x,t,u_n,\nabla u_n)(T_\ell^*(u_n) - \omega_{\mu,j}^i)\nabla u_n \rho_m'(u_n) \exp\Big(\int_0^{u_n} g(s)ds \Big)\, dx\, dt \quad (2)$$

$$+ \int_{\{m \leq u_n \leq m+1\}} \Phi_n(u_n)(T_\ell^*(u_n(u_n) - \omega_{\mu,j}^i)\nabla u_n \rho_m'(u_n) \exp\Big(\int_0^{u_n} g(s)ds \Big)\, dx\, dt \quad (3)$$

$$+ \int_Q \Phi_n(u_n)(\nabla T_\ell^*(u_n) - \nabla \omega_{\mu,j}^i)\rho_m(u_n) \exp\Big(\int_0^{u_n} g(s)ds \Big)\, dx\, dt \quad (4)$$

$$= \int_Q f_n Z_{i,j,n}^{\mu,m}\, dx\, dt - \int_Q g(u_n)\varphi(x, |\nabla u_n|) Z_{i,j,n}^{\mu,m}\, dx\, dt$$

$$= (5) + (6).$$

We will denote $\epsilon(n, j, \mu, i)$ any quantity such that,

$$\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n, j, \mu, i) = 0.$$

Let us recall that for $u_n \in W_0^{1,x}L_\varphi(Q)$, there exists a smooth function $u_{n\sigma}$ such that:

$$u_{n\sigma} \to u_n \text{ for the modular convergence in } W_0^{1,x}L_\varphi(Q) \cap L^2(Q),$$

$$\frac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,x}L\psi(Q) + L^2(Q).$$

$$\begin{aligned}\Big\langle \frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n}\Big\rangle &= \lim_{\sigma\to 0^+}\int_Q (u_{n\sigma})'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_{n\sigma}} g(s)ds\Big)\,dx\,dt\\ &= \lim_{\sigma\to 0^+}\Big(\int_Q (R_m(u_{n\sigma}) - T^*_\ell(u_{n\sigma}))'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\,dx\,dt\\ &\quad + \int_Q T^*_\ell(u_{n\sigma})'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\,dx\,dt\Big)\\ &= \lim_{\sigma\to 0^+}\int_\Omega \Big[(R_m(u_{n\sigma}) - T^*_\ell(u_{n\sigma}))(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})dx\Big]_0^T\\ &\quad - \int_Q (R_m(u_{n\sigma}) - T^*_\ell(u_{n\sigma}))(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})'\,dx\,dt\\ &\quad + \int_Q T^*_\ell(u_{n\sigma})'(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\,dx\,dt = I_1 + I_2 + I_3.\end{aligned}$$

Remark also that,

$R_m(u_{n\sigma}) \geq T^*_\ell(u_{n\sigma})$ if $u_{n\sigma} < \ell$ and $R_m(u_{n\sigma}) > \ell = T^*_\ell(u_{n\sigma}) \geq |\omega^i_{\mu,j})|$ if $u_{n\sigma} \geq \ell$.

$$\begin{aligned}I_1 &= \int_\Omega (R_m(u_{n\sigma})(T) - T^*_\ell(u_{n\sigma})(T))(T^*_\ell(u_{n\sigma})(T) - \omega^i_{\mu,j}(T))dx\\ &- \int_\Omega (R_m(u_{n\sigma})(0) - T^*_\ell(u_{n\sigma})(0))(T^*_\ell(u_{n\sigma})(0) - \omega^i_{\mu,j}(0))dx = I_1^1 + I_1^2.\\ I_1^1 &\geq \int_{\{u_{n\sigma}(T)\leq\ell\}} (R_m(u_{n\sigma})(T) - T^*_\ell(u_{n\sigma})(T))(T^*_\ell(u_{n\sigma})(T) - \omega^i_{\mu,j}(T))dx,\end{aligned}$$

and it is easy to see that,

$$\limsup_{\sigma\to 0^+} I_1^1 \geq \epsilon(n, j, \mu).$$

$$\begin{aligned}I_1^2 &= -\int_{\{u_{n\sigma}(0)\leq\ell\}} (R_m(u_{n\sigma})(0) - T^*_\ell(u_{n\sigma})(0))(T^*_\ell(u_{n\sigma})(0) - T_\ell(\omega_i))dx\\ &- \int_{\{u_{n\sigma}(0)>\ell\}} (R_m(u_{n\sigma})(0) - T^*_\ell(u_{n\sigma})(0))(T^*_\ell(u_{n\sigma})(0) - T_\ell(\omega_i))dx.\end{aligned}$$

For the first part, it is the same as I_1^1 and for the second part, we have

$$I_1^2 \geq \epsilon(n,j,\mu) - \int_{\left\{u_{n\sigma}(0)\geq \ell\right\}} (R_m(u_{n\sigma})(0) - T_\ell^*(u_{n\sigma})(0))(T_\ell^*(u_{n\sigma})(0) - T_\ell(\omega_i))dx.$$

$$\limsup_{\sigma\to 0^+} I_1 \geq \epsilon(n,j,\mu) - \int_{\left\{u_{0n}\geq \ell\right\}} (R_m(u_{0n}) - T_\ell^*(u_{0n})(T_\ell^*(u_{0n}) - T_\ell(\omega_i))dx = J_1.$$

Now letting $n \to \infty$, we have

$$\lim_{n\to+\infty} J_1 = \int_{\left\{u_0\geq \ell\right\}} (R_m(u_0) - T_\ell^*(u_0)(T_\ell^*(u_0) - T_\ell(\omega_i))dx$$

and by letting $i \to \infty$, we obtain

$$\limsup_{\sigma\to 0^+} I_1 \geq \epsilon(n,j,i,\mu).$$

Concerning I_2, we remark that $T_\ell^*(u_{n\sigma})' = 0$ if $u_{n\sigma} > \ell$, then

$$\begin{aligned} I_2 = & -\int_{\left\{u_{n\sigma}\leq \ell\right\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)'\,dx\,dt \\ & + \int_{\left\{u_{n\sigma}> \ell\right\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(\omega_{\mu,j}^i)'\,dx\,dt \\ = & \quad I_2^1 + I_2^2. \end{aligned}$$

As in $I_1, I_1^2 \geq \epsilon(n,j,\mu)$ and

$$\begin{aligned} I_2^2 = & \int_{\left\{u_{n\sigma}> \ell\right\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(\omega_{\mu,j}^i)'\,dx\,dt \\ \geq & \quad \mu\int_{\left\{u_{n\sigma}> \ell\right\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell(v_j) - T_\ell^*(u_{n\sigma}))'\,dx\,dt, \end{aligned}$$

thus by using the fact that

$$(R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell^*(u_{n\sigma} - \omega_{\mu,j}^i)\chi_{\left\{u_{n\sigma}>\ell\right\}} \geq 0.$$

So

$$\limsup_{\sigma\to 0^+} I_2^2 \geq \mu\int_{\left\{u_{n\sigma}> \ell\right\}} (R_m(u_{n\sigma}) - T_\ell^*(u_{n\sigma}))(T_\ell(v_j) - T_\ell^*(u_{n\sigma}))'\,dx\,dt = \epsilon(n,j).$$

About I_3,

$$\begin{aligned} I_3 &= \int_Q T_\ell^*(u_{n\sigma})'(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)\,dx\,dt \\ &= \int_Q (T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)'(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)\,dx\,dt + \int_Q (\omega_{\mu,j}^i)'(T_\ell^*(u_{n\sigma}) - \omega_{\mu,j}^i)\,dx\,dt. \end{aligned}$$

Set $\phi(r) = \frac{r^2}{2}, \qquad \phi \geq 0$, then

$$I_3 = \Big[\int_\Omega \phi(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})dx \Big]_0^T + \mu \int_Q (T_\ell(v_j) - \omega^i_{\mu,j})(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\, dx\, dt.$$

$$\geq \epsilon(n,j,\mu) - \int_\Omega \phi(T^*_\ell(u_{n\sigma})(0) - T_\ell(\omega_i))dx$$

$$+ \mu \int_Q (T_\ell(v_j) - \omega^i_{\mu,j})(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\, dx\, dt \,(\text{as in } I_2).$$

So,

$$\begin{aligned}\limsup_{\sigma\to 0^+} I_3 &\geq \epsilon(n,j,\mu) - \int_\Omega \phi(T^*_\ell(u_{0n}) - T_\ell(\omega_i))dx \\ &\quad + \mu \int_Q (T_\ell(v_j) - T^*_\ell(u_n))(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\, dx\, dt \\ &= - \int_\Omega \phi(T^*_\ell(u_{0n}) - \omega_i)dx + \mu \int_Q (T_\ell(v_j) - \omega^i_{\mu,j})(T^*_\ell(u_{n\sigma}) - \omega^i_{\mu,j})\, dx\, dt \\ &\quad + \epsilon(n,j,\mu),\end{aligned}$$

and we easily deduce

$$\limsup_{\sigma\to 0^+} I_3 \quad \geq \quad \epsilon(n,j,i,\mu).$$

Finally, we conclude that

$$\text{(59)} \quad \Big\langle \frac{\partial u_n}{\partial t}, (T^*_\ell(u_n) - \omega^i_{\mu,j})\rho_m(u_n)\exp(\int_0^{u_n} g(s)ds \Big\rangle \geq \epsilon(n,j,i,\mu).$$

We are interested now with the terms of (1), (2), (4) and (5). Let us remark that

$$\begin{aligned}\nabla T^*_\ell(u) &= \Big(\exp\Big(-\textstyle\int_0^\infty g(s)ds\Big)\Big)\exp\Big(\textstyle\int_0^{T_\ell(u_n)} g(s)ds\Big)\nabla T_\ell(u) \\ &=: \lambda(u)\nabla T_\ell(u).\end{aligned} \tag{60}$$

About (1)

$$\begin{aligned}&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T^*_\ell(u_n) - \nabla\omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\, dx\, dt \\ &= \int_{\{u_n\leq \ell\}} a(x,t,u_n,\nabla u_n)(\nabla T^*_\ell(u_n) - \nabla\omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\, dx\, dt \\ &+ \int_{\{u_n> \ell\}} a(x,t,u_n,\nabla u_n)(\nabla T^*_\ell(u_n) - \nabla\omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\, dx\, dt \\ &= \int_Q a(x,t,T_\ell(u_n),\nabla u_n)(\nabla T^*_\ell(u_n) - \nabla\omega^i_{\mu,j})\exp\Big(\int_0^{u_n} g(s)ds\Big)\, dx\, dt \\ &+ \int_{\{u_n> \ell\}} a(x,t,u_n,\nabla u_n)(\nabla T^*_\ell(u_n) - \nabla\omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\, dx\, dt,\end{aligned}$$

recall that $\rho_m(u_n) = 1$ on $|u_n| \le \ell$.

Let $s > 0$, $Q_s = \{(x,t) \in Q : |\nabla T_\ell(u)| \le s\}$, $Q_s^j = \{(x,t) \in Q : |\nabla T_\ell(v_j)| \le s\}$.

$$\begin{aligned}
&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T_\ell^*(u_n) - \nabla\omega_{\mu,j}^i)\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\
&= \int_Q (a(x,t,T_\ell(u_n),\nabla u_n) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)(\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j)\\
&\times \exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt + \int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)(\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j)\\
&\times \exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\
&+ \int_Q a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))\nabla T_\ell(v_j)\chi_s^j \exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\
&- \int_Q a(x,t,u_n,\nabla u_n)\nabla\omega_{\mu,j}^i\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\
&= J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

We shall go to the limit as n, j, m and $s \to \infty$ in the last three integrals of the last side.

As for the inequality (37), we can prove that

$$T_\ell^*(u_n) \rightharpoonup T_\ell^*(u) \text{ in } W_0^{1,x}L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\psi), \text{ strongly in } L^1(Q), \text{ and a.e. in } Q. \tag{61}$$

Now, Starting with J_2, we have by letting $n \to \infty$,

$$J_2 = \int_Q a(x,t,T_\ell(u),\nabla T_\ell(v_j)\chi_s^j)(\nabla T_\ell^*(u) - \nabla T_\ell(v_j)\chi_s^j)\exp\Big(\int_0^{u} g(s)ds\Big)\,dx\,dt + \epsilon(n).$$

Since

$$a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j) \to a(x,t,T_\ell(u),\nabla T_\ell(v_j)\chi_s^j) \text{ strongly in} (E_\psi(Q))^N,$$

$$a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j) \to a(x,t,T_\ell(u),\nabla T_\ell(u)\chi_s) \text{ strongly in } (E_\psi(Q))^N,$$

and

$$\nabla T_\ell(v_j)\chi_s^j \to \nabla T_\ell^*(u)\chi_s \text{ strongly in } (L_\varphi(Q))^N,$$

then,

$$J_2 = \epsilon(n,j). \tag{62}$$

Following the same way as in J_2 and using (54), one has

$$J_3 = \int_Q \phi_\ell \nabla T_\ell^*(u)\exp\Big(\int_0^{u} g(s)ds\Big)\,dx\,dt + \epsilon(n,j,\mu,s). \tag{63}$$

Concerning the terms J_4:

$$J_4 = -\int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt$$
$$= -\int_{\{|u_n|\le \ell\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt$$
$$-\int_{\{\ell<|u_n|\le m+1\}} a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla\omega^i_{\mu,j}\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt.$$

By letting first n then j and finally μ go to infinity

$$J_4 = -\int_Q \phi_\ell \nabla T^*_\ell(u)\exp\Big(\int_0^{u} g(s)ds\Big)\,dx\,dt + \epsilon(n,j,\mu). \tag{64}$$

We conclude then that

$$\begin{aligned}&\int_Q a(x,t,u_n,\nabla u_n)(\nabla T^*_\ell(u_n)-\nabla\omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\ &\quad=\int_Q (a(x,t,T_\ell(u_n),\nabla u_n)-a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi^j_s) \\ &\quad\times(\nabla T^*_\ell(u_n)-\nabla T_\ell(v_j)\chi^j_s)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt+\epsilon(n,j,\mu,s).\end{aligned}\tag{65}$$

About (2)

$$\Big|\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)(T^*_\ell(u_n)-\omega^i_{\mu,j})\nabla u_n\rho'_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\Big|$$
$$\le C(k)\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt.$$

Then by (48) we deduce that,

$$\Big|\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)(T^*_\ell(u_n)-\omega^i_{\mu,j})\nabla u_n\rho'_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\Big|$$
$$\le \epsilon(n,\mu,m).$$

About (3) and (4), by using, Lebesgue's convergence theorem, we can prove that

$$\Phi_n(u_n)\rho_m(u_n)\to\Phi(u)\rho_m(u) \text{ strongly in } (E_\psi(Q)^N) \text{ as } n\to\infty,$$

and $\Phi_n(u_n)\chi_{\{m\le|u_n|\le m+1\}}(\nabla T^*_\ell(u_n)-\nabla\omega^\mu_{i,j})\rho_m(u_n)\to$
$\Phi(u)\chi_{\{m\le|u_n|\le m+1\}}(\nabla T^*_\ell(u)-\nabla\omega^\mu_{i,j})\rho_m(u)$, strongly in $(E_\psi(Q)^N)$ as $n\to\infty$.

Then by virtue of $\nabla T^*_\ell(u_n)\rightharpoonup\nabla T^*_\ell(u)$ weakly in $(L_\varphi(Q)^N)$, and

$$\nabla u_n\chi_{\{m\le|u_n|\le m+1\}}=\nabla T^*_{m+1}(u_n)\chi_{\{m\le|u_n|\le m+1\}} \text{ a. e. in } Q,$$

one has,

$$\int_Q \Phi_n(u_n)(\nabla T^*_\ell(u_n) - \nabla \omega^\mu_{i,j})\rho_m(u_n)\,dx\,dt \to \int_Q \Phi(u)(\nabla T^*_\ell(u) - \nabla \omega^\mu_{i,j})\rho_m(u)\,dx\,dt,$$

as $n \to \infty$, and

$$\int_{\{m\leq |u_n|\leq m+1\}} \Phi_n(u_n)(T^*_\ell(u_n) - \omega^\mu_{i,j})\nabla u_n \rho'_m(u_n)\,dx\,dt$$
$$\to \int_{\{m\leq |u_n|\leq m+1\}} \Phi(u)(T^*_\ell(u) - \omega^\mu_{i,j})\nabla u \rho'_m(u)\,dx\,dt,$$

as $n \to +\infty$. On the other hand, by using the modular convergence of $(\omega^\mu_{i,j})$ as $j \to +\infty$ and letting μ tend to infinity, we deduce that,

$$\int_Q \Phi_n(u_n)(\nabla T^*_\ell(u_n) - \nabla \omega^i_{\mu,j})\rho_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt = \epsilon(n,j,\mu), \quad (66)$$

and

$$\int_{\{m\leq |u_n|\leq m+1\}} \Phi_n(u_n)(T^*_\ell(u_n(u_n) - \omega^i_{\mu,j})\nabla u_n \rho'_m(u_n)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt$$
$$= \epsilon(n,j,\mu). \quad (67)$$

About (3).

Similarly, by the almost every where convergence of u_n, we have $(T^*_\ell(u_n) - \omega^i_{\mu,j})\rho_m(u_n)\exp(\int_0^{u_n} g(s)ds$ converges to $(T^*_\ell(u) - \omega^i_{\mu,j})\rho_m(u)\exp(\int_0^{u} g(s)ds$ in $L^1(Q)$ weakly * and then,

$$\int_Q f_n(T^*_\ell(u_n) - \omega^\mu_{i,j})\rho_m(u_n)\,dx\,dt \to \int_Q f_n(T^*_\ell(u) - \omega^\mu_{i,j})\rho_m(u)\,dx\,dt.$$

So that,

$$(T^*_\ell(u) - \omega^\mu_{i,j})\rho_m(u) \to (T^*_\ell(u) - T^*_\ell(u)_\mu - \exp(-\mu t)T^*_\ell(w_i),$$

in $L^\infty(Q)$ weakly * as $j \to \infty$, and also,

$$(T^*_\ell(u) - T^*_\ell(u))_\mu - \exp(-\mu t)T^*_\ell(w_i) \to 0 \text{ in } L^\infty(Q) \text{ weak * as } \mu \to \infty.$$

Then, we deduce that,

$$\int_Q f_n(T^*_\ell(u_n) - \omega^\mu_{i,j})\rho_m(u_n)\,dx\,dt = \epsilon(n,j,\mu). \quad (68)$$

Taking now into account the estimation of (1), (2), (3), (4) and (5), we obtain

$$\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)(\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j) \times \exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt = \epsilon(n,j,\mu,i,s,m).$$

On the other hand,

$$\begin{aligned}&\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell^*(u)\chi_s)\\ &\qquad \times (\nabla T_\ell^*(u_n) - \nabla T_\ell^*(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\ &- \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)\\ &\qquad \times (\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\ &= \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n))(\nabla T_\ell(v_j)\chi_s^j - \nabla T_\ell^*(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\ &- \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell^*(u_n)\chi_s)(\nabla T_\ell(v_j)\chi_s^j - \nabla T_\ell *(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\ &+ \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell^*(v_j)\chi_s^j)(\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt.\end{aligned}$$

Each term of the last right hand side is of the form $\epsilon(n,j,s)$, which gives

$$\begin{aligned}&\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell^*(u_n),\nabla T_\ell^*(u)\chi_s)\\ &\qquad \times (\nabla T_\ell^*(u_n) - \nabla T_\ell^*(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\ &= \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j)\\ &\qquad \times (\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt + \epsilon(n,j,s).\end{aligned}$$

Following the same technique used in [13], we have for all $r < s$:

$$\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell^*(u))(\nabla T_\ell^*(u_n) - \nabla T_\ell^*(u))\,dx\,dt \to 0. \tag{69}$$

On the other hand, by using (60), we have

$$(\lambda(u_n) - \lambda(u))\nabla T_\ell(u)\chi_{\{|\nabla T_\ell(u)| \le r\}} \to 0 \text{ strongly in } (E_\varphi(Q))^N,$$

and

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(u)) \rightharpoonup \phi_\ell - a(x,t,T_\ell(u),\nabla T_\ell(u))$$

$$\text{weakly in } (L_\psi(Q))^N,$$

which gives

$$\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(u)))\nabla T_\ell(u)((\lambda(u_n)-\lambda(u))\,dx\,dt \to 0. \tag{70}$$

By using:
-(61),
-the monotonicity condition,
-(60) and the decomposition,
$\nabla T^*_\ell(u_n) - \nabla T^*_\ell(u) = \lambda(u_n)(\nabla T_\ell(u_n) - \nabla T_\ell(u)) + (\lambda(u_n) - \lambda(u))\nabla T_\ell(u)$,
-(69) and (5),
we obtain

$$\lim_{n\to\infty} \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)) - a(x,t,T_\ell(u_n),\nabla T_\ell(u)) \times (\nabla T_\ell(u_n) - \nabla T_\ell(u))\,dx\,dt = 0.$$

Thus, there exists a subsequence also denoted by u_n such that

$$\nabla T_\ell(u_n) \longrightarrow \nabla T_\ell(u) \text{ a.e. in } Q. \tag{71}$$

We deduce then that,

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n) \rightharpoonup a(x,t,T_\ell(u),\nabla T_\ell(u) \text{ in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\varphi,\Pi E_\psi). \tag{72}$$

Step 4: Modular convergence of the truncations

We have proved that

$$\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi^j_s))(\nabla T^*_\ell(u_n) - \nabla T_\ell(v_j)\chi^j_s) \times \exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt \le \epsilon(n,j,\mu,,i,s,m). \tag{73}$$

And we can also deduce that

$$\begin{aligned}&\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n) - a(x,t,T_\ell(u_n),\nabla T_\ell^*(u)\chi_s))\\&\qquad(\nabla T_\ell^*(u_n) - \nabla T_\ell^*(u)\chi_s)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt\\&= \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n) - a(x,t,T_\ell(u_n),\nabla T_\ell(v_j)\chi_s^j))\\&\qquad(\nabla T_\ell^*(u_n) - \nabla T_\ell(v_j)\chi_s^j)\exp\Big(\int_0^{u_n} g(s)ds\Big)\,dx\,dt + \epsilon(n,j,s),\end{aligned}$$

then

$$\begin{aligned}&\int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u_n)\,dx\,dt\\&\quad\le \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u)\chi_s\,dx\,dt\\&\quad+ \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell^*(u)\chi_s)(\nabla T_\ell^*(u_n) - T_\ell(u)\chi_s)\,dx\,dt\\&\quad+ \epsilon(n,j,\mu,,i,s,m),\end{aligned}$$

and

$$\begin{aligned}\limsup_n \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u_n)\,dx\,dt&\\ \le \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u)\chi_s\,dx\,dt&\\ + \lim_n \epsilon(n,j,\mu,,i,s,m),&\end{aligned}$$

then

$$\begin{aligned}\limsup_n \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u_n)\,dx\,dt&\\ \le \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u)\,dx\,dt&\\ \le \liminf_n \int_Q (a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u)\,dx\,dt,&\end{aligned}$$

as $n \to \infty$, we deduce

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell^*(u_n) \to a(x,t,T_\ell(u),\nabla T_\ell(u)\nabla T_\ell^*(u) \text{ in } L^1(Q).$$

Using the same argument as above, we obtain

$$a(x,t,T_\ell(u_n),\nabla T_\ell(u_n)\nabla T_\ell(u_n) \to a(x,t,T_\ell(u),\nabla T_\ell(u)\nabla T_\ell(u) \text{ in } L^1(Q),$$

and Vitali's theorem and (14) gives

$$\nabla T_\ell(u_n) \to \nabla T_\ell(u) \text{ for the modular convergence in } (L_\varphi(Q))^N.$$

Step 5:

In this step we prove that u satisfies (21).

Lemma 5.5. *The limit u of the approximate solution u_n of (28) satisfies. Then*

$$\lim_{m\longrightarrow\infty}\int_{\left\{m\leq|u|\leq m+1\right\}} a(x,t,u,\nabla u)\nabla u dxdt = 0. \tag{74}$$

Proof. To this end, note that for any fixed $m \geq 0$ one has

$$\begin{aligned}
&\int_{\left\{m\leq|u_n|\leq m+1\right\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&= \Big(\int_Q a(x,t,u_n,\nabla u_n)[\nabla T_{m+1}(u_n) - \nabla T_m(u_n)]\,dx\,dt\Big)\\
&= \Big(\int_Q a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\nabla T_{m+1}(u_n)\,dx\,dt\\
&- \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)\,dx\,dt\Big).
\end{aligned} \tag{75}$$

According to (72), one can pass to the limit as n tends to $+\infty$ for fixed $m \geq 0$ to obtain

$$\begin{aligned}
&\lim_{n\longrightarrow\infty}\int_{\left\{m\leq|u_n|\leq m+1\right\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt\\
&= \int_Q a(x,t,T_{m+1}(u),\nabla T_{m+1}(u))\nabla T_{m+1}(u)\,dx\,dt\\
&- \int_Q a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)\,dx\,dt\\
&= \int_{\left\{m\leq|u|\leq m+1\right\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt.
\end{aligned} \tag{76}$$

Taking the limit as m tends to $+\infty$ in (76) and using the estimate (50) it possible to conclude that (75) holds true and the proof of Lemma 5.5 is complete. ☑

Step 6: Passing to the limit

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that supp$S' \subset [-K,K]$. Pointwise multiplication of the approximate equation (28) by $S'(u_n)$ leads to

$$\begin{aligned}\frac{\partial S(u_n)}{\partial t} &- \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)S'(u_n)\Big) + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \\ &- \operatorname{div}\Big(S'(u_n)\Phi(u_n)\Big) + S''(u_n)\Phi(u_n)\nabla u_n + g(u_n)\varphi(x,|\nabla u_n|)S'(u_n) \\ &= f_n S'(u_n). \end{aligned} \tag{77}$$

It what follows we pass to the limit as n tends $+\infty$, in each term of (77).

- Since S is bounded and continuous, then the fact that $u_n \longrightarrow u$ a.e. in Q, implies that $S(u_n)$ converges to $S(u)$ a.e. in Q and L^∞ weak-$*$.

  Consequently,

$$\frac{\partial S(u_n)}{\partial t} \longrightarrow \frac{\partial S(u)}{\partial t} \quad \text{in } D'(Q) \text{ as } n \text{ tends to } +\infty.$$

- Since supp$S' \subset [-K,K]$, we have for $n \geq K$,

$$a(x,t,u_n,\nabla u_n)S'(u_n) = a(x,t,T_K(u_n),\nabla T_K(u_n))S'(u_n) \quad \text{a.e. in } Q.$$

  The pointwise convergence of u_n to u and (71) as n tends to ∞ and the bounded character of S' permit us to conclude that

$$a(x,t,T_K(u_n),\nabla T_K(u_n))S'(u_n) \longrightarrow a(x,t,T_K(u),\nabla T_K(u))S'(u) \quad \text{weakly in} (L_\psi(Q))^N \tag{78}$$

  as n tends to infinity.

- Regarding the 'energy' term, we have

  $S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n = S''(u_n)a(x,t,T_K(u_n),\nabla T_K(u_n))\cdot\nabla T_K(u_n)$ a.e. in Q.

  The pointwise convergence of $S'(u_n) \longrightarrow S'(u)$ and (71) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that

$$S''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \rightharpoonup S''(u)a(x,t,T_K(u),\nabla T_K(u))\cdot\nabla T_K(u) \tag{79}$$

  weakly in $L^1(Q)$.

  Recall that

$$S''(u)a(x,t,T_K(u),\nabla T_K(u))\cdot\nabla T_K(u) = S''(u)a(x,t,u,\nabla u)\cdot\nabla u \text{ a.e. in } Q.$$

- Since *supp*$S' \subset [-K,K]$, we have

$$S'(u_n)\Phi_n(u_n) = S'(u_n)\Phi_n(T_K(u_n)) \text{ a.e. in } Q. \tag{80}$$

As a consequence of (16) and (43), it follows that:

$$S'(u_n)\Phi_n(u_n) \to S'(u)\Phi(T_K(u)) \quad \text{a.e. in } (E_\varphi(Q))^N, \tag{81}$$

we have, $\nabla S''(u_n)$ converges to $\nabla S''(u)$ weakly in $(L_\varphi(Q))^N$ as n tends to $+\infty$, while $\Phi_n(T_K(u_n))$ is uniformly bounded with respect to n and converges a. e. in Q to $\Phi(T_K(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\Phi_n(u_n)\nabla u_n \rightharpoonup S''(u)\Phi(u)\nabla u \quad \text{weakly in } L_\varphi(Q). \tag{82}$$

- Since $suppS' \subset [-K, K]$, we have

$$S'(u_n)g(u_n)\varphi(x, |\nabla u_n|) = S'(u_n)g(T_K(u_n))\varphi(x, |\nabla T_K(u_n)|) \text{ a.e. in } Q. \tag{83}$$

By using lemma (3.1) and (17), we can deduce:

$$S'(u_n)g(u_n)\varphi(x, |\nabla u_n|) \rightharpoonup S'(u)g(T_K(u))\varphi(x, |\nabla T_K(u)|) \text{ weakly in } L^1(Q). \tag{84}$$

- Due to $f_n \to f$ strongly in $L^1(Q)$ and the fact that $u_n \longrightarrow u$ a.e. in Q, we have

$$S'(u_n)f_n \longrightarrow S'(u)f \quad \text{strongly in } L^1(Q). \tag{85}$$

As a consequence of the above convergence results, we are in a position to pass to the limit as n tends to $+\infty$ in equation (77) and to conclude that

$$\begin{aligned} \frac{\partial S(u)}{\partial t} &- \operatorname{div}\Big(a(x,t,u,\nabla u)S'(u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u \\ &- \operatorname{div}\Big(S'(u)\phi(u)\Big) + S''(u)\phi(u)\cdot\nabla u + g(u)\varphi(u,|\nabla u|)S'(u) \\ &= fS'(u). \end{aligned} \tag{86}$$

It remains to show that $S(u)$ satisfies the initial condition. To this end, firstly notice that, since S is bounded, $S(u_n)$ is bounded in $L^\infty(Q)$. Secondly, (77) and the above considerations on the behavior of the terms of this equation show that $\frac{\partial S(u_n)}{\partial t}$ is bounded in $L^1(Q) + V^*$. As a consequence, an Aubin's type lemma (see, e.g, [22]) implies that $S(u_n)$ lies in a compact set of $C^0([0,T], L^1(\Omega))$. It follows that, on the one hand, $S(u_n)(t=0) = S(u_n^0)$ converges to $S(u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S imply that

$$S(u)(t=0) = S(u_0) \quad \text{in } \Omega.$$

As a conclusion of step 1 to step 6, the proof of Theorem 5.1 is complete.

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