Revista Colombiana de Matemáticas
Volumen 45(2011)1, páginas 31-35

Simplified Morasses without Linear Limits

Morasses simplificado sin límites lineales

Franqui Cárdenas

Universidad Nacional de Colombia, Bogotá, Colombia

Abstract. If there is a strongly unfoldable cardinal then there is a forcing extension with a simplified $(\omega_2, 1)$-morass and no simplified $(\omega_1, 1)$-morass with linear limits.

Key words and phrases. Morasses, Square Sequences, Unfoldable cardinals.

2000 Mathematics Subject Classification. 03E35, 03E55.

Resumen. Si hay un cardinal desdoblable entonces hay una extensión forcing con una $(\omega_2, 1)$-morass simplificada y ninguna $(\omega_1, 1)$-morass simplificada con límites lineales.

Palabras y frases clave. Morasses, sucesiones cuadrado, cardinales desdoblables.

1. Introduction

Morasses and its variations have been applied to solving problems of different sources in mathematics like combinatorial (Kurepa, Cantor trees), model theoretic (Chang transfer cardinal theorems) and as a test question for some inner models. We are interested in two kind of morasses: plain morasses and morasses with linear limits. We observe that these two notions do not always agree: If there are simplified morasses with linear limits, then there are morasses but the converse is not generally true.

We will need more than ZFC since Donder [2] has shown that if $V = L$ and $\kappa > \omega$ is a regular but not weakly compact cardinal then there is a simplified $(\kappa, 1)$-morass with linear limits. He also has proved there the following:

Theorem 1 (Lemma 1 in [2])**.** *If there is a simplified $(\kappa, 1)$-morass with linear limits, then κ is not weakly compact.*

Also Stanley in [1] has observed that if there is a supercompact cardinal then there is a simplified $(\omega_2, 1)$-morass but there is no simplified $(\omega_2, 1)$-morass with linear limits. Donder's statement suggests that it should be enough a weakly compact cardinal. In this note, we improve this statement by using just a strongly unfoldable cardinal. Concretely, we prove the following:

Main Theorem. *Let κ be a strongly unfoldable cardinal. Then there is a forcing extension with a simplified $(\omega_2, 1)$-morass but with no a simplified $(\omega_1, 1)$-morass with linear limits.*

For this, we will use the following theorem by Johnstone:

Theorem 2 (See [6])**.** *Let κ be strongly unfoldable cardinal. Then there is a set forcing extension in which the strong unfoldability of κ is indestructible by $< \kappa$-closed, κ-proper forcing of any size. This includes all $< \kappa$-closed posets that are either κ^+-c.c. or $\leq \kappa$-strategically closed.*

Also, we will use Proposition 50 and Proposition 52 in [5]. We summarize these propositions in the following theorem:

Theorem 3. *The forcing which adds a $(\kappa, 1)$-morass is $< \kappa$-closed and has the κ^+-c.c.*

The idea is to add a simplified $(\kappa, 1)$-morass for a strongly unfoldable cardinal κ as above; this partial order is $< \kappa$-closed and has the κ^+-c.c. by Theorem 3 and does not destroys the strongly unfoldability of κ. Then we collapse with the partial order $Col(\omega_1, < \kappa)$. This forcing collapses κ to ω_2 and preserves everything above κ^+, in particular it preserves the simplified morass.

Theorem 4 (Corollary 7.9 in [3])**.** *Let τ regular cardinal and $\kappa > \tau$ weakly compact cardinal. If G is $Col(\tau, < \kappa)$-generic then*

$$V[G] \models \text{“}If\ S \subseteq S^{\tau^+}_{<\tau}\ is\ stationary, there\ is\ an\ \alpha \in S^{\tau^+}_{\tau},\quad with\ S \cap \alpha\ stationary\text{”}.$$

where $S^{\tau^+}_{<\tau} = \{\beta < \tau^+ \mid cof(\beta) < \tau\}$ and $S^{\tau^+}_{\tau} = \{\beta < \tau^+ \mid cof(\beta) = \tau\}$.

Theorem 5 (Fact 2.9 in [4])**.** *If $\square_\tau$ holds and $S \subseteq \tau^+$ is a stationary set, then there exists a stationary $T \subseteq \tau^+$ such that $T \cap \alpha$ is not stationary for every $\alpha < \tau^+$.*

We observe that $\square_{\omega_1}$ fails in this extension.

Theorem 6 (Theorem 3.1 in [1])**.** *If there is a simplified $(\kappa, 1)$-morass with linear limits then $\square_\kappa$ is true.*

We conclude from the previous theorem that there is no simplified $(\omega_1, 1)$-morass with linear limits.

2. Strongly Unfoldable Cardinals

Strongly unfoldable cardinals were introduced by Villaveces in [7], they generalize weakly compact cardinals, preserve to the constructible universe L, but they have some features of strong and supercompact cardinals.

Let $\kappa > \omega$ be a regular cardinal. M is a *κ-model* if $|M| = \kappa$, $\kappa \in M$, $M \models$ ZFC and $M^{<\kappa} \subseteq M$.

Let κ be an inaccessible cardinal, M a κ-model and $\theta \geq \kappa$ be an ordinal. κ is *weakly compact* cardinal if there exists an elementary embedding $j : M \to N$ such that $cp(j) = \kappa$. κ is *θ-strongly unfoldable* if there exists $j : M \to N$ an elementary embedding such that $cp(j) = \kappa$, $j(\kappa) > \theta$ and $V_\theta \subseteq N$. κ is *strongly unfoldable* if for every $\theta > \kappa$, κ is θ-strongly unfoldable. In particular if κ is a strongly unfoldable cardinal, κ is a weakly compact cardinal.

3. Simplified Morasses with Linear Limits

Like $\square_\kappa$ and $\diamondsuit_\kappa$, simplified morasses belong to the family of combinatorial principles true in L, the constructible universe. Morasses were introduced by Jensen in the 1970's in order to solve some cardinal transfer theorems. If there is a $(\kappa^+, 1)$-morass then for every cardinal λ, $(\lambda^{++}, \lambda) \to (\kappa^{++}, \kappa)$, where (λ^{++}, λ) means there is a structure of size λ^{++} with an unary predicate of size λ and the arrow means that if there is a structure $\mathcal{A}$ of type (λ^{++}, λ) then there is a structure $\mathcal{B}$ of type (κ^{++}, κ) such that $\mathcal{A} \equiv \mathcal{B}$.

Let φ, φ' and σ be ordinals such that $\sigma < \varphi$ and $\varphi' = \varphi + (\varphi - \sigma)$. Let $f : \varphi + 1 \to \varphi' + 1$ be an order preserving function. f is a *shift* function with *split point* σ if $f \restriction \sigma = id_\sigma$ and for $\sigma + \delta \leq \varphi$, $f(\sigma + \delta) = \varphi + \delta$.

A *simplified* $(\kappa, 1)$*-morass* is a double sequence: $\big\langle \langle \varphi_\zeta \mid \zeta \leq \kappa \rangle, \langle G_{\zeta\xi} \mid \zeta < \xi \leq \kappa \rangle \big\rangle$ such that

(1) $\langle \varphi_\zeta \mid \zeta \leq \kappa \rangle$ is an increasing sequence of ordinals such that for every $\zeta < \kappa$, $\varphi_\zeta < \kappa$ y $\varphi_\kappa = \kappa^+$.

(2) $G_{\zeta\xi} \subseteq \{f \mid f : \varphi_\zeta + 1 \to \varphi_\xi + 1\}$ is a set of order preserving functions.

(3) For all $\zeta < \xi < \kappa$, $|G_{\zeta\xi}| < \kappa$.

(4) For all $\zeta < \kappa$, $G_{\zeta\zeta+1} = \{id, f\}$, where id is the identity on φ_ζ and f is a shift function with *split* point $\sigma_\zeta < \varphi_\zeta$ so $\varphi_{\zeta+1} = \varphi_\zeta + (\varphi_\zeta - \sigma)$.

(5) For $\zeta < \xi \leq \kappa$, $G_{\zeta\gamma} = \{f \circ g \mid g \in G_{\zeta\xi}, f \in G_{\xi\gamma}\}$.

(6) If $\zeta \leq \kappa$ is a limit ordinal then $\varphi_\zeta = \bigcup_{\xi<\zeta} \{f'' \varphi_\xi \mid f \in G_{\xi\zeta}\}$.

(7) For all γ limit ordinal, $\gamma \leq \kappa$ and for all $\zeta_1, \zeta_2 \leq \gamma$ y $f_1 \in G_{\zeta_1\gamma}$, $f_2 \in G_{\zeta_2\gamma}$, there are ξ, $\zeta_1, \zeta_2 < \xi < \gamma$ and $f_1' \in G_{\zeta_1\xi}$, $f_2' \in G_{\zeta_2\xi}$, $g \in G_{\xi\gamma}$ such that $f_1 = g \circ f_1'$, and $f_2 = g \circ f_2'$.

Let M be a simplified $(\kappa,1)$-morass. M is a *simplified* $(\kappa,1)$ *morass with linear limits* if there is additionally a double sequence $\left\langle \langle \beta^\alpha_\delta, f^\alpha_\delta \rangle : \delta < \tau^\alpha \right\rangle$ for every $\alpha < \kappa$, α a limit ordinal, such that

(1) If $\delta < \gamma < \tau^\alpha$ then $\beta^\alpha_\delta < \beta^\alpha_\gamma$ and there is a $g \in G_{\beta^\alpha_\delta \beta^\alpha_\gamma}$ such that $f^\alpha_\delta = f^\alpha_\gamma \circ g$.

(2) If $\beta < \alpha$ and $f \in G_{\beta\alpha}$ then there exists $\delta < \tau^\alpha$ such that $\beta < \beta^\alpha_\delta$ and there exists $g \in G_{\beta\beta^\alpha_\delta}$ such that $f = f^\alpha_\delta \circ g$.

(3) Suppose $\gamma < \tau^\alpha$ and γ is a limit ordinal. Let $\overline{\alpha} = \beta^\alpha_\gamma$. Then $\overline{\alpha}$ is a limit ordinal, $\tau^{\overline{\alpha}} = \gamma$, and for all $\delta < \gamma$ $\beta^{\overline{\alpha}}_\delta = \beta^\alpha_\delta$ and $f^\alpha_\delta = f^\alpha_\gamma \circ f^{\overline{\alpha}}_\delta$.

If there is a simplified $(\kappa,1)$-morass with linear limits then there is a $\Box_\kappa$-sequence (see [1]) and there is κ-Kurepa tree with no λ-Aronszajn subtrees for any regular infinite $\lambda < \kappa$ and no ν-Cantor subtree for any infinite $\nu < \kappa$ (see [1]). We will use the first statement to prove our main Theorem by showing that $\Box_{\omega_1}$ fails in the final forcing extension, so there cannot be a simplified $(\omega_1,1)$-morass with linear limits.

Proof Main Theorem. Since the forcing $\mathbb{P}$ which adds a simplified $(\kappa,1)$-morass is $<\kappa$-closed and $\kappa^{<\kappa} = \kappa$, $\mathbb{P}$ satisfies the κ^+-c.c., we can apply the Theorem 2. So there is a forcing extension where there is a strongly unfoldable cardinal κ and a simplified $(\kappa,1)$-morass. To finish the proof we collapse κ to ω_2 with the partial order $Col(\omega_1,<\kappa)$, where for τ a regular cardinal $Col(\tau,<\lambda)$ is the set $\big\{p \mid p \text{ function } |p| < \tau, dom(p) \subseteq \lambda\times\tau, \forall(\alpha,\zeta) \in dom(p)(\alpha > 0 \to p(\alpha,\zeta) \in \alpha)\big\}$ order by $p \leq q$ if $q \subseteq p$.

Since every strongly unfoldable cardinal is weakly compact cardinal and if we collapse a weakly compact cardinal to ω_2 there is no $\Box_{\omega_1}$-sequence due to Theorems 4 and 5, so using Theorem 6 we can't have in this forcing extension a simplified $(\omega_1,1)$-morass with linear limits. However we do have a simplified $(\omega_2,1)$-morass since being ordinal and order preserving function (and hence split function) is absolute. ☑

References

[1] Velleman Dan, *Simplified Morasses with Linear Limits*, J. Symbolic Logic **4** (1984), 1001–1021.

[2] Donder Hans-Dieter, *Another Look at Gap-1 Morasses*, Proc. Sympos. Pure Math. **42** (1985), 223–236.

[3] Baumgartner James, *A New Class of Order Types*, Ann. Math. Logic **9** (1976), 187–222.

[4] Cummings James, *Large Cardinal Properties of Small Cardinals*, In Proceedinds of the 1996 Barcelona Set theory, Kluwer Academic Publisher, 1996, pp. 23–39.

[5] Brooke Taylor, *Large Cardinals and L-Like Combinatorics*, Ph.D. Thesis, Universität Wien, 2007.

[6] Johnstone Thomas, *Strongly Unfoldable Cardinals made Indestructible*, J. Symbolic Logic **73** (2008), no. 4, 1215–1248.

[7] Andrés Villaveces, *Chains of Elementary end Extensiond of Models of Set Theory*, J. Symbolic Logic **63** (1998), no. 3, 1116–1136.

(Recibido en junio de 2010. Aceptado en abril de 2011)

Departamento de Matemáticas
Facultad de Ciencias
Universidad Nacional de Colombia
Carrera 30, calle 45
Ciudad Universitaria
Bogotá, Colombia
e-mail: fscardenasp@unal.edu.co