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Convolution of Distribution-Valued Functions. Applications.

Convolución de funciones con valores distribuciones. Aplicaciones.

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Abstract. In this article we examine products and convolutions of vector-valued functions. For nuclear normal spaces of distributions Proposition 25 in [31, p. 120] yields a vector-valued product or convolution if there is a continuous product or convolution mapping in the range of the vector-valued functions. For specific spaces, we generalize this result to hypocontinuous bilinear maps at the expense of generality with respect to the function space. We consider holomorphic, meromorphic and differentiable vector-valued functions and state propositions that contain assertions on products and convolutions of distribution-valued functions in literature as particular cases. Moreover we consider the general convolution of analytic distribution-valued functions and give an approach different to [22].

Key words and phrases. Distributions, Convolution, Multiplication.

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Resumen. En este artículo se investigan los productos y convoluciones de las funciones con valores vectoriales. Para espacios nucleares y normales de distribuciones se obtiene de la Proposition 25 en [31, p. 120] una multiplicación o una convolución con valores vectoriales si existe una multiplicación o una convolución continua en los espacios de las imágenes de las funciones con valores vectoriales. Para espacios particulares se generaliza este resultado a las aplicaciones bilineales hipocontinuas a expensas de la generalidad relativo a los espacios funcionales. Se examinan funciones holomorfas, meromorfas y diferenciables con valores vectoriales y se formulan proposiciones que contienen proposiciones encontradas en la literatura sobre multiplicación y convolución de funciones con valores distribuciones. Además se contempla la convolución

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general de las funciones analíticas con valores distribuciones y se da un enfoque distinto del presentado en [22].

Palabras y frases clave. Distribuciones, convolución, multiplicación.

1. Introduction

For classical function spaces $\mathcal{K}(\Omega)$, as for example $\mathcal{H}(\Omega)$ or $\mathcal{E}^m(\Omega)$, and Hausdorff locally convex topological vector spaces E, F and G we would like to consider convolution maps (in the notation of [29]) of the form

$$\overset{\cdot}{*} : \mathcal{K}(\Omega; E) \times \mathcal{K}(\Omega; F) \to \mathcal{K}(\Omega; G), \quad \Big(\vec{f}, \vec{g}\Big) \mapsto \Big[x \mapsto \vec{f}(x) * \vec{g}(x)\Big].$$

As for decomposed elements $g \otimes \vec{e} \in \mathcal{K}(\Omega) \otimes E$ and $h \otimes \vec{f} \in \mathcal{K}(\Omega) \otimes F$ the property $\overset{\cdot}{*}(g \otimes \vec{e}, h \otimes \vec{f}) = (gh) \otimes (\vec{e} * \vec{f})$ is satisfied, these maps are vector-valued multiplications in the sense of [31], at least in the case of nuclear normal spaces of distributions, where Proposition 25 in [31, p. 120] yields such a convolution map if $* : E \times F \to G$ is continuous. For holomorphic distribution-valued functions such convolution maps are considered in [9], [10], [16], [22] and [24], furthermore the convolution of distribution-valued meromorphic functions in [24].

For concrete spaces $\mathcal{K}(\Omega)$ we generalize these vector-valued convolution maps to hypocontinuous convolutions $* : E \times F \to G$ at the expense of generality with respect to the choice of the space $\mathcal{K}(\Omega)$, i.e., we consider hypocontinuous bilinear maps

$$\overset{\cdot}{b} : \mathcal{K}(\Omega; E) \times \mathcal{K}(\Omega; F) \to \mathcal{K}(\Omega; G), \quad \Big(\vec{f}, \vec{g}\Big) \mapsto \Big[x \mapsto b\Big(\vec{f}(x), \vec{g}(x)\Big)\Big],$$

for specific non-nuclear or non-normal spaces of distributions $\mathcal{K}(\Omega)$. We apply these propositions on hypocontinuous bilinear maps of distribution-valued functions to products and convolutions considered in literature. By doing so, we state propositions that contain the corresponding ones given in the literature as special cases. Moreover we prove the continuity, respectively the hypocontinuity, of these convolution maps which is not observed in the literature cited above.

Let us recall some notation. Throughout we use the notation of L. Schwartz' theory of (vector-valued) distributions in [29], [30] and [32]. Let E be a Hausdorff locally convex topological vector space, we denote by E'_c its dual space endowed with the topology of uniform convergence on compact absolutely convex subsets of E. In particular we use L. Schwartz' notation for spaces of distributions, i.e., we denote by $\mathcal{D}(\Omega)$ the space of smooth functions with compact support and by $\mathcal{E}^m(\Omega)$, where $m \in \mathbb{N}_0 \cup \{\infty\}$, the space of m times continuously differentiable functions with the topology of uniform convergence on compact subsets with respect to all differentiations. If we do not specify the argument of a distribution space it is always $\mathbb{R}^n$ or $\mathbb{R}^{2n}$. Let $\Lambda \subset \mathbb{C}^n$ be an open set, we denote by $\mathcal{H}(\Lambda)$ the space of holomorphic functions on Λ. Let M and N be two separated locally convex topological vector spaces, we denote by $M \,\varepsilon\, N \cong \mathcal{L}_\varepsilon(M'_c, N)$ the ε-product defined by L. Schwartz in [30]. For different topologies on $M \otimes N$ refer to [30], [31] or [11]. If M or N is nuclear we do not specify the topology of $M \otimes N$ as the ε- and the π-topology coincide and we do not consider other topologies on $M \otimes N$. $M \widehat{\otimes}_\varepsilon N$ and $M \widehat{\otimes}_\pi N$ denote the completed tensor product of M and N with respect to the ε- and π-topology whereas $M \widehat{\otimes}_\varepsilon N$ denotes the quasi-completion of $M \otimes_\varepsilon N$.

2. Distribution-Valued Holomorphic Functions

As we consider vector-valued holomorphic functions we first repeat some important properties of the spaces $\mathcal{H}(\Lambda)$ of holomorphic functions and $\mathcal{H}(\Lambda; E)$, E as above, of vector-valued holomorphic functions. For the convenience of the reader we repeat the definition of the "approximation property" [30, Définition, p. 5]:

Definition 1. A separated locally convex topological vector space E satisfies the *approximation property*, if the identity map $\mathrm{id} : E \to E$ is adherent to $E' \otimes E$ in the space $\mathcal{L}_c(E, E)$ of linear operators on E endowed with the topology of uniform convergence on compact absolutely convex subsets of E.

If E satisfies the approximation property the identity $E \widehat{\otimes}_\varepsilon F = E \,\varepsilon\, F$ holds for every complete separated topological vector space F by Corollary 1 in [30, p. 47].

Proposition 1. *The space $\mathcal{H}(\Lambda)$ of holomorphic functions on an open set $\Lambda \subset \mathbb{C}^n$ is nuclear, satisfies the approximation property and has the ε-property. For every separated complete locally convex topological vector space it holds*

$$\mathcal{H}(\Lambda; E) \cong \mathcal{H}(\Lambda) \widehat{\otimes} E = \mathcal{H}(\Lambda) \,\varepsilon\, E$$

as topological vector spaces. Hence $\mathcal{H}(\Lambda) \otimes E \subset \mathcal{H}(\Lambda; E)$ is a dense subspace.

Proof. $\mathcal{H}(\Lambda)$ is nuclear according to [11, p. II.56] or 8. Corollary in [18, p. 499] and therefore satisfies the approximation property by Théorème 6 [11, p. II.34].

(1) We now show that $\mathcal{H}(\Lambda)$ satisfies the ε-property [30, p. 53] analogously to the proof of Proposition 3.2 in [29, p. 10]. $\mathcal{H}(\Lambda) \subset \mathcal{E}(\Lambda)$ is a subspace with the induced topology and $\mathcal{E}(\Lambda)$ satisfies the ε-property [30, p. 55]. Let E be a quasi-complete separated locally convex topological vector space and $\vec{T} \in \mathcal{D}'(E)$ where $\langle \vec{T}, e' \rangle \in \mathcal{H}(\Lambda)$ for all $e' \in E'$. Hence we have for the transpose ${}^t\vec{T}(E') \subset \mathcal{H}(\Lambda)$ and ${}^t\vec{T} : E'_c \to \mathcal{E}(\Lambda)$ is continuous. Now ${}^t\vec{T} : E'_c \to \mathcal{H}(\Lambda)$ is continuous as $\mathcal{H}(\Lambda) \subset \mathcal{E}(\Lambda)$ has the induced topology and ${}^t\vec{T}(E') \subset \mathcal{H}(\Lambda)$.

(2) According to [27, p. 106] we have

$$\mathcal{E}(\Lambda; E) \cong \mathcal{E}(\Lambda) \widehat{\otimes} E = \mathcal{E}(\Lambda) \,\varepsilon\, E,$$

where E is a quasi-complete locally convex topological vector space. A function $\vec{f} : \Lambda \to E$ is holomorphic, i.e., $\vec{f} \in \mathcal{H}(\Lambda; E)$ if and only if for all $e' \in E'$ it holds $\langle \vec{f}, e' \rangle \in \mathcal{H}(\Lambda)$ according to Théorème 1 in [9, p. 37]. Now

$$\vec{f} \in \mathcal{H}(\Lambda; E) \Leftrightarrow \forall e' \in E' : \left\langle \vec{f}, e' \right\rangle \in \mathcal{H}(\Lambda) \Leftrightarrow \vec{f} \in \mathcal{H}(\Lambda) \,\varepsilon\, E,$$

i.e., $\mathcal{H}(\Lambda; E) = \mathcal{H}(\Lambda) \,\varepsilon\, E$ algebraically. The topology of $\mathcal{H}(\Lambda) \,\varepsilon\, E$ is defined by the semi-norms

$$\sup_{e' \in H} \sup_{z \in K} \left| \left\langle \vec{f}(z), e' \right\rangle \right|,$$

where $H \subset E'$ is an equicontinuous subset and $K \subset \Lambda$ is compact. For every equicontinuous subset H there is a continuous semi-norm on E where the above semi-norm is equivalent to

$$\sup_{z \in K} p\Big(\vec{f}(z)\Big),$$

as the topology of uniform convergence on equicontinuous subset of E' coincides with the topology of E by Proposition 7 in [14, p. 200]. Hence we have $\mathcal{H}(\Lambda; E) = \mathcal{H}(\Lambda) \,\varepsilon\, E$ in the sense of topological vector spaces. ☑

2.1. Bilinear Hypocontinuous Maps of Holomorphic Functions

Using the properties of vector-valued holomorphic functions studied above, we now state two propositions on hypocontinuous maps of holomorphic functions.

Proposition 2. *Let E, F and G be three quasi-complete separated locally convex topological vector spaces and*

$$b : E \times F \to G$$

a hypocontinuous bilinear map. Let $\Lambda_1 \subset \mathbb{C}^n$ and $\Lambda_2 \subset \mathbb{C}^m$ be open subsets.

There is a hypocontinuous bilinear map

$$\underset{b}{\otimes} : \mathcal{H}(\Lambda_1; E) \times \mathcal{H}(\Lambda_2; F) \to \mathcal{H}(\Lambda_1 \times \Lambda_2; G)$$

satisfying the consistency property

$$\underset{b}{\otimes} \left(g \otimes \vec{e}, h \otimes \vec{f} \right) = (g \otimes h) \otimes b\Big(\vec{e}, \vec{f}\Big),$$

if $g \in \mathcal{H}(\Lambda_1)$, $h \in \mathcal{H}(\Lambda_2)$, $\vec{e} \in E$ and $\vec{f} \in F$. The map $\underset{b}{\otimes}$ is given by

$$\underset{b}{\otimes} \left(\vec{f}, \vec{g} \right) : \Lambda_1 \times \Lambda_2 \to \mathbb{C}, \quad (z, w) \mapsto \underset{b}{\otimes} \left(\vec{f}, \vec{g} \right)(z, w) := b\Big(\vec{f}(z), \vec{g}(w)\Big).$$

If E, F and G are complete $\underset{b}{\otimes}$ is the unique partially continuous bilinear map satisfying the above consistency property.

Proof.

(1) If E, F and G are complete $\mathcal{H}(\Lambda_1)\otimes E$ and $\mathcal{H}(\Lambda_2)\otimes F$ are dense subsets of $\mathcal{H}(\Lambda_1;E)$ and $\mathcal{H}(\Lambda_2;F)$, respectively according to Proposition 1. Let $\underset{b}{\otimes}{}_1$ and $\underset{b}{\otimes}{}_2$ be two hypocontinuous bilinear maps fulfilling

$$\underset{b}{\otimes}{}_1\Big(g\otimes\vec{e},h\otimes\vec{f}\,\Big)=(g\otimes h)\otimes b\Big(\vec{e},\vec{f}\,\Big)=\underset{b}{\otimes}{}_2\Big(g\otimes\vec{e},h\otimes\vec{f}\,\Big)$$

for $g\in\mathcal{H}(\Lambda_1)$, $h\in\mathcal{H}(\Lambda_2)$ and $(e,f)\in E\times F$. Hence the mappings $\underset{b}{\otimes}{}_1$ and $\underset{b}{\otimes}{}_2$ coincide on a dense subset of $\mathcal{H}(\Lambda_1;E)\times\mathcal{H}(\Lambda_2;F)$. As they are partially continuous they coincide.

(2) Now we show that the mapping $\underset{b}{\otimes}$ is well-defined. Since $\Lambda_1\times\Lambda_2$ is a subset of $\mathbb{C}^{m+n}$, holomorphy is equivalent to weak holomorphy by Théorème 1 in [9, p. 37]. According to Hartogs' theorem [17, Th. 2.2.8, p. 28] holomorphy is equivalent to partial holomorphy. Hence we only have to prove that $(z,w)\mapsto b\Big(\vec{f}(z),\vec{g}(w)\Big)$ is weakly partially holomorphic which amounts to assume $G=\mathbb{C}$ without loss of generality.

Let $1\le j\le n+m$, without restriction let $j\le n$. The j-th partial derivative is

$$\begin{aligned}\partial_j b\Big(\vec{f}(z),\vec{g}(w)\Big)&=\lim_{h\to 0}\frac{1}{h}\Big(b\Big(\vec{f}(z+he_j),\vec{g}(w)\Big)-b\Big(\vec{f}(z),\vec{g}(w)\Big)\Big)\\&=\lim_{h\to 0}b\bigg(\frac{1}{h}\Big(\vec{f}(z+he_j)-\vec{f}(z)\Big),\vec{g}(w)\bigg)\\&=b\Big(\partial_j\vec{f}(z),\vec{g}(w)\Big)\end{aligned}$$

since b is bilinear and partially continuous and $\vec{f}$ is holomorphic. Hence $\underset{b}{\otimes}\Big(\vec{f},\vec{g}\Big)$ is holomorphic.

(3) Next let us prove that, given a continuous semi-norm r_K on the space $\mathcal{H}(\Lambda_1\times\Lambda_2;G)$ and a bounded subset $A\subset\mathcal{H}(\Lambda_2;F)$, the map

$$\mathcal{H}(\Lambda_1;E)\to\mathbb{R}_+,\quad \vec{f}\mapsto\sup_{\vec{g}\in A}r_K\Big(\underset{b}{\otimes}\Big(\vec{f},\vec{g}\Big)\Big)$$

is a continuous semi-norm on $\mathcal{H}(\Lambda_1;E)$. Hence $\underset{b}{\otimes}$ is hypocontinuous with respect to bounded subsets of $\mathcal{H}(\Lambda_2;F)$ according to Proposition 3(c) in [4, p. III.30].

Let $K\subset\Lambda_1\times\Lambda_2$ be a compact subset. There are compact sets $K_1\subset\Lambda_1$ and $K_2\subset\Lambda_2$ where $K\subset K_1\times K_2$. The semi-norms on $\mathcal{H}(\Lambda_1\times\Lambda_2;G)$ are of the form $r_K(\vec{h})=\sup_{(z,w)\in K}q\Big(\vec{h}(z,w)\Big)$, where q is a continuous semi-norm on G.

$$\begin{aligned}\sup_{\vec g\in A} r_K\Big(\overset{\otimes}{b}\big(\vec f,\vec g\big)\Big) &= \sup_{\vec g\in A}\sup_{(z,w)\in K} q\Big(b\Big(\vec f(z),\vec g(z)\Big)\Big)\\ &\le \sup_{z\in K_1}\sup_{\vec g\in A}\sup_{w\in K_2} q\Big(b\Big(\vec f(z),\vec g(z)\Big)\Big)\\ &\le \sup_{z\in K_1}\sup_{x\in\bigcup\limits_{\vec g\in A}\vec g(K_2)} q\Big(b\Big(\vec f(z),x\Big)\Big).\end{aligned}$$

Since $A\subset\mathcal{H}(\Lambda_2;F)$ is bounded, $\bigcup_{\vec g\in A}\vec g(K_2)\subset F$ is a bounded subset of F [10, p. 80]. Hence by Proposition 3(c) in [4, p. III.30] there is a continuous semi-norm p on E where

$$\sup_{x\in\bigcup\limits_{\vec g\in A}\vec g(K_2)} q\big(b(y,x)\big)\le p(y)$$

for all $y\in E$ since b is hypocontinuous with respect to bounded subsets of F. Consequently

$$\sup_{\vec g\in A} r_K\Big(\overset{\otimes}{b}\big(\vec f,\vec g\big)\Big)\le\sup_{z\in K_1} p\Big(\vec f(z)\Big),$$

i.e., $\vec f\mapsto\sup_{\vec g\in A} r_K\Big(\overset{\otimes}{b}\big(\vec f,\vec g\big)\Big)$ is a continuous semi-norm on $\mathcal{H}(\Lambda_1;E)$.

This proves the hypocontinuity in the first component with respect to bounded subsets in the second variable. Analogously it is shown that for all bounded subsets $B\subset\mathcal{H}(\Lambda_1;E)$ and all continuous semi-norms r_K on $\mathcal{H}(\Lambda_1\times\Lambda_2;G)$ the map

$$\mathcal{H}(\Lambda_2;F)\to\mathbb{R}_+,\quad \vec g\mapsto\sup_{\vec f\in B} r_K\Big(\overset{\otimes}{b}\big(\vec f,\vec g\big)\Big)$$

is a continuous semi-norm. Hence $\overset{\otimes}{b}$ is hypocontinuous. ☑

***Remarks* 1.**

(1) If the bilinear map $b:E\times F\to G$ is even continuous then so is $\overset{\otimes}{b}$. As b is continuous, there are continuous semi-norms p on E, and q on F and $K_1\subset\Lambda_1$, $K_2\subset\Lambda_2$ compact, where $K\subset K_1\times K_2$ and a constant $C>0$ such that

$$\sup_{(z,w)\in K} r\Big(b\Big(\vec f(z),\vec g(w)\Big)\Big)\le C\sup_{z\in K_1} p\Big(\vec f(z)\Big)\sup_{w\in K_2} q\Big(\vec g(w)\Big).$$

Hence $\overset{\otimes}{b}$ is continuous.

(2) Let E, F and G be complete. If we apply Proposition 3 in [31, p. 37] to

$$\otimes : \mathcal{H}(\Lambda_1) \times \mathcal{H}(\Lambda_2) \to \mathcal{H}(\Lambda_1 \times \Lambda_2), \quad (g, h) \mapsto g \otimes h$$

we get a unique continuous bilinear map (in the notation of [31])

$$\otimes_\pi : \mathcal{H}(\Lambda_1; E) \times \mathcal{H}(\Lambda_2; F) \to \mathcal{H}(\Lambda_1 \times \Lambda_2; E\widehat{\otimes}_\pi F),$$

where $\otimes_\pi(g \otimes \vec{e}, h \otimes \vec{f}) = (g \otimes h) \otimes (\vec{e} \otimes \vec{f})$ for all $g \in \mathcal{H}(\Lambda_1)$, $h \in \mathcal{H}(\Lambda_2)$, $\vec{e} \in E$ and $\vec{f} \in F$. If $b : E \times F \to G$ is a continuous bilinear map, there is a unique continuous linear map $\tilde{b} : E\widehat{\otimes}_\pi F \to G$, where $\tilde{b}(\vec{e} \otimes \vec{f}) = b(\vec{e}, \vec{f})$ for all $\vec{e} \in E$ and $\vec{f} \in F$. Hence the continuous bilinear map

$$\otimes_\pi \circ \tilde{b} : \mathcal{H}(\Lambda_1; E) \times \mathcal{H}(\Lambda_2; F) \to \mathcal{H}(\Lambda_1 \times \Lambda_2; G)$$

satisfies $(\otimes_\pi \circ \tilde{b})(g \otimes \vec{e}, h \otimes \vec{f}) = (g \otimes h) \otimes b(\vec{e}, \vec{f})$ and therefore it coincides with the map

$$\underset{b}{\otimes} : \mathcal{H}(\Lambda_1; E) \times \mathcal{H}(\Lambda_2; F) \to \mathcal{H}(\Lambda_1 \times \Lambda_2; G),$$
$$\Big(\vec{g}, \vec{h}\Big) \mapsto \Big[(z, w) \mapsto b\Big(\vec{g}(z), \vec{h}(w)\Big)\Big]$$

defined in Proposition 2.

If b is not continuous but hypocontinuous, Proposition 3 in [31, p. 37] cannot be applied, but Proposition 2 is still valid.

By restricting the map defined in Proposition 2 to the diagonal $(z, z) \in \Lambda \times \Lambda$ we get the following result, that corresponds to Proposition 25 in [31, p. 120] as both satisfy the same consistency property for decomposed elements. In our setting, Proposition 25 in [31, p. 120] cannot be applied, as $\mathcal{H}(\Lambda)$ is not a normal space of distributions. The main reason to assume that a space should be a normal space of distributions in [31] is that a multiplication can be defined as a map that coincides in $\mathcal{D}(\Omega)$ with the classical multiplication. Hence the main difference is, as above, that b is assumed to be hypocontinuous and not necessarily continuous.

Proposition 3. *Let E, F and G be three quasi-complete separated locally convex topological vector spaces, $b : E \times F \to G$ a hypocontinuous bilinear map. Let $\Lambda \subset \mathbb{C}^n$ be an open subset.*

The hypocontinuous bilinear map

$$\underset{b}{\cdot} : \mathcal{H}(\Lambda; E) \times \mathcal{H}(\Lambda; F) \to \mathcal{H}(\Lambda; G), \quad \Big(\vec{f}, \vec{g}\Big) \mapsto \Big[z \mapsto b\Big(\vec{f}(z), \vec{g}(z)\Big)\Big]$$

satisfies the consistency property

$$\underset{b}{\cdot}\Big(h(z)\vec{e}, g(z)\vec{f}\Big) = (gh)(z) \otimes b\Big(\vec{e}, \vec{f}\Big)$$

for decomposed elements $g(z)\otimes\vec{e}\in\mathcal{H}(\Lambda)\otimes E$ and $h(z)\otimes\vec{f}\in\mathcal{H}(\Lambda)\otimes F$. If E, F and G are complete $\underset{b}{\cdot}$ is the uniquely determined partially continuous bilinear map satisfying the above consistency property.

Proof. According to Proposition 2 there is a unique hypocontinuous bilinear map

$$\underset{b}{\otimes} : \mathcal{H}(\Lambda;E)\times\mathcal{H}(\Lambda;F)\to\mathcal{H}(\Lambda\times\Lambda;G).$$

Since

$$\Lambda\to\Lambda\times\Lambda,\quad z\mapsto(z,z)$$

is holomorphic and for every continuous semi-norm p it holds

$$\sup_{z\in K} p\Big(\vec{f}(z,z)\Big)\leq\sup_{z\in K}\sup_{w\in K} p\Big(\vec{f}(z,w)\Big)=\sup_{(z,w)\in K\times K} p\Big(\vec{f}(z,w)\Big),$$

the map

$$\mathcal{H}\big(\Lambda\times\Lambda;G\big)\to\mathcal{H}\big(\Lambda;G\big),\quad \vec{f}\mapsto\Big[z\mapsto\vec{f}(z,z)\Big]$$

is well-defined and continuous. Hence the bilinear map

$$\underset{b}{\cdot} : \mathcal{H}(\Lambda;E)\times\mathcal{H}(\Lambda;F)\to\mathcal{H}(\Lambda;G),\quad \Big(\vec{f},\vec{g}\Big)\mapsto\Big[z\mapsto\underset{b}{\otimes}\Big(\vec{f},\vec{g}\Big)(z,z)\Big]$$

is hypocontinuous with respect to bounded subsets of $\mathcal{H}(\Lambda,E)$ and $\mathcal{H}(\Lambda,F)$, respectively. As $\underset{b}{\otimes}$ satisfies the consistency property, $\underset{b}{\cdot}$ does. Therefore its uniqueness is shown analogously to Proposition 2. If b is continuous Remark 1 yields the continuity of $\underset{b}{\cdot}$. ☑

***Remark* 2.** Comparing Proposition 3 to Remark 4 in [9, p. 40] we see that the assumptions on the space E and the bilinear map are slightly more general there but the conclusion is only that the mapping is well-defined. Moreover Proposition 3 proves the uniqueness and hypocontinuity of the map.

2.2. Application 1: Vector-Valued Products and Convolutions

Now we state propositions on distribution-valued holomorphic functions following from Proposition 3 and containing the corresponding propositions in [16] and [22]. Moreover we prove the hypocontinuity or continuity of the mappings, a property which in [16] and [22] is missing.

Let $\mathcal{K}(\Omega_1)$, $\mathcal{K}(\Omega_2)$ and $\mathcal{K}(\Omega_3)$ be spaces of scalar-valued functions and $u:\mathcal{K}(\Omega_1)\times\mathcal{K}(\Omega_2)\to\mathcal{K}(\Omega_3)$ a hypocontinuous bilinear mapping. Additionally let E, F and G be three complete separated locally convex topological vector spaces and $b:E\times F\to G$ a bilinear map. In the following we use the terminology that the map

$$\underset{b}{\overset{u}{}} : \Big(\mathcal{K}(\Omega_1)\widehat{\otimes}_\varepsilon E\Big)\times\Big(\mathcal{K}(\Omega_2)\widehat{\otimes}_\varepsilon F\Big)\to\mathcal{K}(\Omega_3)\widehat{\otimes}_\varepsilon G,$$

if it exists, satisfies the consistency property for decomposed elements if the equation

$$\overset{u}{\underset{b}{}}\Big(g \otimes \vec{e}, h \otimes \vec{f}\Big) = u(g,h) \otimes b\Big(\vec{e}, \vec{f}\Big),$$

holds for all $g \in \mathcal{K}(\Omega_1)$, $h \in \mathcal{K}(\Omega_2)$, $\vec{e} \in E$ and $\vec{f} \in F$.

Proposition 4 (Distributions of Finite Order [16, (2.1.6) Prop. (i), p. 79, (2.1.9) Prop. (i), p. 81 and (2.1.11) Prop., p. 83])**.** *Let $m, s, t \in \mathbb{N}_0 \cup \{\infty\}$.*

(1) The continuous multiplication map

$$\dot{\cdot} : \mathcal{H}\big(\Lambda; \mathcal{D}'^m(\Omega)\big) \times \mathcal{H}\big(\Lambda; \mathcal{E}^m(\Omega)\big) \to \mathcal{H}\big(\Lambda; \mathcal{D}'^m(\Omega)\big),$$
$$\Big(\vec{f}, \vec{g}\Big) \mapsto \Big[\lambda \mapsto \vec{f}(\lambda) \cdot \vec{g}(\lambda)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the product $\lambda \mapsto \vec{T}(\lambda)\vec{a}(\lambda)$ of a distribution-valued holomorphic function $\vec{T} \in \mathcal{H}\big(\Lambda; \mathcal{D}'^m(\Omega)\big)$ and a vector-valued holomorphic function $\vec{a} \in \mathcal{H}\big(\Lambda; \mathcal{E}^m(\Omega)\big)$ is again holomorphic.

(2) The continuous map

$$\dot{\otimes} : \mathcal{H}\big(\Lambda; \mathcal{D}'^s(\Xi)\big) \times \mathcal{H}\big(\Lambda; \mathcal{D}'^t(H)\big) \to \mathcal{H}\big(\Lambda; \mathcal{D}'^{s+t}(\Xi \times H)\big),$$
$$\Big(\vec{f}, \vec{g}\Big) \mapsto \Big[\lambda \mapsto \vec{f}(\lambda) \otimes \vec{g}(\lambda)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the tensor product $\lambda \mapsto \vec{S}(\lambda) \otimes \vec{T}(\lambda)$ of two distribution-valued holomorphic functions $\vec{S} \in \mathcal{H}\big(\Lambda; \mathcal{D}'^s(\Xi)\big)$, and $\vec{T} \in \mathcal{H}\big(\Lambda; \mathcal{D}'^t(H)\big)$, is again holomorphic.

(3) The hypocontinuous convolution map

$$\dot{*} : \mathcal{H}\big(\Lambda; \mathcal{E}'^s(\Omega)\big) \times \mathcal{H}\big(\Lambda; \mathcal{D}'^t(\Omega)\big) \to \mathcal{H}\big(\Lambda; \mathcal{D}'^{s+t}(\Omega)\big),$$
$$\Big(\vec{f}, \vec{g}\Big) \mapsto \Big[\lambda \mapsto \vec{f}(\lambda) * \vec{g}(\lambda)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular, the convolution $\lambda \mapsto \vec{S}(\lambda) * \vec{T}(\lambda)$ of two distribution-valued holomorphic functions $\vec{S} \in \mathcal{H}\big(\Lambda; \mathcal{E}'^s(\Omega)\big)$ and $\vec{T} \in \mathcal{H}\big(\Lambda; \mathcal{D}'^t(\Omega)\big)$ is again holomorphic.

Proof.

(1) $\mathcal{D}'^m(\Omega)$ and $\mathcal{E}^m(\Omega)$ are complete and the multiplication map

$$\mathcal{D}'^m(\Omega) \times \mathcal{E}^m(\Omega) \to \mathcal{D}'^m(\Omega), \quad (T, \alpha) \mapsto \alpha T$$

is continuous by Proposition 6 in [14, p. 362]. Hence Proposition 3 can be applied.

(2) $\mathcal{D}'^s(\Xi)$ is complete and the map

$$\mathcal{D}'^s(\Xi) \times \mathcal{D}'^t(H) \to \mathcal{D}'^{s+t}(\Xi \times H), \quad (S, T) \mapsto S \otimes T$$

is continuous according to Proposition 7 in [14, p. 377]. Hence Proposition 3 can be applied.

(3) The spaces $\mathcal{E}'^s(\Omega)$ and $\mathcal{D}'^t(\Omega)$ are complete and the convolution map

$$\mathcal{E}'^s(\Omega) \times \mathcal{D}'^t(\Omega) \to \mathcal{D}'^{s+t}(\Omega), \quad (S, T) \mapsto S * T$$

is hypocontinuous according to Proposition 7 in [14, p. 388]. Hence Proposition 3 can be applied. ☑

Proposition 5 (Temperate Distributions [16, (2.1.7), p. 80 and (2.1.12), p. 83])**.**

(1) The hypocontinuous multiplication map

$$\dot{\cdot} : \mathcal{H}(\Lambda; \mathcal{S}') \times \mathcal{H}(\Lambda; \mathcal{O}_M) \to \mathcal{H}(\Lambda; \mathcal{S}'), \quad \left(\vec{f}, \vec{g}\right) \mapsto \left[\lambda \mapsto \vec{f}(\lambda) \cdot \vec{g}(\lambda)\right]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the product $\lambda \mapsto \vec{T}(\lambda)\vec{a}(\lambda)$ of a distribution-valued holomorphic function $\vec{T} \in \mathcal{H}(\Lambda; \mathcal{S}')$ and a vector-valued holomorphic function $\vec{a} \in \mathcal{H}(\Lambda; \mathcal{O}_M)$ is again holomorphic.

(2) The hypocontinuous convolution map

$$\dot{*} : \mathcal{H}(\Lambda; \mathcal{S}') \times \mathcal{H}(\Lambda; \mathcal{O}'_C) \to \mathcal{H}(\Lambda; \mathcal{S}'), \quad \left(\vec{f}, \vec{g}\right) \mapsto \left[\lambda \mapsto \vec{f}(\lambda) * \vec{g}(\lambda)\right]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the convolution $\lambda \mapsto \vec{S}(\lambda) * \vec{T}(\lambda)$ of two distribution-valued holomorphic functions $\vec{S} \in \mathcal{H}(\Lambda; \mathcal{S}')$ and $\vec{T} \in \mathcal{H}(\Lambda; \mathcal{O}'_C)$ is again holomorphic.

Proof.

(1) $\mathcal{S}'$ and $\mathcal{O}_M$ are complete and the multiplication map

$$\mathcal{S}' \times \mathcal{O}_M \to \mathcal{S}', \quad (T, \alpha) \mapsto \alpha T$$

is hypocontinuous according to Théorème X in [32, p. 246]. Hence Proposition 3 can be applied.

(2) $\mathcal{S}'$ and $\mathcal{O}'_C$ are complete and the multiplication map

$$\mathcal{S}' \times \mathcal{O}'_C \to \mathcal{S}', (S, T) \mapsto S * T$$

is hypocontinuous according to Théorème XI in [32, p. 247]. Hence Proposition 3 can be applied. ☑

Proposition 6 (p-Integrable Distributions [22, Prop. 4 (i), p. 373])**.**

(1) The hypocontinuous multiplication map

$$\dot{\cdot} : \mathcal{H}(\Lambda; \mathcal{D}_{L^p}) \times \mathcal{H}(\Lambda; \mathcal{D}'_{L^q}) \to \mathcal{H}(\Lambda; \mathcal{D}'_{L^r}),$$
$$\left(\vec{f}, \vec{g}\right) \mapsto \left[\lambda \mapsto \vec{f}(\lambda) \cdot \vec{g}(\lambda)\right],$$

where $r \geq 1$ and $\frac{1}{r} \leq \frac{1}{q} + \frac{1}{p}$, is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the product $\lambda \mapsto \vec{T}(\lambda)\vec{a}(\lambda)$ of a distribution-valued holomorphic function $\vec{T} \in \mathcal{H}(\Lambda; \mathcal{D}'_{L^q})$ and a vector-valued holomorphic function $\vec{a} \in \mathcal{H}(\Lambda; \mathcal{D}_{L^p})$ is again holomorphic.

(2) The continuous convolution map

$$\dot{*} : \mathcal{H}(\Lambda; \mathcal{D}'_{L^p}) \times \mathcal{H}(\Lambda; \mathcal{D}'_{L^q}) \to \mathcal{H}(\Lambda; \mathcal{D}'_{L^r}),$$
$$\left(\vec{f}, \vec{g}\right) \mapsto \left[\lambda \mapsto \vec{f}(\lambda) * \vec{g}(\lambda)\right],$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$, is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the convolution $\lambda \mapsto \vec{S}(\lambda) * \vec{T}(\lambda)$ of two distribution-valued holomorphic functions $\vec{S} \in \mathcal{H}(\Lambda; \mathcal{D}'_{L^p})$ and $\vec{T} \in \mathcal{H}(\Lambda; \mathcal{D}'_{L^q})$ is again holomorphic.

Proof.

(1) The spaces $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^p}$ are complete and

$$\mathcal{D}_{L^p} \times \mathcal{D}'_{L^p} \to \mathcal{D}'_{L^r}, \quad (S, T) \mapsto S \cdot T,$$

where $r \geq 1$ and $\frac{1}{r} \leq \frac{1}{q} + \frac{1}{p}$, is hypocontinuous according to Théorème XXV in [32, p. 201].

(2) The mapping

$$\mathcal{D}'_{L^p} \times \mathcal{D}'_{L^p} \to \mathcal{D}'_{L^r}, \quad (S,T) \mapsto S * T,$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$, is continuous according to Théorème XXV in [32, p. 201]. ☑

Proposition 7 (Distributions With Support in a Cone)**.** *Let* $\Gamma \subset \mathbb{R}^n$ *be a closed convex acute cone with non-empty interior and* $\mathcal{D}'_{+\Gamma}$ *the space of distributions with support in a translate of* Γ*. The hypocontinuous convolution map*

$$\overset{\otimes}{*} : \mathcal{H}(\Lambda_1; \mathcal{D}'_{+\Gamma}) \times \mathcal{H}(\Lambda_2; \mathcal{D}'_{+\Gamma}) \to \mathcal{H}(\Lambda_1 \times \Lambda_2; \mathcal{D}'_{+\Gamma}),$$
$$\left(\vec{f}, \vec{g}\right) \mapsto \left[(\lambda, \mu) \mapsto \vec{f}(\lambda) * \vec{g}(\mu)\right],$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the space of holomorphic $\mathcal{D}'_{+\Gamma}$*-valued functions* $\mathcal{H}(\Lambda; \mathcal{D}'_{+\Gamma})$ *is an algebra with respect to convolution as the second law of composition (which is hypocontinuous).*

Proof. The space $\mathcal{D}'_{+\Gamma}$ is a complete separated ultrabornological Montel-space by Corollary 3.4 in [26, p. 360] and even nuclear by [34, p. 403]. Hence it is sufficient to show the partial continuity of the convolution map $* : \mathcal{D}'_{+\Gamma} \times \mathcal{D}'_{+\Gamma} \to \mathcal{D}'_{+\Gamma}$, since every ultrabornological space is barrelled [14, p. 287]. As the pre-dual $\mathcal{F}(\Gamma+)$ of $\mathcal{D}'_{+\Gamma}$ is a complete separated locally convex space and $\mathcal{D}'_{+\Gamma}$ is bornological we only have to show that for fixed $S \in \mathcal{D}'_{+\Gamma}$ the mapping $\mathcal{D}'_{+\Gamma} \to \mathcal{D}'_{+\Gamma}, T \mapsto S * T$ maps weakly bounded subsets into weakly bounded subsets according to Theorem 4 and Proposition 1 in [14, p. 210 and p. 220]. Let $u \in \mathring{\Gamma}$ and $B \subset \mathcal{D}'_{+\Gamma}$ a weakly bounded subset, i.e., there is $k \in \mathbb{N}$ such that $\operatorname{supp}(T) \subset -k \cdot u + \Gamma$ for all $T \in B$ and for all $\psi \in \mathcal{E}(\mathbb{R}^n)$, where $\operatorname{supp}(\psi) \cap (-k \cdot u + \Gamma)$ is a relatively compact set, there is a constant C_ψ such that $\sup_{T \in B} \left|\langle \psi, T\rangle\right| \leq C_\psi$. Let $\operatorname{supp} S \subset -l \cdot u + \Gamma$, then $\operatorname{supp}(S * T) \subset -(l+k) \cdot u + \Gamma$ according to [35, p. 64] and $-l \cdot u + \Gamma$ and $-k \cdot u + \Gamma$ fulfill condition (Σ) of Definition 2 [14, p. 384]. Let $\psi \in \mathcal{E}(\mathbb{R}^n)$ where $\operatorname{supp} \psi \cap (-(l+k) \cdot u + \Gamma)$ is a relatively compact set. Hence there is a compact subset $K \subset \mathbb{R}^n$ such that $\operatorname{supp} \psi \cap (-(l+k) \cdot u + \Gamma) \subset K$. We choose $\alpha \in \mathcal{D}(\mathbb{R}^n)$ where $\alpha(x) = 1$ for all $x \in K$. Hence we obtain the equality

$$\sup_{T \in B} \left|\langle \psi, S * T\rangle\right| = \sup_{T \in B} \left|\langle \alpha\psi, S * T\rangle\right| = \sup_{T \in B} \left|\langle (\alpha\psi)^\Delta, S \otimes T\rangle\right|.$$

Let $\rho \in \mathcal{D}\big(\mathbb{R}^{2n}\big)$ where $\rho(x) = 1$ for all $x \in \operatorname{supp}(\alpha\psi)^\Delta \cap \big((-l \cdot u + \Gamma) \times (-k \cdot u + \Gamma)\big)$. Hence

$$\sup_{T \in B} \left|\langle \psi, S * T\rangle\right| = \sup_{T \in B} \left|\langle \rho(\alpha\psi)^\Delta, S \otimes T\rangle\right| = \sup_{T \in B} \left|\big\langle \langle \rho(\alpha\psi)^\Delta, S\rangle, T\big\rangle\right| \leq C_\psi,$$

as $\langle \rho(\alpha\psi)^{\Delta}, S\rangle \in \mathcal{D}(\mathbb{R}^n) \subset \mathcal{F}(-k \cdot u + \Gamma)$ and B is $\sigma\big(\mathcal{D}'_{+\Gamma}, \mathcal{F}(\Gamma+)\big)$-bounded. Therefore $\mathcal{D}'_{+\Gamma} \times \mathcal{D}'_{+\Gamma} \to \mathcal{D}'_{+\Gamma}, (S,T) \mapsto S * T$ is hypocontinuous. ☑

***Remarks* 3.**

(1) The spaces $\mathcal{S}'$, $\mathcal{O}'_C$ and $\mathcal{O}_M$ are nuclear normal spaces of distributions and the multiplication map $\cdot : \mathcal{H}(\Lambda) \times \mathcal{H}(\Lambda) \to \mathcal{H}(\Lambda)$ is continuous. Hence the existence of the mappings in Proposition 5 can also be shown by applying Proposition 25 in [31, p. 120] for the multiplication and Proposition 34 in [31, p. 151] for the convolution. As $\mathcal{D}'^m$, $\mathcal{E}^m$, for finite m, $\mathcal{D}_{L^p}$ and $\mathcal{D}'_{L^q}$ are not nuclear, the Propositions 4 and 6 are not special cases neither of Proposition 34 nor of Proposition 25 in [31, pp. 151, 120].

(2) Note that the space $\mathcal{F}(\Gamma+)$ is isomorphic to $\big(s^{(\mathbb{N})}\big)^{\mathbb{N}}$ according to Theorem 2.3 in [25, p. 418] whereas the space $\mathcal{D}(\Gamma+) = \mathcal{D}_{+\Gamma}$ is isomorphic to $\big(s^{\mathbb{N}}\big)^{(\mathbb{N})}$ (Corollary 1.1 in [33, p. 315] or [34, p. 403]). This agrees with the assertion in [25, p. 415]:

> "Therefore the strong dual of $\mathcal{D}(\Gamma+)$ is not isomorphic to a space of type $\big(\mathcal{D}'(\Gamma+), \beta\big(\mathcal{D}'(\Gamma+), \mathcal{F}(\Gamma+)\big)\big)$."

Example 1 (Hyperbolic M. Riesz Kernels [24, Ex. 2.4.1, p. 46] or [32, Ex. 4, p. 49])**.** Let $f \in L^1_{\mathrm{loc}}$ be a locally integrable function. Denoting by $(f)^{\alpha}_{+}$ the function $x \mapsto \mathrm{Y}(f(x))(f(x))^{\alpha}$, we consider the holomorphic function

$$Z : \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -2\} \to \mathcal{D}'_{+\Gamma}\big(\mathbb{R}^{n+1}_{x,t}\big) \cap L^1_{\mathrm{loc}}\big(\mathbb{R}^{n+1}_{x,t}\big),$$
$$\lambda \mapsto \frac{\mathrm{Y}(t)}{\pi^{\frac{n-1}{2}} 2^{\lambda-1} \Gamma\big(\frac{\lambda}{2}\big)\Gamma\big(\frac{\lambda+1-n}{2}\big)} \big(t^2 - |x|^2\big)_{+}^{(\lambda-n-1)/2}.$$

This function can be extended to an entire function

$$Z : \mathbb{C} \to \mathcal{D}'_{+\Gamma}\big(\mathbb{R}^{n+1}_{x,t}\big), \quad \lambda \mapsto \frac{\mathrm{Y}(t)}{\pi^{\frac{n-1}{2}} 2^{\lambda-1} \Gamma\big(\frac{\lambda}{2}\big)\Gamma\big(\frac{\lambda+1-n}{2}\big)} \big(t^2 - |x|^2\big)_{+}^{(\lambda-n-1)/2}$$

with values in $\mathcal{D}'_{+\Gamma}$ [32, p. 177], where $\Gamma = \big\{(x,t) \in \mathbb{R}^{n+1} \;\big|\; |x| \le t\big\}$. Hence $Z \in \mathcal{H}(\mathbb{C}; \mathcal{D}'_{+\Gamma})$. According to Proposition 7, the convolution of two hyperbolic M. Riesz kernels is well-defined and holomorphic. A classical result [32, p. 177] is $Z(\lambda) * Z(\mu) = Z(\lambda + \mu)$.

2.3. Application 2: Convolvability Conditions for Distribution-Valued Holomorphic Functions

We aim at giving an approach to the question of holomorphy of convolutions which is different from that of [22].

Definition 2 ([28, exposé n° 22, p. 1], [13, (1), p. 185], [12, p. 8]). Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we define $\varphi^\Delta \in \mathcal{B}(\mathbb{R}^{2n})$ by $\varphi^\Delta(x,y) := \varphi(x+y)$. Two distributions $S, T \in \mathcal{D}'(\mathbb{R}^n)$ are convovable if condition (Γ) is fulfilled:

$$(\Gamma)\ \forall \varphi \in \mathcal{D}(\mathbb{R}^n) : \varphi^\Delta (S \otimes T) \in \mathcal{D}'_{L^1}(\mathbb{R}^n).$$

In [22], N. Ortner introduced the condition

$$(\Gamma_\Lambda)\ \forall \varphi \in \mathcal{D} : \Lambda \to \mathcal{D}'_{L^1}, \quad \lambda \mapsto \big(\varphi * \check{S}(\lambda)\big)T(\lambda) \quad \text{is weakly continuous.}$$

If (Γ_Λ) is satisfied, the distribution-valued convolution mapping $\Lambda \to \mathcal{D}', \lambda \mapsto S(\lambda) * T(\lambda)$ is holomorphic by Proposition 3 in [22, p. 373]. Proposition 6 in [23, p. 330] proves the equivalence of this condition with the more symmetric one

$$(\Gamma'_\Lambda)\ \forall \varphi \in \mathcal{D} : \Lambda \to \mathcal{D}'_{L^1}, \quad \lambda \mapsto \varphi^\Delta\big(S(\lambda) \otimes T(\lambda)\big) \text{ is weakly continuous.}$$

By means of Proposition 3 we give another proof for the sufficiency of condition (Γ'_Λ) for the map $\Lambda \to \mathcal{D}', \lambda \mapsto \vec{f}(\lambda) * \vec{g}(\lambda)$ to be holomorphic.

Definition 3 ([28, exposé n° 22, p. 1], [6, p.186]). A sequence $(\eta_k)_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{R}^n)$ is called an approximate unit if

(i) The sequence $(\eta_k)_{k\in\mathbb{N}} \subset \mathring{\mathcal{B}}(\mathbb{R}^n)$ is bounded.

(ii) The sequence $(\eta_k)_{k\in\mathbb{N}}$ converges to 1 in $\mathcal{E}(\mathbb{R}^n)$.

We now show that for all $\psi \in \mathcal{B}$ and all approximate units $(\eta_k)_{k\in\mathbb{N}}$, the product $\psi\eta_k$ converges to ψ if $k \to \infty$ in $\mathcal{B}_c$.

Lemma 8. *The multiplication map $\mathcal{B}_c \times \mathcal{B}_c \to \mathcal{B}_c, (\psi, \varphi) \mapsto \psi \cdot \varphi$ is continuous. In particular if $(\eta_k)_{k\in\mathbb{N}}$ is an approximate unit and $\psi \in \mathcal{B}$ then $\eta_k\psi$ converges to ψ in $\mathcal{B}_c$.*

Proof. P. Dierolf and S. Dierolf showed in [5], that the topology of $\mathcal{B}_c(\mathbb{R}^n)$ can be defined by the family of semi-norms

$$\sup_{|\alpha|\le m} \big\| f\partial^\alpha \varphi \big\|_\infty,$$

where $f \in \mathcal{C}_0(\mathbb{R}^n) \setminus \{0\}$. The space $\mathring{\mathcal{B}}(\mathbb{R}^n) \subset \mathcal{B}_c(\mathbb{R}^n)$ is contained as a subspace with a finer topology and on bounded subsets the topology of $\mathcal{B}_c(\mathbb{R}^n)$ and the topology induced by $\mathcal{E}(\mathbb{R}^n)$ coincide [32, p. 203]. Hence $(\eta_k)_{k\in\mathbb{N}}$ converges to 1 in $\mathcal{B}_c(\mathbb{R}^n)$. The multiplication in $\mathcal{B}_c(\mathbb{R}^n)$ is continuous, since

$$\begin{aligned}\sup_{|\alpha|\le m} \big\| f\partial^\alpha(\varphi\psi)\big\|_\infty &\le C \sup_{|\alpha|\le m}\sup_{|\beta|\le m}\sup_{x\in\mathbb{R}^n} \sqrt{|f(x)|}\,|(\partial^\alpha\varphi)(x)|\sqrt{|f(x)|}\,\big|(\partial^\beta\psi)(x)\big| \\ &\le C \sup_{|\alpha|\le m}\big\|g\partial^\alpha\varphi\big\|_\infty \sup_{|\beta|\le m}\big\|g\partial^\beta\psi\big\|_\infty,\end{aligned}$$

where $g(x) := \sqrt{|f(x)|} \in \mathcal{C}_0(\mathbb{R}^n)$. Hence if $(\eta_k)_{k\in\mathbb{N}}$ is an approximate unit the product $\psi\eta_k \to \psi$ converges in $\mathcal{B}_c$ for all $\psi \in \mathcal{B}$. ☑

By means of this lemma and the definition of the topology of $\mathcal{B}_c$ we obtain the following sufficient criterion for a holomorphic $\mathcal{D}'$-valued function $\vec{f} : \Lambda \to \mathcal{D}'$ $\big($whose image $\vec{f}(\Lambda) \subset \mathcal{D}'_{L^1}$ is a subset of $\mathcal{D}'_{L^1}\big)$ to be a $\mathcal{D}'_{L^1}$-valued holomorphic function.

Proposition 9. *Let $\Lambda \subset \mathbb{C}^n$ be an open subset and $\vec{f} : \Lambda \to \mathcal{D}'$ holomorphic with $\vec{f}(\Lambda) \subset \mathcal{D}'_{L^1}$. If $\vec{f}$ maps compact subsets of Λ into relatively compact subsets of $\mathcal{D}'_{L^1}$ then it is a $\mathcal{D}'_{L^1}$-valued holomorphic function.*

Proof. Let $(\eta_k)_{k\in\mathbb{N}}$ an approximate unit and $\psi \in \mathcal{B}$, then $\psi\eta_k \to \psi$ in $\mathcal{B}_c$ and $\psi\eta_k \in \mathcal{D}$. Hence the map

$$\Lambda \to \mathbb{C}, z \mapsto \big\langle \psi\eta_k, \vec{f}(z)\big\rangle$$

is holomorphic for all $k \in \mathbb{N}$. We now show that this sequence converges to $z \mapsto \langle \psi, \vec{f}(z)\rangle$ in $\mathcal{C}(\Lambda) = \mathcal{E}^0(\Lambda)$, hence $\vec{f} : \Lambda \to \mathcal{D}'_{L^1}$ is holomorphic. Let $K \subset \Lambda$ be a compact subset, then

$$\begin{aligned}\sup_{z\in K} \Big|\big\langle \psi\eta_k, \vec{f}(z)\big\rangle - \big\langle \psi, \vec{f}(z)\big\rangle\Big| &= \sup_{z\in K} \Big|\big\langle \psi\eta_k - \psi, \vec{f}(z)\big\rangle\Big| \\ &\le \sup_{S\in K_1} \big|\langle \psi\eta_k - \psi, S\rangle\big| \to 0,\end{aligned}$$

since $\psi\eta_k \to \psi$ in $\mathcal{B}_c = \big(\mathcal{D}'_{L^1}\big)'_c$ and $K_1 \subset \mathcal{D}'_{L^1}$ compact. ☑

***Remark* 4.** As one of the referees pointed out the assertion of Proposition 9 follows from a more abstract result:

Let E be a locally complete space and let $\vec{f} : \Omega \to E$ be a holomorphic function on a domain $\Omega \subset \mathbb{C}^n$. Let $F \subset E$ be a dense subset with possibly a finer locally complete topology. If $\vec{f}(\Omega) \subset F$ and $\vec{f}(K)$ is bounded in F for each compact subset K in Ω then $\vec{f}$ is holomorphic as a function with values in F. A proof can be given using Theorem 1 in [7, p. 399] or the generalization in [1, p. 237].

Proposition 10 (cf. [22, Prop. 3, p. 372])**.** *Let $\Lambda_1 \subset \mathbb{C}^m$ and $\Lambda_2 \subset \mathbb{C}^n$ open subsets, $\vec{f} \in \mathcal{H}(\Lambda_1; \mathcal{D}')$ and $\vec{g} \in \mathcal{H}(\Lambda_2; \mathcal{D}')$ holomorphic distribution-valued functions. If the mapping*

$$\Lambda_1 \times \Lambda_2 \to \mathcal{D}'_{L^1}, \quad (\lambda, \mu) \mapsto \varphi^{\Delta}\Big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\Big)$$

maps compact subsets of $\Lambda_1 \times \Lambda_2$ into relatively compact subsets of $\mathcal{D}'_{L^1}$ then the distribution-valued convolution

$$\Lambda_1 \times \Lambda_2 \to \mathcal{D}', \quad (\lambda, \mu) \mapsto \vec{f}(\lambda) * \vec{g}(\mu)$$

is also holomorphic.

Proof. Since the multiplication maps

$$\mathcal{D}'(\Omega_1) \times \mathcal{D}'(\Omega_2) \to \mathcal{D}'(\Omega_1 \times \Omega_2), \quad (S, T) \mapsto S \otimes T$$

and

$$\mathcal{E}(\Omega) \times \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega), \quad (\alpha, T) \mapsto \alpha T$$

are continuous, respectively hypocontinuous [32, Th. VI, p. 110, respectively Th. III, p. 119], the mapping

$$\begin{aligned} \mathcal{H}\big(\Lambda_1; \mathcal{D}'(\mathbb{R}^n)\big) \times \mathcal{H}\big(\Lambda_2; \mathcal{D}'(\mathbb{R}^n)\big) \to \mathcal{H}\big(\Lambda_1 \times \Lambda_2; \mathcal{D}'(\mathbb{R}^{2n})\big), \\ \Big(\vec{f}, \vec{g}\Big) \mapsto \Big[(\lambda, \mu) \mapsto \varphi^{\Delta}\Big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\Big)\Big], \end{aligned}$$

is continuous for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Hence

$$\Lambda_1 \times \Lambda_2 \to \mathcal{D}', \quad (\lambda, \mu) \mapsto \varphi^{\Delta}\Big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\Big)$$

is holomorphic for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Since

$$\Lambda_1 \times \Lambda_2 \to \mathcal{D}'_{L^1}, \quad (\lambda, \mu) \mapsto \varphi^{\Delta}\Big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\Big)$$

maps compact subsets of $\Lambda_1 \times \Lambda_2$ into relatively compact subsets of $\mathcal{D}'_{L^1}$, it is holomorphic according to Proposition 9. Hence

$$\Lambda_1 \times \Lambda_2 \to \mathcal{D}', \quad (\lambda, \mu) \mapsto \vec{f}(\lambda) * \vec{g}(\mu) = \Big[\varphi \mapsto \Big\langle 1, \varphi^{\Delta}\Big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\Big)\Big\rangle\Big]$$

is holomorphic. ☑

***Remarks* 5.**

(1) The sufficiency of the property

$$(\Gamma'_{\Lambda})\ \Lambda \to \mathcal{D}'_{L^1}, \quad \lambda \mapsto \varphi^{\Delta}\Big(\vec{f}(\lambda) \otimes \vec{g}(\lambda)\Big) \text{ is weakly continuous.}$$

For the distribution-valued convolution

$$\Lambda \to \mathcal{D}', \quad \lambda \mapsto \vec{f}(\lambda) * \vec{g}(\lambda)$$

to be holomorphic in [23] can be shown analogously to Proposition 10.

(2) As the space $\mathcal{D}'_{L^1}$ satisfies the Schur-property according to Corollary (3.5) in [5, p. 71] and every point $z \in \Lambda$ has a countable fundamental system of neighborhoods, $\vec{f}$ is continuous if and only if it is weakly continuous.

The preceding propositions show, that for a distribution-valued holomorphic function $\vec{f} : \Lambda \to \mathcal{D}'$ on an open set $\Lambda \subset \mathbb{C}^n$ with $\vec{f}(\Lambda) \subset \mathcal{D}'_{L^1}$, the following assertions are equivalent:

(a) $\vec{f}:\Lambda\to\mathcal{D}'_{L^1}$ is (weakly) continuous.

(b) $\vec{f}$ maps compact subsets of Λ into relatively compact subsets of $\mathcal{D}'_{L^1}$.

(c) $\vec{f}:\Lambda\to\mathcal{D}'_{L^1}$ is (weakly) holomorphic.

Example 2 ([24, Ex. 2.3.2, p. 42])**.** Let $\Lambda=\mathbb{C}$; we consider the "Cauchy-Riemann-polynomial" $z=x_1+\mathrm{i}x_2$ and the "Cauchy-kernel"

$$C:\Lambda\to\mathcal{S}',\quad \lambda\mapsto\mathcal{F}^{-1}\big(z^\lambda\big):=\mathcal{F}^{-1}\big(\,\mathrm{e}^{\lambda\log z}\,\big),$$

defined by analytic continuation. According to [24, p. 43 and Ex. 3.4.3, p. 102] it is an entire distribution-valued function, i.e. $C\in\mathcal{H}(\mathbb{C};\mathcal{S}')$ and the convolution of two Cauchy-kernels $C(\lambda)$ and $C(\mu)$ is well-defined if the real part $\mathrm{Re}(\lambda+\mu)>-2$. By Proposition 10, the function

$$\big\{(\lambda,\mu)\in\mathbb{C}^2\mid\mathrm{Re}(\lambda+\mu)>-2\big\}\to\mathcal{S}'(\mathbb{R}^2),\quad(\lambda,\mu)\mapsto C(\lambda)*C(\mu)$$

is holomorphic. In [24, p. 103] the equality

$$C(\lambda)*C(\mu)=C(\lambda+\mu)\quad\text{for}\quad\mathrm{Re}\,(\lambda+\mu)>-2$$

is shown.

3. Distribution-Valued Meromorphic Functions

Let us now extend our considerations to distribution-valued meromorphic functions. The locally convex topology on the space $\mathcal{M}(\Lambda)$ of meromorphic functions on a complex domain Λ, discussed in [8], has the disadvantage that, according to Theorem 5 in [8, p. 296], multiplication is not continuous. Additionally for a locally convex space E, according to Proposition 6 in [2, p. 357], there are strong restrictions for the equation $\mathcal{M}(\Lambda;E)=\mathcal{M}(\Lambda)\,\varepsilon\,E$ to hold algebraically; furthermore, according to Theorem 4.6 in [19, p. 286], the topology of $\mathcal{M}(\Lambda;E)$ coincides with the topology of $\mathcal{M}(\Lambda)\,\varepsilon\,E$ if and only if E is finite dimensional. Therefore we only consider a subspace $\mathcal{M}(\Lambda;D)\subset\mathcal{M}(\Lambda)$ with pole restrictions. We denote by $\widehat{\mathbb{C}}$ the extended complex plane.

Definition 4 ([8, p. 290], [19, p. 275])**.** Let $\Lambda\subset\widehat{\mathbb{C}}$ be a domain. A positive divisor is a map

$$D:\Lambda\to\mathbb{N}_0,$$

where $P_D=\operatorname{supp}D=\{\alpha\in\Lambda\mid D(\alpha)\neq0\}$ is a discrete subset of Λ. Let $f:\Lambda\to\widehat{\mathbb{C}}$ be a meromorphic function; we denote by $o_\alpha(f)$ the order of the pole of f in α. Given a positive divisor D, we denote by

$$\mathcal{M}(\Lambda;D):=\big\{f:\Lambda\to\widehat{\mathbb{C}}\;\big|\;f\text{ meromorphic and }o_\alpha(f)\leq D(\alpha)\text{ for }\alpha\in\Lambda\big\}$$

the space of meromorphic functions with poles in P_D and $o_\alpha(f) \leq D(\alpha)$. We consider $\mathcal{M}(\Lambda; D) \subset \mathcal{H}(\Lambda \setminus P_D)$ as a subspace of the space of holomorphic functions on $\Lambda \setminus P_D$ carrying the induced topology. Let E be a complete separated locally convex vector space. We denote by

$$\mathcal{M}(\Lambda; D, E) := \big\{\vec{f} : \Lambda \to E \mid \vec{f} \text{ meromorphic and } o_\alpha(\vec{f}) \leq D(\alpha)\big\}$$

the space of E-valued meromorphic functions with poles in P_D and $o_\alpha(\vec{f}) \leq D(\alpha)$. We consider $\mathcal{M}(\Lambda; D, E)$ with the topology of uniform convergence on compact subsets of $\Lambda \setminus P_D$, i.e., with the topology induced by $\mathcal{H}(\Lambda \setminus P_D; E)$.

In Proposition 3.1 in [19, p. 275] E. Jordá showed that for all positive divisors D on Λ and all complete separated locally convex topological vector spaces the spaces $\mathcal{M}(\Lambda; D, E)$ and $\mathcal{H}(\Lambda; E)$ are isomorphic as topological vector spaces. As for concrete calculations there are differences between the case of holomorphic functions and of meromorphic functions we nevertheless consider vector-valued meromorphic functions with pole restrictions.

Proposition 11. *Let $\Lambda \subset \mathbb{C}$ be a complex domain and D a positive divisor. $M(\Lambda; D)$ is a nuclear Fréchet space and therefore has the approximation property. If E is a complete separated locally convex topological vector space, it holds*

$$\mathcal{M}(\Lambda; D, E) \cong \mathcal{M}(\Lambda; D)\widehat{\otimes}E$$

in the sense of topological vector spaces.

Proof. The isomorphism $\mathcal{H}(\Lambda; E) \cong \mathcal{M}(\Lambda; D, E)$ is an isomorphism of topological vector spaces for all positive divisors D on Λ according to [19, Prop. 3.1, p. 275]. Hence $\mathcal{M}(\Lambda; D, E)$ satisfies the properties stated above as $\mathcal{H}(\Lambda; E)$ does. ☑

As $o_\alpha\Big(\Big[z \mapsto b\Big(\vec{f}(z), \vec{g}(z)\Big)\Big]\Big) \leq o_\alpha(\vec{f}\,) + o_\alpha(\vec{g})$, Proposition 3 is equivalent to the following one.

Proposition 12. *Let E, F and G be complete separated locally convex topological vector spaces and $b : E \times F \to G$ a hypocontinuous bilinear map. The hypocontinuous bilinear map*

$$\dot{b} : \mathcal{M}(\Lambda; D_1, E) \times \mathcal{M}(\Lambda; D_2, F) \to \mathcal{M}(\Lambda; D_1 + D_2; G),$$
$$(\vec{f}, \vec{g}) \mapsto \Big[z \mapsto b\Big(\vec{f}(z), \vec{g}(z)\Big)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements. If b is even continuous, so is $\dot{b}$.

We now state propositions on multiplication and convolution of distribution-valued meromorphic functions that are special cases of Proposition 12 and correspond to propositions given in [24]. Moreover we prove the hypocontinuity, respectively the continuity, of the bilinear maps which is not considered in [24]. In the following, let $\Lambda \subset \mathbb{C}$ be a domain and $\Omega \subset \mathbb{R}^n$ an open set.

Proposition 13 (Distributions of Finite Order, cf. [24, Prop. 1.6.3, p. 28 and Prop. 1.6.4, p. 29]). *Let $m, s, t \in \mathbb{N}_0 \cup \{\infty\}$.*

(1) The hypocontinuous multiplication map

$$\because : \mathcal{M}\big(\Lambda; D_1, \mathcal{D}'^m(\Omega)\big) \times \mathcal{M}\big(\Lambda; D_2, \mathcal{E}^m(\Omega)\big) \to \mathcal{M}\big(\Lambda; D_1 + D_2, \mathcal{D}'^m(\Omega)\big),$$
$$\Big(\vec{f}, \vec{g}\Big) \mapsto \Big[z \mapsto \vec{f}(z) \cdot \vec{g}(z)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

(2) The hypocontinuous convolution map

$$\because : \mathcal{M}(\Lambda; D_1, \mathcal{D}'^s(\Omega)) \times \mathcal{M}(\Lambda; D_2, \mathcal{E}'^t(\Omega)) \to \mathcal{M}(\Lambda; D_1 + D_2, \mathcal{D}'^{s+t}(\Omega)),$$
$$\Big(\vec{f}, \vec{g}\Big) \mapsto \Big[z \mapsto \vec{f}(z) * \vec{g}(z)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

Proposition 14 (Temperate Distributions, cf. [24, Prop. 1.6.3, p. 28 and Prop. 1.6.4, p. 29]).

(1) The hypocontinuous multiplication map

$$\because : \mathcal{M}(\Lambda; D_1, \mathcal{S}') \times \mathcal{M}(\Lambda; D_2, \mathcal{O}_M) \to \mathcal{M}(\Lambda; D_1 + D_2, \mathcal{S}'),$$
$$\Big(\vec{f}, \vec{g}\Big) \mapsto \Big[z \mapsto \vec{f}(z) \cdot \vec{g}(z)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

(2) The hypocontinuous convolution map

$$\because : \mathcal{M}(\Lambda; D_1, \mathcal{S}' \times \mathcal{M}(\Lambda; D_2, \mathcal{O}'_C) \to \mathcal{M}(\Lambda; D_1 + D_2, \mathcal{S}'),$$
$$\Big(\vec{f}, \vec{g}\Big) \mapsto \Big[z \mapsto \vec{f}(z) * \vec{g}(z)\Big]$$

is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

Proposition 15 (p-Integrable Distributions, cf. [24, Prop. 1.6.3, p. 28 and Prop. 1.6.4, p. 29])**.**

(1) The hypocontinuous multiplication map

$$\because : \mathcal{M}(\Lambda; D_1, \mathcal{D}_{L^p}) \times \mathcal{M}(\Lambda; D_2, \mathcal{D}'_{L^q}) \to \mathcal{M}(\Lambda; D_1 + D_2, \mathcal{D}'_{L^r}),$$
$$\left(\vec{f}, \vec{g}\right) \mapsto \left[z \mapsto \vec{f}(z) \cdot \vec{g}(z)\right],$$

where $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{q}$, is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

(2) The continuous convolution map

$$\because : \mathcal{M}(\Lambda; D_1, \mathcal{D}'_{L^p}) \times \mathcal{M}(\Lambda; D_2, \mathcal{D}'_{L^q}) \to \mathcal{M}(\Lambda; D_1 + D_2, \mathcal{D}'_{L^r}),$$
$$\left(\vec{f}, \vec{g}\right) \mapsto \left[z \mapsto \vec{f}(z) * \vec{g}(z)\right],$$

where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

Let us consider the general convolution of two distribution-valued meromorphic functions. We state a condition similar to (Γ'_Λ) for the convolution of the distribution-valued meromorphic functions being again a meromorphic function. As the spaces $\mathcal{M}\big(\Lambda; D, \mathcal{D}'(\mathbb{R}^n)\big)$ and $\mathcal{H}\big(\Lambda; \mathcal{D}'(\mathbb{R}^n)\big)$ are isomorphic, we can use the propositions on the convolution of distribution-valued holomorphic functions.

Proposition 16. *Let $\vec{f} \in \mathcal{M}(\Lambda_1; D_1, \mathcal{D}')$ and $\vec{g} \in \mathcal{M}(\Lambda_2; D_2, \mathcal{D}')$ two distribution-valued meromorphic functions where $\varphi^\Delta\big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\big) \in \mathcal{D}'_{L^1}$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and all $(\lambda, \mu) \in (\Lambda_1 \setminus P_{D_1}) \times (\Lambda_2 \setminus P_{D_2})$. If additionally*

$$(\Lambda_1 \setminus P_{D_1}) \times (\Lambda_2 \setminus P_{D_2}) \to \mathcal{D}'_{L^1}, \quad (\lambda, \mu) \mapsto \varphi^\Delta\Big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\Big)$$

maps compact subset of $\Lambda_1 \setminus P_{D_1} \times \Lambda_2 \setminus P_{D_2}$ into relatively compact subsets of $\mathcal{D}'_{L^1}$ then the convolution $\big[(\lambda, \mu) \mapsto \vec{f}(\lambda) * \vec{g}(\mu)\big] \in \mathcal{M}(\Lambda_1 \times \Lambda_2; \mathcal{D}')$ is meromorphic with poles possibly in a non-discrete set.

Proof. For the divisors D_1 and D_2 according to [19, p. 275] there are scalar valued holomorphic function $h_i \in \mathcal{H}(\Lambda_i)$ where for all $\alpha \in P_{D_i}$ it holds $h_i(z) = 0$, $\lim_{z \to \alpha} \frac{h_i(z)}{(z-\alpha)^{D_i(\alpha)}} \neq 0$ and $h_i(z) \neq 0$ for all $z \in \Lambda_i \setminus P_{D_i}$. Hence $\Lambda_1 \times \Lambda_2 \to \mathcal{D}'_{L^1}, (\lambda, \mu) \mapsto \varphi^\Delta\big(\big(h_1 \vec{f}\,\big)(\lambda) \otimes \big(h_2 \vec{g}\big)(\mu)\big)$ is holomorphic for all $\varphi \in \mathcal{D}$ according to Proposition 10. Therefore by multiplication with the function $\frac{1}{h_1} \otimes \frac{1}{h_2}$ we get $\big[(\lambda, \mu) \mapsto \varphi^\Delta\big(\vec{f}(\lambda) \otimes \vec{g}(\mu)\big)\big] \in \mathcal{M}(\Lambda_1 \times \Lambda_2; \mathcal{D}'_{L^1})$, i.e., their convolution $\big[(\lambda, \mu) \mapsto \vec{f}(\lambda) * \vec{g}(\mu)\big] \in \mathcal{M}(\Lambda_1 \times \Lambda_2; , \mathcal{D}')$ is a meromorphic function. ☑

***Remark* 6.** For $\vec{f} \in \mathcal{M}(\Lambda; D_1, \mathcal{D}')$ and $\vec{g} \in \mathcal{M}(\Lambda; D_2, \mathcal{D}')$ where for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and all $\lambda \in \Lambda \setminus P_{D_1+D_2}$ the image $\varphi^{\Delta}\big(\vec{f}(\lambda) \otimes \vec{g}(\lambda)\big) \in \mathcal{D}'_{L^1}$ is an integrable distribution, the following convolvability condition is shown analogous to Proposition 16:

If the map

$$\Lambda \setminus P_{D_1+D_2} \to \mathcal{D}'_{L^1}, \lambda \mapsto \varphi^{\Delta}\Big(\vec{f}(\lambda) \otimes \vec{g}(\lambda)\Big)$$

maps compact subsets of $\Lambda \setminus P_{D_1+D_2}$ into relatively compact subsets of $\mathcal{D}'_{L^1}$ then the distribution-valued convolution

$$\Big[\lambda \mapsto \vec{f}(\lambda) * \vec{g}(\lambda)\Big] \in \mathcal{M}(\Lambda; D_1 + D_2, \mathcal{D}')$$

is again a meromorphic function.

Example 3 (Elliptic M. Riesz kernels [15, p. 176], [22, p. 369])**.** We consider the elliptic M. Riesz kernels R_λ in dimension n. If $0 < \operatorname{Re}\lambda < n$, R_λ is defined by

$$R_\lambda = \frac{\Gamma\big(\frac{n-\lambda}{2}\big)}{2^\lambda \pi^{n/2}\Gamma\big(\frac{\lambda}{2}\big)}|x|^{\lambda-n} \in L^1_{\text{loc}}.$$

Let

$$D : \mathbb{C} \to \mathbb{N}_0, \quad \lambda \mapsto \begin{cases} 0, & \text{if } \lambda \in \mathbb{C} \setminus \{-n - 2\mathbb{N}_0\}; \\ 1, & \text{if } \lambda \in -n - 2\mathbb{N}_0. \end{cases}$$

By analytic continuation we get $R \in \mathcal{M}(\mathbb{C}; D, \mathcal{S}'(\mathbb{R}^n))$. According to Theorem 6 in [21, p. 31] the mapping

$$\Lambda \to \mathcal{D}', \quad (\lambda, \mu) \mapsto R_\lambda * R_\mu$$

is well defined if and only if one of the following assumptions is satisfied:

(1) $\Lambda = \Lambda_1 \times \Lambda_2$ where $\Lambda_1 = -2\mathbb{N}_0$ and $\Lambda_2 = \mathbb{C}$ or $\Lambda_1 = \mathbb{C}$ and $\Lambda_2 = -2\mathbb{N}_0$.

(2) $\Lambda = \big\{(\lambda, \mu) \in \mathbb{C}^2 \mid \operatorname{Re}(\lambda + \mu) < \mathrm{n}\big\}$.

In both cases, Proposition 16 is not applicable directly; in the first case one of the sets is not open, in the second one it has not the form of a Cartesian product. Exhausting the set Λ in the second case by sets of the type

$$\Lambda_k = \big\{(\lambda, \mu) \in \mathbb{C}^2 \mid \operatorname{Re}\lambda < k, \operatorname{Re}\mu < n - k\big\}, \quad k \in \mathbb{N}$$

and applying Proposition 16 to

$$\Lambda_{k,1} \times \Lambda_{k,2} \to \mathcal{S}', \quad (\lambda, \mu) \mapsto R_\lambda * R_\mu,$$

where $\Lambda_{k,1} = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < k\}$ and $\Lambda_{k,2} = \{\mu \in \mathbb{C} \mid \operatorname{Re}\mu < n - k\}$, we conclude that

$$\big[(\lambda, \mu) \mapsto R_\lambda * R_\mu\big] \in \mathcal{M}(\Lambda; \mathcal{S}').$$

Example 4. Let

$$D : \mathbb{C} \to \mathbb{N}_0, \quad \lambda \mapsto \begin{cases} 0, & \text{if } \lambda \in \mathbb{C} \setminus \{-n - 2\mathbb{N}_0\}; \\ 2, & \text{if } \lambda \in -n - 2\mathbb{N}_0. \end{cases}$$

The function

$$L : \mathbb{C} \to \mathcal{S}'(\mathbb{R}^n), \quad \lambda \mapsto |x|^\lambda \log |x|$$

is a distribution-valued meromorphic function $L \in \mathcal{M}(\mathbb{C}; D, \mathcal{S}')$ with second order poles in $-n - 2\mathbb{N}_0$ as it is the derivative of $\lambda \mapsto |x|^\lambda$ with respect to λ.

4. Distribution-Valued Differentiable Functions

L. Schwartz treated in [27] different classes of differentiable distribution-valued functions (but not holomorphic ones). Therefore we now consider differentiable vector-valued functions.

We state a proposition on hypocontinuous vector-valued products similar to Proposition 2 and a proposition similar to Propositions 3 and 12. For m finite, $\mathcal{E}^m$ is not nuclear [29, p. 70]. Therefore L. Schwartz' Theorems on products of vector-valued distributions in [31] cannot be applied.

Proposition 17. *Let E, F and G be three quasi-complete separated locally convex topological vector spaces, $b : E \times F \to G$ a hypocontinuous bilinear map, $\Omega_1 \subset \mathbb{R}^d$ and $\Omega_2 \subset \mathbb{R}^n$ open subsets and $m \in \mathbb{N}_0 \cup \{\infty\}$. The hypocontinuous bilinear map*

$$\underset{b}{\otimes} : \mathcal{E}^m(\Omega_1; E) \times \mathcal{E}^m(\Omega_2; F) \to \mathcal{E}^m(\Omega_1 \times \Omega_2; G),$$
$$(\vec{f}, \vec{g}) \mapsto \Big[(x, y) \mapsto b\Big(\vec{f}(x), \vec{g}(y)\Big)\Big]$$

is the only one satisfying the consistency property for decomposed elements. If b is continuous so is $\underset{b}{\otimes}$.

Proof.

(1) For a quasi-complete separated locally convex topological vector space L the equality

$$\mathcal{E}^m(\Omega; L) \cong \mathcal{E}^m(\Omega) \widehat{\otimes}_\varepsilon L$$

holds according to [27, p. 106]. $\mathcal{E}^m(\Omega)$ satisfies the strict approximation property according to [30, Cor., p. 10]; hence $\mathcal{E}^m(\Omega) \otimes_\varepsilon L \subset \mathcal{E}^m(\Omega; L)$ is a dense subset. Therefore $\underset{b}{\otimes}$ is unique, since it is partially continuous and satisfies the consistency property for decomposed elements.

(2) In order to show that $\underset{b}{\otimes}$ is well-defined, we have to prove that $\partial^\alpha\Big(\underset{b}{\otimes}\Big(\vec{f}, \vec{g}\Big)\Big)$ exists and is a continuous function for $|\alpha| \le m$. (cf. [3, p. 19])

(a) Let $\vec{f} : \Omega_1 \to E$ and $\vec{g} : \Omega_2 \to F$ be continuous. We show the continuity of $(x, y) \mapsto b\big(\vec{f}(x), \vec{g}(y)\big)$ in $(x_0, y_0) \in \Omega_1 \times \Omega_2$. As in $\Omega_1 \times \Omega_2$ every point has a countable fundamental system of neighborhoods, it is sufficient to show that the map is sequentially continuous. Let $(x_k, y_k)_{k\in\mathbb{N}}$ be a sequence converging to (x_0, y_0). We get

$$b\Big(\vec{f}(x_k), \vec{g}(y_k)\Big) - b\Big(\vec{f}(x_0), \vec{g}(y_0)\Big) = \\ b\Big(\vec{f}(x_k) - \vec{f}(x_0), \vec{g}(y_k)\Big) + b\Big(\vec{f}(x_0), \vec{g}(y_k) - \vec{g}(x_0)\Big).$$

As $(y_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence and $\vec{g}$ is continuous, $\vec{g}(y_k)$ is a Cauchy-sequence too and hence bounded. Therefore the right-hand-side of the above equation converges to zero as b is hypocontinuous. This proves the continuity of $(x, y) \mapsto b\big(\vec{f}(x), \vec{g}(y)\big)$.

(b) Next we show the existence and continuity of $(x, y) \mapsto \partial_j b\big(\vec{f}(x), \vec{g}(y)\big)$. Let $\vec{f} \in \mathcal{E}^1(\Omega_1; E)$ and $\vec{g} \in \mathcal{E}^1(\Omega_2; F)$. Without loss of generality we assume $1 \le j \le d$.

$$\begin{aligned} \partial_j b\Big(\vec{f}(x), \vec{g}(y)\Big) &= \lim_{h\to 0} \frac{1}{h}\Big[b\Big(\vec{f}(x + he_j), \vec{g}(y)\Big) - b\Big(\vec{f}(x), \vec{g}(y)\Big)\Big] \\ &= b\Big(\partial_j \vec{f}(x), \vec{g}(y)\Big) \end{aligned}$$

as b is partially continuous.
As $\partial_j \vec{f}$ and $\vec{g}$ are continuous so is

$$(x, y) \mapsto b\Big(\Big(\partial_j \vec{f}\,\Big)(x), \vec{g}(y)\Big),$$

according to 2(a).

(c) Existence and continuity of

$$(x, y) \mapsto \Big(\partial^\alpha \overset{\otimes}{_b} \Big(\vec{f}, \vec{g}\Big)\Big)(x, y)$$

for $|\alpha| \le m$ follow by induction.

(3) It is easy to check that $\overset{\otimes}{_b}$ is consistent with respect to decomposed elements.

(4) Now we show the hypocontinuity of $\overset{\otimes}{_b}$.

(a) Let $A \subset \mathcal{E}^m(\Omega_1; E)$ be a subset. We prove that A is bounded if and only if for all compact subsets $K \subset \Omega_1$ the set

$$B_A := \bigcup_{\vec{f}\in A} \bigcup_{|\alpha|\le m} \Big\{\Big(\partial^\alpha \vec{f}\,\Big)(x) \mid x \in K\Big\} \subset E$$

is bounded:
$A \subset \mathcal{E}^m(\Omega_1; E)$ is bounded if and only if for all continuous semi-norms p on E and for all compact subsets $K \subset \Omega_1$ there exists a constant $C_{A,K,p} > 0$ such that

$$\sup_{x \in K} \sup_{|\alpha| \le m} p\Big(\big(\partial^\alpha \vec{f}\,\big)(x)\Big) \le C_{A,K,p}$$

for all $\vec{f} \in A$. This is equivalent to $p(u) \le C_{A,K,p}$ for all $u \in B_A$, i.e., $B_A \subset E$ is bounded.

(b) Now let $A \subset \mathcal{E}^m(\Omega_1; F)$ be a bounded subset and

$$r_K\big(\vec{f}\,\big) = \sup_{|\alpha| \le m} \sup_{(x,y) \in K} q\Big(\vec{f}(x,y)\Big)$$

a continuous semi-norm on $\mathcal{E}^m(\Omega_1 \times \Omega_2; G)$, where $K \subset \Omega_1 \times \Omega_2$ is a compact subset and q a continuous semi-norm on G. There are compact subsets $K_1 \subset \Omega_1$ and $K_2 \subset \Omega_2$ such that $K \subset K_1 \times K_2$. As b is hypocontinuous and B_A is bounded, we obtain

$$\begin{aligned}
&\sup_{\vec{g} \in A} r_K\Big(\underset{b}{\otimes}\,(\vec{f}, \vec{g})\Big) \\
&\qquad = \sup_{\vec{g} \in A} \sup_{(x,y) \in K} \sup_{|\alpha| \le m} q\Big(\partial^\alpha b\big(\vec{f}(x), \vec{g}(y)\big)\Big) \\
&\qquad = \sup_{(x,y) \in K} \sup_{|\alpha| \le m} \sup_{\vec{g} \in A} q\Big(b\big(\big(\partial^{\alpha_I} \vec{f}\,\big)(x), \big(\partial^{\alpha_{II}} \vec{g}\big)(y)\big)\Big) \\
&\qquad \le \sup_{x \in K_1} \sup_{y \in K_2} \sup_{|\beta| \le m} \sup_{|\gamma| \le m} \sup_{\vec{g} \in A} q\Big(b\big(\big(\partial^{\beta} \vec{f}\,\big)(x), \big(\partial^{\gamma} \vec{g}\big)(y)\big)\Big) \\
&\qquad \le \sup_{x \in K_1} \sup_{|\beta| \le m} \sup_{z \in B_A} q\Big(b\big(\big(\partial^{\beta} \vec{f}\,\big)(x), z\big)\Big) \\
&\qquad \le C \sup_{x \in K_1} \sup_{|\beta| \le m} p\Big(\big(\partial^{\beta} \vec{f}\,\big)(x)\Big) = C\, p_{K_1}\big(\vec{f}\,\big),
\end{aligned}$$

where $\alpha = (\alpha_I, \alpha_{II}) \in \mathbb{N}_0^{d+n}$, $\beta \in \mathbb{N}_0^d$, $\gamma \in \mathbb{N}_0^n$ and p is a continuous semi-norm on E. Hence

$$\vec{f} \mapsto \sup_{\vec{g} \in A} r_K\Big(\underset{b}{\otimes}\,(\vec{f}, \vec{g})\Big)$$

is a continuous semi-norm on $\mathcal{E}^m(\Omega_1; E)$.

(c) It can be shown analogously that

$$\vec{g} \mapsto \sup_{\vec{f} \in B} r_K\Big(\underset{b}{\otimes}\,(\vec{f}, \vec{g})\Big)$$

is a continuous semi-norm on $\mathcal{E}^m(\Omega_2; F)$ if $B \subset \mathcal{E}^m(\Omega_1; E)$ is bounded.

Hence the map $\overset{\otimes}{b}$ is hypocontinuous.

(5) If b is continuous we get

$$\begin{aligned}
r_K\Big(\overset{\otimes}{b}(\vec{f},\vec{g})\Big) &= \sup_{(x,y)\in K}\sup_{|\alpha|\le m} r\Big(b\big((\partial^{\alpha_I}\vec{f}\,)(x),(\partial^{\alpha_{II}}\vec{g})(y)\big)\Big)\\
&\le \sup_{(x,y)\in K}\sup_{|\beta|\le m}\sup_{|\gamma|\le m} r\Big(b\big((\partial^{\beta}\vec{f}\,)(x),(\partial^{\gamma}\vec{g})(y)\big)\Big)\\
&\le C\sup_{x\in K_1}\sup_{|\beta|\le m} p\Big((\partial^{\beta}\vec{f}\,)(x)\Big)\sup_{y\in K_2}\sup_{|\gamma|\le m} q\Big((\partial^{\gamma}\vec{g})(y)\Big)\\
&= C\,p_{K_1}(\vec{f}\,)q_{K_2}(\vec{g}),
\end{aligned}$$

where $\alpha=(\alpha_I,\alpha_{II})\in\mathbb{N}_0^{d+n}$, $\beta\in\mathbb{N}_0^d$ and $\gamma\in\mathbb{N}_0^n$. Hence the map $\overset{\otimes}{b}$ is continuous. ☑

***Remark* 7.** If $m=\infty$, Proposition 17 is a particular case of Proposition 3 in [31, p. 37].

As in the case of holomorphic functions we obtain a proposition corresponding to Proposition 25 in [31, p. 120] by restricting the map defined above to the diagonal $(x,x)\in\Omega\times\Omega$.

Proposition 18. *Let E, F and G be three quasi-complete separated locally convex topological vector spaces, $b: E\times F\to G$ a hypocontinuous bilinear map, $\Omega\subset\mathbb{R}^n$ an open subset and $m\in\mathbb{N}_0\cup\{\infty\}$. The hypocontinuous bilinear map*

$$\begin{aligned}
\overset{\cdot}{b}:\mathcal{E}^m(\Omega;E)\times\mathcal{E}^m(\Omega;F)&\to\mathcal{E}^m(\Omega;G),\\
(\vec{f},\vec{g})&\mapsto\Big[x\mapsto b\Big(\vec{f}(x),\vec{g}(x)\Big)\Big]
\end{aligned}$$

is the only one satisfying the consistency property for decomposed elements. If b is continuous so is $\overset{\cdot}{b}$.

Proof. The map $\Omega\to\Omega\times\Omega, x\mapsto(x,x)$ is a smooth function. The inequality

$$\sup_{x\in K}\sup_{|\alpha|\le m} q\big(\partial^{\alpha}\vec{h}(x,x)\big)\le C\sup_{(x,y)\in K\times K}\sup_{|\tilde{\alpha}|\le m} q\big(\partial^{\tilde{\alpha}}\vec{h}(x,y)\big)=q_{K\times K}(\vec{h}),$$

where $K\subset\Omega$ is a compact subset and $\tilde{\alpha}\in\mathbb{N}_0^{2n}$, yields the continuity of the map

$$\mathcal{E}^m(\Omega\times\Omega;G)\to\mathcal{E}^m(\Omega;G),\quad \vec{h}\mapsto\Big[x\mapsto\vec{h}(x,x)\Big].$$

Hence the map

$$\begin{aligned}
\overset{\cdot}{b}:\mathcal{E}^m(\Omega;E)\times\mathcal{E}^m(\Omega;F)&\to\mathcal{E}^m(\Omega;G),\\
(\vec{f},\vec{g})&\mapsto\Big[x\mapsto b\Big(\vec{f}(x),\vec{g}(x)\Big)\Big]
\end{aligned}$$

is hypocontinuous, respectively continuous, due to Proposition 17. It is uniquely determined as it is consistent with respect to decomposed elements and the space $\mathcal{E}^m(\Omega)\otimes_\varepsilon L \subset \mathcal{E}^m(\Omega; L)$, where $L = E$ or $L = F$, is a dense subspace. ☑

Proposition 19 (p-Integrable Distributions). *Let $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^d$ be open subsets and $m \in \mathbb{N}_0 \cup \{\infty\}$.*

(1) The hypocontinuous multiplication map

$$\overset{\otimes}{\cdot} : \mathcal{E}^m(\Omega_1; \mathcal{D}_{L^p}) \times \mathcal{E}^m(\Omega_2; \mathcal{D}'_{L^q}) \to \mathcal{E}^m(\Omega_1 \times \Omega_2; \mathcal{D}'_{L^r}),$$
$$(\vec{f}, \vec{g}) \mapsto \Big[(x, y) \mapsto \vec{f}(x) \cdot \vec{g}(y)\Big],$$

where $r \geq 1$ and $\frac{1}{r} \leq \frac{1}{q} + \frac{1}{p}$, is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the product $(x, y) \mapsto \vec{T}(x) \otimes \vec{a}(y)$ of a distribution-valued differentiable function $\vec{T} \in \mathcal{E}^m(\Omega_1; \mathcal{D}'_{L^q})$ and a vector-valued differentiable function $\vec{a} \in \mathcal{E}^m(\Omega_2; \mathcal{D}_{L^p})$ is again m-times continuously differentiable.

(2) The continuous convolution map

$$\overset{\otimes}{*} : \mathcal{E}^m\left(\Omega_1; \mathcal{D}'_{L^p}\right) \times \mathcal{E}^m\left(\Omega_2; \mathcal{D}'_{L^q}\right) \to \mathcal{E}^m\left(\Omega_1 \times \Omega_2; \mathcal{D}'_{L^r}\right),$$
$$(\vec{f}, \vec{g}) \mapsto \Big[(x, y) \mapsto \vec{f}(x) * \vec{g}(y)\Big],$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$, is the uniquely determined partially continuous bilinear map consistent with respect to decomposed elements.

In particular the convolution $(x, y) \mapsto \vec{S}(x) * \vec{T}(y)$ of two distribution-valued differentiable functions $\vec{S} \in \mathcal{E}^m(\Omega_1; \mathcal{D}'_{L^p})$ and $\vec{T} \in \mathcal{E}^m(\Omega_2; \mathcal{D}'_{L^q})$ is again differentiable.

Example 5. Let $K : \mathbb{R}^n \times (\mathbb{R} \setminus \{0\}) \to \mathbb{C}$ be a "summability-kernel" fulfilling the following conditions.

(1) There is a constant $A > 0$ such that for all $y \neq 0$ the integral

$$\int_{\mathbb{R}^n} |K(x, y)|\, dx \leq A$$

is bounded. Hence, in particular $K(\cdot, y) \in L^1(\mathbb{R}^n)$ for all $y \in \mathbb{R} \setminus \{0\}$.

(2) $\int_{\mathbb{R}^n} K(x, y)\, dx = 1$ for all $y \neq 0$.

(3) For fixed $\eta > 0$, the integral $\int_{|x|\geq\eta} |K(x, y)|\, dx$ converges to 0 as $y \to 0$.

(4) The map $\mathbb{R} \setminus \{0\} \to L^1(\mathbb{R}^n)$; $y \mapsto K(\cdot, y)$ is continuous.

Then $K \in \mathcal{E}^0(\mathbb{R}; \mathcal{D}'_{L^1})$ if we set $K(\cdot, 0) := \delta$ as $K(\cdot, y) \to \delta$ in $\mathcal{D}'_{L^1}$ if $y \to 0$ due to Theorem 4 in [20, p. 12].

Proposition 19 shows that the convolution

$$\big[(y, z) \mapsto K(\cdot, y) * K(\cdot, z)\big] \in \mathcal{E}^0\big(\mathbb{R}; \mathcal{D}'_{L^1}\big)$$

is again a continuous distribution-valued function.

An example of such a kernel is the Poisson-kernel

$$P(x, y) = \begin{cases} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{|y|}{(|x|^2+y^2)^{\frac{n+1}{2}}}, & \text{if } y \neq 0; \\ \delta, & \text{if } y = 0; \end{cases}$$

where $P(\cdot, y) \in \mathcal{D}'_{L^1}(\mathbb{R}^n)$ for all $y \in \mathbb{R}$. Assumptions 1. – 3. are fulfilled according to [20, Ex., p. 12]. Therefore we only have to show the continuity of $\mathbb{R} \setminus \{0\} \to L^1(\mathbb{R}^n), y \mapsto K(\cdot, y)$. Let $y \neq 0$ and $(y_k)_{k\in\mathbb{N}}$ be a sequence converging to y. As $y \neq 0$ there are constants $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all $k \geq N$ it holds

$$|P(x, y_k)| = c_n \left| \frac{y_k}{\big(|x|^2 + y_k^2\big)^{\frac{n+1}{2}}} \right| \leq \left| \frac{\varepsilon}{\big(|x|^2 + \varepsilon^2\big)^{\frac{n+1}{2}}} \right| \in L^1\big(\mathbb{R}^n\big).$$

Hence the mapping $\mathbb{R} \setminus \{0\} \to L^1(\mathbb{R}^n), y \mapsto P(\cdot, y)$ is continuous according to Lebesgue's theorem on dominated convergence. Therefore $P \in \mathcal{E}^0(\mathbb{R}; \mathcal{D}'_{L^1})$.

Another interesting example $K \in \mathcal{E}^0\big([0, \infty), \mathcal{D}'_{L^1}\big)$ is the heat kernel

$$K(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \, \mathrm{e}^{-|x|^2/4t}, & \text{if } t > 0; \\ \delta, & \text{if } t = 0. \end{cases}$$

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