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Embedded CMC Hypersurfaces on Hyperbolic Spaces

Hipersuperficies encajadas con CMC en el espacio hiperbólico

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Abstract. In this paper we will prove that for every integer $n > 1$, there exists a real number $H_0 < -1$ such that every $H \in (-\infty, H_0)$ can be realized as the mean curvature of an embedding of $H^{n-1} \times S^1$ in the $n+1$-dimensional space H^{n+1}. For $n = 2$ we explicitly compute the value H_0. For a general value n, we provide a function ξ_n defined on $(-\infty, -1)$, which is easy to compute numerically, such that, if $\xi_n(H) > -2\pi$, then, H can be realized as the mean curvature of an embedding of $H^{n-1} \times S^1$ in the $(n+1)$-dimensional space H^{n+1}.

Key words and phrases. Principal curvatures, Hyperbolic spaces, Constant mean curvature, CMC, Embeddings.

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Resumen. En este artículo demostramos que para cada número entero $n > 1$, existe un número real $H_0 < -1$, tal que todo $H \in (-\infty, H_0)$ puede obtenerse como la curvatura media de un encaje de la variedad $H^{n-1} \times S^1$ en el espacio hiperbólico $n+1$ dimensional H^{n+1}. Para $n = 2$ calcularemos explícitamente el valor H_0. Para otros valores de n, daremos una función ξ_n definida en el intervalo $(-\infty, -1)$, la cual es fácil de calcular numéricamente, con la propiedad de que si $\xi_n(H) > -2\pi$, entonces el número H puede obtenerse como la curvatura media de un encaje de la variedad $H^{n-1} \times S^1$ en el espacio hiperbólico $n+1$ dimensional H^{n+1}.

Palabras y frases clave. Curvaturas principales, espacio hiperbólico, curvatura media constante, CMC, encajes.

1. Introduction and Preliminaries

Here we will be considering the following model of the hyperbolic space,

$$H^{n+1} = \{x \in \mathbf{R}^{n+2} \mid x_1^2 + \cdots + x_{n+1}^2 - x_{n+2}^2 = -1\}$$

where the space $\mathbf{R}^{n+2}$ is endowed with the following inner product

$$\langle v, w \rangle = v_1 w_1 + \cdots + v_{n+1} w_{n+1} - v_{n+2} w_{n+2}$$

for $v = (v_1, \ldots, v_{n+2})$ and $w = (w_1, \ldots, w_{n+2})$.

In [2] we proved the following theorem that shows that $S^{n-1} \times \mathbf{R}$ can be embedded in the hyperbolic space with constant mean curvature (CMC).

Theorem 1. *Let* $g_{C,H} : \mathbf{R} \to \mathbf{R}$ *be a positive solution of the equation*

$$(g')^2 + g^{2-2n} + (H^2 - 1)g^2 + 2Hg^{2-n} = C \tag{1}$$

associated with a non negative H *and a positive constant* C*. If* $\mu, \lambda, r, \theta : \mathbf{R} \to \mathbf{R}$ *are defined by*

$$\begin{aligned} r &= \frac{g_{C,H}}{\sqrt{C}}, \\ \lambda &= H + g_{C,H}^{-n}, \\ \mu &= nH - (n-1)\lambda = H - (n-1)g_{C,H}^{-n} \end{aligned}$$

and

$$\theta(u) = \int_0^u \frac{r(s)\lambda(s)}{1 + r^2(s)}\, ds$$

then, the map $\phi : S^{n-1} \times \mathbf{R} \to H^{n+1}$ *given by*

$$\phi(y, u) = \big(r(u)y,\ \sqrt{1 + r(u)^2}\sinh(\theta(u)),\ \sqrt{1 + r(u)^2}\cosh(\theta(u))\big) \tag{2}$$

defines an embedded hypersurface in H^{n+1} *with constant mean curvature* H*. Moreover, if* $H^2 > 1$*, the embedded manifold defined by* (2) *admits* $O(n) \times Z$ *in its group of isometries, where* Z *is the group of integers.*

The existence of the previous examples just as immersions were studied in [3] as Delaunay-type hypersurfaces of the hyperbolic space and also in [1] as rotational hypersurfaces of spherical type. Also, a classification of immersed complete hypersurfaces in hyperbolic spaces with constant mean curvature and two principal curvatures was made in [4] by Wu. Having in mind the previous work, we can say that the results in this paper shows that some of the known complete immersions with constant mean curvature from $H^{n-1} \times \mathbf{R}$ to H^{n+1}

define *embedded* examples with CMC from $H^{n-1} \times S^1$ to H^{n+1}. We also provided examples of immersed examples from $H^{n-1} \times S^1$ to H^{n+1} that has the cyclic group Z_m in the group of isometries.

For the proof of the main result of this paper (Theorem 4), we first provide explicit immersions $\phi_{C,H}$ of $H^{n-1} \times \mathbf{R}$ in H^{n+1} with constant mean curvature H. Then, we find four functions $C_0 = C_0(n, H)$, $\widetilde{C} = \widetilde{C}(n, H)$, $K = K(n, C, H)$ and $\xi_n = \xi_n(H)$ with the property that if $C \in (C_0, \widetilde{C})$ and $K = -2\pi$, then, the immersion $\phi_{C,H}$ defines an embedding from $H^{n-1} \times S^1$ to H^{n+1}. Finally, we prove that for every H such that $\xi_n(H) > -2\pi$, there exists a value $C \in (C_0, \widetilde{C})$ such that $K(C, H) = -2\pi$, such values of H exist because for every n, $\lim_{H\to-\infty} \xi_n(H) = -\pi$. In some steps of the theorem we use the intermediate value theorem and the following corollary of Lemma 5.1 in [2],

Corollary 2. *Let ϵ be a positive real number and let $f, g : (v_0 - \epsilon, v_0 + \epsilon) \times (c_0 - \epsilon, c_0 + \epsilon) \to \mathbf{R}$ be two smooth functions, such that $\frac{\partial f}{\partial c}(v_0, c_0) > 0$, $f(v_0, c_0) = \frac{\partial f}{\partial v}(v_0, c_0) = 0$ and $\frac{\partial^2 f}{\partial v^2}(v_0, c_0) = -2a < 0$. If for any small $c > c_0$, $t_1(c) < t_0 < t_2(c)$ are such that $f(t_1(c)) + c = 0 = f(t_2(c)) + c$, then*

$$\lim_{c\to c_0^+} \int_{t_1(c)}^{t_2(c)} \frac{g(v, c)}{\sqrt{f(v, c)}}\, dt = \frac{g(v_0, c_0)\, \pi}{\sqrt{a}}.$$

2. Embedded Hyperbolic Type Rotational Surfaces in H^3

It is not difficult to show that the function

$$\xi : (-\infty, -1) \to \mathbf{R}, \quad \text{given by} \quad \xi(H) = \int_0^{\pi} \frac{\sqrt{2}H}{\sqrt{2H^2 + \sin(2t) - 1}}\, dt$$

is decreasing, $\lim_{H\to-\infty} \xi(H) = -\pi$ and $\xi(H) < -2\pi$ for values of H close to -1. The previous observations guarantee the existence of a unique H_0 such that $\xi(H_0) = -2\pi$. A numerical computation shows that

$$H_0 \simeq -1.0158136657178574.$$

In this section we will show that every $H < H_0$ can be realized as the mean curvature of a hyperbolic type rotational embedded constant mean curvature surface in the hyperbolic three dimensional space. Let us state and prove the only theorem in this section.

Theorem 3. *For any $H < -1$ and $C \in (C_1, 0)$ where $C_1 = 2(H + \sqrt{-1 + H^2})$, let us define $f : \mathbf{R} \to \mathbf{R}$ by*

$$f(t) = \sqrt{\frac{C - 2H + \sqrt{4 + C^2 - 4CH}\,\sin(2\sqrt{H^2 - 1}t)}{2H^2 - 2}}$$

If we define,

$$r(t)=\frac{f(t)}{\sqrt{-C}} \quad \textit{and} \quad \lambda(t)=H+\big(f(t)\big)^{-2}$$

then, the function $\frac{\lambda(t)r(t)}{r^2(t)-1}$ *is a smooth function everywhere and if we define*

$$\theta(t)=\int_0^t \frac{\lambda(s)r(s)}{r^2(s)-1}\,ds$$

then, the map

$$\phi(u,v)= \Big(\sqrt{r(u)^2-1}\cos(\theta(u)),\sqrt{r(u)^2-1}\sin(\theta(u)),r(u)\sinh(v),r(u)\cosh(v)\Big) \quad (3)$$

defines and immersion from $\mathbf{R}^2$ *to* H^3. *We also have that for every* $H<-1$ *there exist infinitely many choices of* C *such that the immersion* ϕ *is periodic in the variable* u *and therefore it defines immersions from* $\mathbf{R}\times S^1$ *to* H^3. *Moreover, we have that for every* $H<H_0$, *there exists a value* C *such that* ϕ *defines an embedding from* $\mathbf{R}\times S^1$ *to* H^3.

Proof. Since $H<-1$ and $C\in(C_1,0)$, we have that the function f is a real-value T-periodic function that oscillates from t_1 to t_2 where

$$t_1,t_2=\sqrt{\frac{C-2H\pm\sqrt{4+C^2-4CH}}{2H^2-2}} \quad \text{and} \quad T=\frac{\pi}{\sqrt{H^2-1}}.$$

A direct computation shows that

$$(f')^2+f^{-2}+(H^2-1)f^2+2H=C.$$

The equation above shows that the function $r(t)$ satisfies the identity

$$(r')^2+\lambda^2\,r^2=r^2-1 \quad (4)$$

This equation shows that $r(t)\geq 1$, moreover, it shows that $r(t^\star)=1$, if and only if $\lambda(t^\star)=0$ and $r'(t^\star)=0$. By the definition of the function λ, we have, in this situation, that $t^\star$ is a root of λ and r^2-1 with the same multiplicity. Since the function r is analytic, we get that the function $\frac{\lambda(s)r(s)}{r^2(s)-1}$ is smooth near $t^\star$, therefore it is smooth everywhere. A direct computation shows that

$$\frac{\partial\phi}{\partial u}=\frac{r\,r'}{\sqrt{r^2-1}}\big(\cos(\theta),\sin(\theta),0,0\big)+ \frac{r\,\lambda}{\sqrt{r^2-1}}\big(-\sin(\theta),\cos(\theta),0,0\big)+r'\big(0,0,\sinh(v),\cosh(v)\big)$$

and

$$\frac{\partial \phi}{\partial v} = r(u)\big(0,0,\cosh(v),\sinh(v)\big).$$

It is not difficult to prove that the map

$$\nu = -r\lambda\big(0,0,\sinh(v),\cosh(v)\big) - \frac{r^2\,\lambda}{\sqrt{r^2-1}}\big(\cos(\theta),\sin(\theta),0,0\big) + \frac{r'}{\sqrt{r^2-1}}\big(-\sin(\theta),\cos(\theta),0,0\big)$$

is a Gauss map of the immersion ϕ. It follows that the immersion ϕ has constant mean curvature H by noticing that

$$\frac{\partial \nu}{\partial v} = -\lambda \frac{\partial \phi}{\partial v} \quad \text{and} \quad \frac{\partial \nu}{\partial u} = -(2H-\lambda)\frac{\partial \phi}{\partial u}.$$

Let us define the function K that depends on H and C by

$$K(C,H) = \int_0^T \frac{\lambda(s)\,r(s)}{r^2(s)-1}\,ds.$$

A direct computation shows that for every fixed H we have,

$$\lim_{C\to C_1} K(C,H) = -\pi\sqrt{2-\frac{2H}{\sqrt{H^2-1}}} = b_2(H) \quad \text{and} \quad \lim_{C\to 0} K(C,H) = 0. \quad (5)$$

Since $H < -1$ we have that $b_2(H) < -2\pi$. Using the limits in (5) we get that for any fixed value $H < -1$ and for every positive integer m, there exists a real number $C^\star$ between C_1 and 0 such that $K(C^\star, H) = -\frac{2\pi}{m}$. Since the function θ satisfies that,

for any integer j and $u \in [jT,(j+1)T]$ we have that $\theta(u) = jK + \theta(u-jT)$,

we get that if we choose the value $C^\star$, we get that $\theta(mT) = -2\pi$ and therefore the immersion $\phi(u,v)$ will be mT-periodic in the variable u and it will define an immersion from $\mathbf{R}\times S^1$ to H^3. Let us prove that for every $H < H_0$ there exists an embedding from $\mathbf{R}\times S^1$ to H^3. By using the definition of the function λ and the expression for the bounds t_1 and t_2 of the function f, we have that for a given H, the function $\lambda < 0$ if and only if $C_1 < C < \frac{1}{H}$. Notice that if λ is always negative, then the function θ is strictly decreasing, and in particular it is one to one. A direct computation shows that

$$K\left(\frac{1}{H},H\right) = \int_0^T \frac{H\sqrt{2H^2-2}}{\sqrt{2H^2-1+\sin\left(2\sqrt{H^2-1}\,s\right)}}\,ds = \int_0^\pi \frac{H\sqrt{2}}{\sqrt{2H^2-1+\sin(2t)}}\,dt$$

As pointed out at the beginning of this section, the function $\xi(H) = K(\frac{1}{H}, H)$ is decreasing and the limit when $H \to -\infty$ is $-\pi$. Therefore for any $H < H_0$ there exists a $C^\star$ between C_1 and $\frac{1}{H}$ such that $K(C^\star, H) = -2\pi$. By the way we picked $C^\star$ we get that the function θ is strictly decreasing and $\theta(T) = -2\pi$, these two conditions guarantee that the immersion $\phi(u, v)$ is T-periodic and injective in $\mathbf{R} \times (0, T)$, therefore ϕ defines an embedding from $\mathbf{R} \times S^1$ to H^3. This completes the proof of the theorem. ☑

2.1. Graph of Some Profile Curves

The examples described in Theorem 3 are obtained by doing a hyperbolic rotation of the profile curve

$$\alpha(t) = \big(\sqrt{r^2(t) - 1}\cos(\theta(t)), \sqrt{r^2(t) - 1}\sin(\theta(t))\big)$$

We show the graphs of a profile curve that corresponds to an embedded example (Figure 1) and two profile curves corresponding to immersed examples (Figure 2 and Figure 3), all of them represent examples with constant mean curvature $H = -1.1$. To finish the section we will show one of the numerical difficulties in order to do the graph. This difficulty is the fact that the angle function θ moves a lot in a small variation of the parameter t, during this small variation of parameter t, the radius function $\sqrt{r^2(t) - 1}$ is very close to zero (Figure 4). We will show this fact by graphing the function $\theta'(t)$, first by limiting the codomain to some values close to zero, and then by showing the whole graph of θ' (Figure 5). Notice that even though the graph of the profile curve may look like a non differentiable curve, the curve as well as the immersion are indeed smooth since there are explicit expressions for these functions in term of smooth function and integral of smooth functions.

3. Embedded Solutions in Hyperbolic Spaces

It is well known that the existence of CMC hypersurfaces in hyperbolic spaces with two principal curvatures relies on the existence of solutions of the following differential equation,

$$(g')^2 + g^{2-2n} + (H^2 - 1)g^2 + 2Hg^{2-n} = C.$$

It is not difficult to check that, when $H < -1$, it is possible to obtain solutions of this equation associated with negative values of C. These CMC examples produced by these solutions when $C < 0$ correspond to those named as rotational hyperbolic type in [1]. Similar arguments as those shown in [2] will give us explicit immersions for such a choice of the constant C. The following sequences of statements tell us how to pick the negative values of C to obtain solutions in the case that $H < -1$ and several other properties that will be useful in the proof of Theorem 4 in this paper which consists in generalizing

FIGURE 1. Profile curve for a surface with CMC $H = -1.1$, in this case the surface is embedded and $C = -0.9091743461769703$ and $K = -2\pi$.

FIGURE 2. Profile curve for a surface with CMC $H = -1.1$, in this case $C = -0.6835660909345689$ and $K = -\frac{2\pi}{5}$.

the results of the previous section to dimensions greater than two by finding explicit immersions of CMC hypersurfaces in the hyperbolic space of Delaunay type and by showing that some of these explicit examples are embedded. The

Figure 3. Profile curve for a surface with CMC $H = -1.1$, in this case $C = -0.19607165524075582$ and $K = -\frac{2\pi}{10}$.

Figure 4. Graph of the function θ' associated with the embedded example whose profile curve is shown above, in this case just part of the graph is shown.

latter part, the proof showing the embeddeness of the immersions requires a very careful analysis.

***Remark* 1.** The function $q : (0, \infty) \to \mathbf{R}$ defined by $q(v) = C - v^{2-2n} + (1 - H^2)v^2 - 2Hv^{2-n}$, where $H < -1$ and $C < 0$, has the following properties:

FIGURE 5. Graph of the function θ' example associated with the embedded which profile curve is shown above.

1) The positive real number v_0 given by

$$v_0 = \left(\frac{H(n-2) + \sqrt{4 - 4n + H^2n^2}}{2H^2 - 2}\right)^{\frac{1}{n}} = \left(\frac{2(n-1)}{\sqrt{4 - 4n + H^2n^2} - H(n-2)}\right)^{\frac{1}{n}}$$

is the only positive critical point of q.

2) $p(v) = v^{2n-2}q(v)$ is a polynomial with even degree, negative leading coefficient and $p(0) = -1$.

3) Since $q'(v) > 0$ if $v < v_0$, $q'(v) < 0$ if $v > v_0$, and $q(v_0) = C - C_0$ where

$$C_0 = n\frac{H^2n - 2 + H\sqrt{4 - 4n + H^2n^2}}{\left(H(n-2) + \sqrt{4 - 4n + H^2n^2}\right)^{\frac{2n-2}{n}}}(2H^2 - 2)^{\frac{n-2}{n}} \tag{6}$$

then, q has exactly 2 roots whenever $0 > C > C_0$.

4) The functions $t_1, t_2 : (C_0, 0) \times (-\infty, -1) \to (0, \infty)$ defined by the equations

$$q\big(t_1(C,H)\big) = 0 \quad q\big(t_2(C,H)\big) = 0 \quad \text{with} \quad t_1(C,H) < t_2(C,H) \tag{7}$$

are smooth, $t_1(C,H)$ is decreasing with respect to C, $t_2(C,H)$ is increasing with respect to C and the limit of both functions when $C \to C_0$ is v_0.

5) Since the roots of q when $C = 0$ are $v_1 = \frac{1}{(1-h)^{\frac{1}{n}}}$ and $v_2 = \frac{1}{(-1-h)^{\frac{1}{n}}}$ then for any fixed H the derivative of the functions t_1 and t_2 defined on $(C_0, 0)$ never vanish and

$$\lim_{C\to 0} t_1(C) = v_1, \quad \lim_{C\to 0} t_2(C) = v_2 \quad \text{and} \quad \lim_{C\to C_0} t_1(C) = \lim_{C\to C_0} t_2(C) = v_0.$$

6) The following identities are true:

$$\lambda_1 = H + v_0^{-n} = \frac{nH + \sqrt{H^2n^2 - 4(n-1)}}{2(n-1)} < 0; \qquad \lambda_2 = H + v_1^{-n} = 1$$

7) For a fixed $H < -1$, the previous two items guarantee the existence of a unique $\widetilde{C}(H) \in (C_0, 0)$ such that $t_1^\star(H) = t_1\big(\widetilde{C}(H), H\big)$ satisfies that

$$H + \big(t_1^\star(H)\big)^{-n} = 0.$$

The equality above defines a smooth function $\widetilde{C} : (-\infty, -1) \to \mathbf{R}$.

8) We can explicitly compute the function $\widetilde{C}$ by noticing first that for that special value of C, the number $t_1 = (-H)^{\frac{-1}{n}}$ must be a root of the function q, therefore $q\Big((-H)^{\frac{-1}{n}}\Big)$ must be zero, i.e,

$$q\Big((-H)^{\frac{-1}{n}}\Big) = C + (-H)^{-\frac{2}{n}} = 0.$$

Therefore $\widetilde{C}(H) = -(-H)^{-\frac{2}{n}}$.

9) The function $\widetilde{q}(v) = -\frac{1}{C} q\big(\sqrt{-C}v\big)$ has the following expression:

$$\widetilde{q}(v) = -1 - (-C)^{-n} v^{2-2n} + v^2\big(1 - H^2 - 2H\big(\sqrt{-C}v\big)^{-n}\big).$$

Moreover, by the definition of $\widetilde{q}$ and the properties of the function q we have that, for any $C \in (C_0, 0)$, the only 2 positive roots of $\widetilde{q}$ are

$$\widetilde{t_1}(C, H) = \frac{t_1(C, H)}{\sqrt{-C}} \quad \text{and} \quad \widetilde{t_2}(C, H) = \frac{t_2(C, H)}{\sqrt{-C}}.$$

Therefore we have that $\widetilde{t_1}(\widetilde{C}, H) = \frac{(-H)^{-\frac{1}{n}}}{\sqrt{(-H)^{-\frac{2}{n}}}} = 1$.

10) A direct computation shows that when $C = \widetilde{C}$, the polynomial $\widetilde{q}$ reduces to the polynomial Q given by

$$Q = -1 + v^2 - H^2v^2 - H^2v^{2-2n} + 2H^2v^{2-n}.$$

It is not difficult to check that, when $n > 2$, for any positive ϵ, $\lim_{H\to-\infty} Q(1+\epsilon) = -\infty$. Therefore we have that

$$\widetilde{t_1}\big(\widetilde{C}, H\big) = 1 \quad \text{and} \quad \lim_{H\to-\infty} \widetilde{t_2}\big(\widetilde{C}, H\big) = 1. \tag{8}$$

11) Let us define the function $h : (1, \infty) \to \mathbf{R}$ by

$$h(v) = \frac{2Hv^{1-n}(-1+v^n)}{v^2-1} = 2Hv^{1-n}\frac{1+v+\cdots+v^{n-1}}{1+v}$$

and the function $\xi_n : (-\infty, -1) \to \mathbf{R}$ by

$$\xi_n(H) = \int_1^{\widetilde{t_2}\left(\widetilde{C},H\right)} \frac{h_n(v)}{\sqrt{Q(v)}}\, dv.$$

12) A direct computation shows that

$$\widetilde{a} = -\frac{1}{2}Q''(1) = n^2H^2 - 1. \tag{9}$$

Therefore, using Corollary 2, we get that

$$\lim_{H\to-\infty} \xi_n(H) = -\pi. \tag{10}$$

Notice that we have not applied Corollary 2 at $v = 1$ because $\frac{\partial Q}{\partial v}(1) = 2 \neq 0$. We are applying Corollary 2 to $v = v^\star(h)$ where $v^\star(H)$ is a number between $t_1\big(\widetilde{C}, H\big) = 1$ and $t_2\big(\widetilde{C}, H\big)$ such that $\frac{\partial Q}{\partial v}(v^\star) = 0$. Since $t_2\big(\widetilde{C}, H\big) \to 1$, as $H \to -\infty$, then $v^\star(H) \to 1$ as $H \to -\infty$ and $\frac{\partial^2 Q}{\partial v^2}(v^\star) \to \frac{\partial^2 Q}{\partial v^2}(1)$ as $H \to -\infty$.

Theorem 4. *Let $g : \mathbf{R} \to \mathbf{R}$ be a positive solution of the equation*

$$(g')^2 + g^{2-2n} + (H^2-1)g^2 + 2Hg^{2-n} = C \tag{11}$$

associated with a negative constant C. If $\mu, \lambda, r, \theta : \mathbf{R} \to \mathbf{R}$ are defined by

$$\begin{aligned} r &= \frac{g}{\sqrt{-C}}, \\ \lambda &= H + g^{-n}, \\ \mu &= nH - (n-1)\lambda = H - (n-1)g^{-n} \end{aligned}$$

and

$$\theta(u) = \int_0^u \frac{r(s)\lambda(s)}{r^2(s)-1}\, ds$$

then, the map $\phi_{C,H} : H^{n-1} \times \mathbf{R} \to H^{n+1}$ given by

$$\phi_{C,H}(y,u) = \big(\sqrt{r(u)^2-1}\cos(\theta(u)), \sqrt{r(u)^2-1}\sin(\theta(u)), r(u)y\big) \tag{12}$$

defines an immersed hypersurface in H^{n+1} with constant mean curvature H. We also have that when $H < -1$, the function g is periodic and if we denote its period by T, then, $\phi_{C,H}$ defines an immersion from $H^{n-1} \times S^1$ to H^n whenever

$$K(C,H) = \int_0^T \frac{r(s)\lambda(s)}{r^2(s)-1}\, ds = -\frac{2k\pi}{m}, \text{ for some pair of integers } k \text{ and } m. \quad (13)$$

Moreover, we have that anytime $\xi_n(H_1) > -2\pi$, where ξ_n is the function defined in item 11 in Remark 1, then, there exists a constant C such that the immersion ϕ_{C,H_1} defines an embedding from $H^{n-1} \times S^1$ to H^{n+1}.

Proof. A direct computation shows the following identities:

$$(r')^2 + r^2\lambda^2 = r^2 - 1; \qquad \lambda r' + r\lambda' = \mu r'.$$

Let us define

$$B_2(u) = \big(\cos(\theta(u)), \sin(\theta(u)), 0, \dots, 0\big) \quad \text{and}$$
$$B_3(u) = \big(-\sin(\theta(u)), \cos(\theta(u)), 0, \dots, 0\big).$$

Notice that $\langle B_2, B_2\rangle = 1$, $\langle B_3, B_3\rangle = 1$, $\langle B_2, B_3\rangle = 0$, $B_2' = \frac{r\lambda}{r^2-1} B_3$ and $B_3' = -\frac{r\lambda}{r^2-1} B_2$, moreover, we have that the map $\phi = \phi_{C,H}$ can be written as

$$\phi = r(0,0,y) + \sqrt{r^2-1}B_2$$

A direct verification shows that $\langle \phi, \phi\rangle = -1$ and that

$$\frac{\partial \phi}{\partial u} = r'(0,0,y) + \frac{rr'}{\sqrt{r^2-1}}B_2 + \frac{r\lambda}{\sqrt{r^2-1}}B_3$$

is a unit vector, i.e, $\langle \frac{\partial\phi}{\partial u}, \frac{\partial\phi}{\partial u}\rangle = 1$. We have that the tangent space of the immersion at (y,u) is given by

$$T_{\phi(y,u)} = \left\{ (v,0,0) + s\frac{\partial \phi}{\partial u} \;\Big|\; \langle v, y\rangle = 0 \quad \text{and} \quad s \in \mathbf{R} \right\}.$$

A direct verification shows that the map

$$\nu = -r\lambda(0,0,y) - \frac{r^2\lambda}{\sqrt{r^2-1}}B_2 + \frac{r'}{\sqrt{r^2-1}}B_3$$

satisfies that $\langle \nu, \nu\rangle = 1$, $\langle \nu, \frac{\partial\phi}{\partial u}\rangle = 0$ and, for any $v \in \mathbf{R}^n$ with $\langle v, y\rangle = 0$, we have that $\langle \nu, (v,0,0)\rangle = 0$. It then follows that ν is a Gauss map of the immersion ϕ. The fact that the immersion ϕ has constant mean curvature H follows because, for any unit vector v in $\mathbf{R}^n$ perpendicular to y, we have that

$$\beta(t) = \big(\cos(\theta(u)), \sin(\theta(u)), r\cosh(t)y + r\sinh(t)v\big) + \sqrt{r^2-1}B_2 = \phi\big(\cosh(t)y + \sinh(t)v, u\big)$$

satisfies that $\beta(0) = \phi(y,u)$, $\beta'(0) = rv$ and

$$\frac{d\nu(\beta(t))}{dt}\bigg|_{t=0} = d\nu(rv) = -r\lambda\, v$$

Therefore, λ is a principal curvature with multiplicity $n-1$. Now, since $\big\langle \frac{\partial\nu}{\partial u}, (v,0,0)\big\rangle = 0$ for every $(v,0,0) \in T_{\phi(y,u)}$, we have that $\frac{\partial\phi}{\partial u}$ defines a principal direction, i.e., we have that $\frac{\partial\nu}{\partial u}$ must be a multiple of $\frac{\partial\phi}{\partial u}$. A direct verification shows that,

$$\left\langle \frac{\partial\nu}{\partial u}, y\right\rangle = -\lambda' r - \lambda r' = -\mu r' = -\big(nH - (n-1)\lambda\big)r'$$

We also have that $\langle \frac{\partial\phi}{\partial u}, y\rangle = r'$, therefore,

$$\frac{\partial\nu}{\partial u} = d\nu\left(\frac{\partial\phi}{\partial u}\right) = -\mu\frac{\partial\phi}{\partial u} = -\big(nH-(n-1)\lambda\big)\frac{\partial\phi}{\partial u}.$$

It follows that the other principal curvature is $nH-(n-1)\lambda$. Therefore ϕ defines an immersion with constant mean curvature H. This proves the first item in the theorem. The fact that the map defines an immersion from $H^{n-1}\times S^1$ whenever $K(C,H) = -\frac{2k\pi}{m}$, follows from the property,

for any integer j and $u \in [jT, (j+1)T]$ we have that $\theta(u) = jK + \theta(u - jT)$,

which implies that the map ϕ is periodic in the variable u, with period mT. Let us prove the embedding part of the theorem. In this part of the proof we will be using the functions and constants

$$q,\ \widetilde{q},\ Q,\ \xi_n,\ t_1,\ t_2,\ \widetilde{t_1},\ \widetilde{t_2},\ \widetilde{C}, C_0, \quad \text{and} \quad v_0$$

defined in Remark 1. Let us start by noticing that the differential equations for the functions g and r can be written as

$$(g')^2 = q(g) \quad \text{and} \quad (r')^2 = \widetilde{q}(r)$$

It follows that, in order to obtain a solution g of this differential equation, we need that $C > C_0$ and, once we have the solution g associated with the number C and H, this solution g varies from $t_1(C,H)$ to $t_2(C,H)$. Since we know the maximum and the minimum of the function g in terms of C and H, we can verify that anytime $C < \widetilde{C} = -(-H)^{-\frac{2}{n}}$, the function λ is negative, we also have that when $C = \widetilde{C}$, 0 is the maximum of the function λ. The previous affirmation guarantees that anytime $C \in (C_0, \widetilde{C})$, the function θ is one to one. By doing the substitution $v = g(s)$ in the integral $K(C,H)$, we get that

$$K(C,H) = \int_{t_1(C,H)}^{t_2(C,H)} \frac{2\sqrt{-C}(1+Hv^n)v^{1-n}}{(C+v^2)\sqrt{q(v)}}\, dv.$$

In the previous expression we have used the symmetry of the function g, and therefore the symmetries of the functions r and λ, to express K as

$$K = 2\int_0^{\frac{T}{2}} \frac{r(s)\lambda(s)}{r^2(s)-1}\, ds.$$

When $C = C_0$, we have that $q(v_0) = 0 = q'(v_0)$, then, we can apply the Corollary 2 to obtain that

$$\lim_{C\to C_0} K(C,H) = -\pi\sqrt{2}\sqrt{1 - \frac{nH}{\sqrt{n^2H^2 - 4(n-1)}}} = lb.$$

Notice that for any $n \geq 2$ and any $H < -1$, the bound $lb < -2\pi$. By doing the substitution $v = r(s)$ in the integral $K(C,H)$, we get that

$$K(C,H) = \int_{\widetilde{t_1}(C,H)}^{\widetilde{t_2}(C,H)} \frac{2v\Big(H + \big(\sqrt{-C}v\big)^{-n}\Big)}{(v^2-1)\sqrt{\widetilde{q}(v)}}\, dv.$$

When we replace C by $\widetilde{C}$ the integral above reduces to,

$$K(\widetilde{C}, H) = \xi_n(H).$$

Using the intermediate value theorem we conclude the theorem because anytime $\xi_n(H) > -2\pi$ there exists a $C^\star \in (C_0, \widetilde{C})$ such that $K(C^\star, H) = -2\pi$; therefore the map $\phi_{C^\star,H}$ is periodic in the u variable, and since $C < \widetilde{C}$ the function θ is injective and therefore the map $\phi_{C^\star,H}$ is an embedding. ☑

Corollary 5. *For any integer $n > 1$ there exists an $H_0 \leq -1$ such that for any $H < H_0$ there exists an embedding with constant mean curvature H from $H^{n-1} \times S^1$ to H^{n+1}.*

Proof. The corollary follows from the fact that $\lim_{H\to-\infty} \xi_n(H) = -\pi$. See item 12 in Remark 1. ☑

***Remark* 2.** The integral ξ_n is easy to evaluate numerically, for example

$$\xi_3(-1) = -5.97106763713693$$
$$\xi_4(-1) = -4.599155062889069$$
$$\xi_5(-1) = -4.13016242612799$$

The following graphs suggest that for $n = 3, 4, 5$, there exist embeddings for all $H < -1$.

FIGURE 6. Graph of the function ξ_3 on $[-50, -1]$.

FIGURE 7. Graph of the function ξ_4 on $[-50, -1]$.

FIGURE 8. Graph of the function ξ_5 on $[-50, -1]$.

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