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On Spherical Invariance

Sobre invariancia esférica

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Abstract. In 1964 Pommerenke introduced the notion of linear invariant family for locally injective analytic functions defined in the unit disk of the complex plane. Following Ma and Minda (who extended this notion to spherical geometry), we consider in this paper locally injective meromorphic functions in the unit disk. More precisely, we study families of such functions for which a certain invariant, called spherical order, is finite. Several consequences on the finiteness of the spherical order are explored, in particular the connection with the Schwarzian and normal orders, and with uniform perfectness.

Key words and phrases. Spherical invariance, Spherical order, Schwarzian derivative, Normal function, Uniformly perfect.

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Resumen. En 1964 Pommerenke introdujo la noción de familia linealmente invariante para funciones analíticas localmente inyectivas definidas en el disco unidad del plano complejo. Siguiendo las ideas de Ma y Minda (quienes extendieron ésta noción a la geometría esférica), en este artículo consideramos funciones meromorfas localmente inyectivas definidas en el disco unidad. Más precisamente, estudiamos familias de tales funciones para las cuales un cierto invariante, llamado orden esférico, es finito. Varias consecuencias sobre la finitud del orden esférico son exploradas, en particular la conexión con los órdenes schwarziano y normal, y con dominios cuya frontera es uniformemente perfecta.

Palabras y frases clave. Invariancia esférica, orden esférico, derivada schwarziana, función normal, uniformemente perfecto.

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1. Introduction

Let Ω be a hyperbolic domain in the complex plane $\mathbb{C}$. We say that a family $\mathcal{F}$ of locally injective meromorphic functions defined on Ω is *spherically invariant* or, in short, *s-invariant*, if

$$f \in \mathcal{F}, \sigma \in \mathrm{Rot}\big(\widehat{\mathbb{C}}\big), \tau \in \mathrm{Aut}(\Omega) \Longrightarrow \sigma \circ f \circ \tau \in \mathcal{F},$$

where $\mathrm{Aut}(\Omega)$ is the group of conformal automorphisms of Ω and $\mathrm{Rot}\big(\widehat{\mathbb{C}}\big)$ denotes the group of rotations on the Riemann sphere $\widehat{\mathbb{C}}$. Spherical invariance was first considered by Ma and Minda [8] as an analog in spherical geometry to (euclidean) linear invariance, a concept introduced and extensively studied by Pommerenke [11]. We define the *spherical order (s-order)* of a locally injective meromorphic function f on Ω by

$$\big\|A_f^{\#}\big\|_{\Omega} := \sup_{z\in\Omega} \big|A_f^{\#}(z)\big|,$$

where

$$A_f^{\#}(z) = \frac{1}{\lambda_{\Omega}(z)} \frac{\partial}{\partial z}\left(\log \frac{f^{\#}}{\lambda_{\Omega}}\right)(z).$$

Here $f^{\#}$ stands for the spherical derivative of f, namely, $|f'|/(1+|f|^2)$ and λ_{Ω} is the density of the hyperbolic metric on Ω. We recall that the hyperbolic density λ_{Ω} is given by

$$\lambda_{\Omega}(z) = \frac{1}{\big(1-|w|^2\big)|\pi'(w)|}, \qquad z \in \Omega,$$

where $z = \pi(w)$ is any holomorphic covering of $\mathbb{D}$ onto Ω. When $\Omega = \mathbb{D}$, $A_f^{\#}$ takes the form

$$A_f^{\#}(z) = \frac{\big(1-|z|^2\big)}{2}\frac{f''(z)}{f'(z)} - \overline{z} - \frac{\big(1-|z|^2\big)\overline{f(z)}f'(z)}{1+|f(z)|^2}. \tag{1}$$

The s-order of a s-invariant family $\mathcal{F}$ is

$$\mathcal{O}_s(\mathcal{F}) := \sup_{f\in\mathcal{F}} \big\|A_f^{\#}\big\|_{\Omega}.$$

Ma and Minda [8] proved that the s-order of a locally injective meromorphic function on $\mathbb{D}$ is not less than 1, and it is 1 precisely when the mapping is *spherically convex (s-convex)*, that is, when it maps $\mathbb{D}$ conformally onto a proper domain in $\widehat{\mathbb{C}}$ for which any two points can be joined inside the domain by the smaller arc of the great circle between the two points. Ma and Minda also proved that the finiteness of the spherical order for a function f is equivalent to the existence of a positive number ρ such that for all a in $\mathbb{D}$, the image

$f(D_h(a,\rho))$ is s-convex, where $D_h(a,\rho)$ is the hyperbolic disk in $\mathbb{D}$ centered at a with hyperbolic radius ρ. A function that satisfies this last condition is referred as *uniformly locally spherically convex.*

This paper is organized as follows. In Section 2 we introduce the spherical preschwarzian of a locally injective meromorphic function f defined on a domain Ω in $\widehat{\mathbb{C}}$. Basically, the spherical preschwarzian mesures the deviation of f from a rotation of the Riemann sphere. More precisely, the spherical preschwarzian for f identically vanishes if and only if $f \in \mathrm{Rot}\big(\widehat{\mathbb{C}}\big)$ (Theorem 1). The boundedness on the spherical preschwarzian leads to a two-point distortion theorem for locally injective meromorphic functions defined on a spherically starlike domain (Theorem 2). Section 3 is devoted to study the main properties of the differential operator $A_f^{\#}$. The boundedness on this operator is equivalent to a two-point distortion condition on the function f (Theorem 3). Section 4 is the main section of this paper. Here we study the relation between Schwarzian, normal and spherical orders. In particular, we extend to locally injective meromorphic functions on $\mathbb{D}$ an inequality due to Wirths [15] for spherically convex functions (Theorem 4). The result of Ma and Minda [8, Theorem 8] mentioned above is the key ingredient in the proof. As a consequence, every locally injective meromorphic function on $\mathbb{D}$ with finite s-order has finite Schwarzian and normal orders. For any hyperbolic domain Ω in $\mathbb{C}$ these orders are given, respectively, by

$$\|S_f\|_\Omega = \sup_{z\in\Omega} \frac{1}{\lambda_\Omega(z)^2}|S_f(z)|, \qquad \|N_f\|_\Omega = \sup_{z\in\Omega} \frac{1}{\lambda_\Omega(z)}|f^{\#}(z)|, \tag{2}$$

where

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of f. It turns out (by a result of Pommerenke) that if f is any locally injective meromorphic function with finite s-order on a hyperbolic domain Ω, then the boundary of Ω, $\partial\Omega$, must be uniformly perfect (Theorem 4) and $\partial\Omega$ is uniformly perfect if and only if every holomorphic covering from $\mathbb{D}$ onto Ω has finite s-order (Theorem 5). In virtue of [14, Theorem 1], the finiteness on the s-order for a covering is equivalent to the finiteness on the Schwarzian order. Once we know that a mapping has finite s-order we can give an estimate in terms of the Schwarzian and normal orders. This is the content of Theorem 6. In Section 5 we provide what we think is a basic example for a s-invariant family with finite s-order; namely, the family of all locally injective meromorphic functions f on $\mathbb{D}$ with the property that $f(\mathbb{D})$ has no antipodal points and the Schwarzian order for f is finite. If, in addition, f is univalent, the s-order for f is less or equal than 2. The value 2 is sharp. This particular family was studied by Kühnau [5]. Probably the Kühnau class plays within this context the role of the entire family of univalent analytic functions on $\mathbb{D}$ in the euclidean setting. This last family has infinite s-order (Proposition 2).

2. Spherical Preschwarzian

Our approach to spherically invariant families uses the following differential operator:

Definition 1. Let Ω be a domain in $\mathbb{C}$ and let $f : \Omega \to \widehat{\mathbb{C}}$ be a locally injective meromorphic function. We define the *spherical preschwarzian* of f by

$$T_f(z) = \frac{\partial}{\partial z} \log \frac{\big(1+|z|^2\big)|f'(z)|}{1+|f(z)|^2} = \frac{1}{2}\frac{f''(z)}{f'(z)} - \frac{f'(z)\overline{f(z)}}{1+|f(z)|^2} + \frac{\overline{z}}{1+|z|^2}. \tag{3}$$

It is easy to see that $T_f = T_{1/f}$ so that T_f is defined and continuous at the poles of f. The following two properties for T_f can be checked with no difficulty.

(i) If $f \in \operatorname{Rot}\big(\widehat{\mathbb{C}}\big)$ then $T_f \equiv 0$.

(ii) (*Chain rule*) Let g be a locally injective meromorphic function defined on a domain $\Omega \subset \mathbb{C}$ and let f be a locally injective meromorphic function defined on $g(\Omega)$. Then

$$T_{f\circ g} = (T_f \circ g)g' + T_g. \tag{4}$$

We will see shortly that T_f identically vanishes only when f is a rotation of the Riemann sphere. Therefore it follows that

$$T_{f\circ g} = T_g \quad \text{if and only if} \quad f \in \operatorname{Rot}\big(\widehat{\mathbb{C}}\big). \tag{5}$$

Also, if $g \in \operatorname{Rot}\big(\widehat{\mathbb{C}}\big)$, then $T_{f\circ g} = T_f(g)g'$. In particular, $T_{f\circ 1/z}(z) = -\frac{1}{z^2}T_f(1/z)$. In the case $\infty \in \Omega$ this equality allows to define T_f at infinity in a continuous manner by setting $T_f(\infty) = 0$.

Theorem 1. *Let f be a locally injective meromorphic function on a domain $\Omega \subset \widehat{\mathbb{C}}$. Then $T_f \equiv 0$ if and only if $f \in \operatorname{Rot}\big(\widehat{\mathbb{C}}\big)$.*

We may say then that the sup-norm of T_f measures the deviation of f from a rotation of the Riemann sphere. It is also interesting to point out that, unlike the euclidean preschwarzian, T_f is meromorphic only when $f \in \operatorname{Rot}\big(\widehat{\mathbb{C}}\big)$.

Proof. The sufficiency is the above property (i). For the necessity we may assume, in virtue of properties (i) and (ii) and suitable rotations of $\widehat{\mathbb{C}}$, that $0 \in \Omega$ and f has the expansion

$$f(z) = \alpha z + a_2 z^2 + a_3 z^3 + \cdots, \quad \alpha > 0,$$

at the origin. So, we must show that $f(z) = z$. The hypothesis gives immediately $\frac{1}{\alpha}a_2 = T_f(0) = 0$ and therefore, after some calculations

$$T_f(z) = \frac{3a_3}{\alpha}z + \big(1-\alpha^2\big)\overline{z} + O\big(|z|^2\big)$$

near the origin. Now we equate this expression to zero, divide by z and let $z \to 0$ to obtain $a_3 = 0$ and $\alpha = 1$. Next, assume inductively that $a_4 = a_5 = \cdots = a_k = 0$ and let us show that $a_{k+1} = 0$. The calculations now give

$$T_f(z) = \frac{k(k+1)a_{k+1}}{2} z^{k-1} + O\big(|z^{k+1}|\big)$$

near the origin. Similarly as above, we equate this expression to zero, divide by z^{k-1} and let $z \to 0$ to obtain $a_{k+1} = 0$. This finishes the proof. ☑

We recall that the *spherical distance* between $a, b \in \widehat{\mathbb{C}}$ is defined by

$$d^{\#}(a,b) := \inf \int_{\gamma} \frac{|dz|}{1+|z|^2},$$

where the infimum is taken over all paths γ on $\widehat{\mathbb{C}}$ which join a and b.

A domain $\Omega \subset \widehat{\mathbb{C}}$ is called *spherically starlike* with respect to $w_0 \in \Omega$ if, for any point $w \in \Omega$, the smaller arc of the greatest circle (spherical geodesic) between w_0 and w also lies in Ω.

Theorem 2. *Suppose $\Omega \subset \widehat{\mathbb{C}}$ is a domain spherically starlike with respect to a point $z_0 \in \Omega$ and let f be a locally injective meromorphic function on Ω.*

(i) If $|T_f(z)| \leq \alpha < \infty$, then

$$\exp\big(-2\alpha \tan d^{\#}(z_0,z)\big) \leq \frac{\big(1+|z|^2\big)f^{\#}(z)}{\big(1+|z_0|^2\big)f^{\#}(z_0)} \leq \exp\big(2\alpha \tan d^{\#}(z_0,z)\big).$$

(ii) If the previous inequalities holds, then $|T_f(z_0)| \leq \alpha$.

In particular, if Ω is spherically convex then, for all $z \in \Omega$, $|T_f(z)| \leq \alpha$ if and only if

$$\exp\big(-2\alpha \tan d^{\#}(w,z)\big) \leq \frac{\big(1+|z|^2\big)f^{\#}(z)}{\big(1+|w|^2\big)f^{\#}(w)} \leq \exp\big(2\alpha \tan d^{\#}(w,z)\big)$$

for all $z, w \in \Omega$.

Proof. The geometric condition on Ω and the quantities involved remain invariant under composition with rotations of the Riemann sphere. Therefore we may assume that $z_0 = 0$ and Ω is euclidean starlike with respect to the origin. To prove (i) we must then show the inequalities

$$\exp(-2\alpha|z|) \leq \big(1+|z|^2\big)\frac{f^{\#}(z)}{f^{\#}(0)} \leq \exp(2\alpha\,|z|).$$

Fix $z \in \Omega$ and let γ be the radial segment from 0 to z parameterized by euclidean arc length. Set $h(s) = \big(1 + |\gamma(s)|^2\big) f^{\#}\big(\gamma(s)\big)$. Then

$$\frac{d}{ds} \log h(s) = \operatorname{Re}\left[\frac{d}{ds} \log \big(1 + |\gamma(s)|^2\big) f^{\#}\big(\gamma(s)\big)\right] = 2\operatorname{Re}\big[T_f\big(\gamma(s)\big)\dot{\gamma}(s)\big].$$

By hypothesis we obtain

$$\left|\frac{d}{ds} \log h(s)\right| \leq 2\alpha,$$

or equivalently

$$-2\alpha \leq \frac{d}{ds} \log h(s) \leq 2\alpha.$$

Now we integrate this inequality with respect to s over γ to get

$$-2\alpha|z| \leq \log \frac{h(|z|)}{h(0)} \leq 2\alpha|z|.$$

This proves part (i). To complete the proof of the theorem, set $u(z) = \log\big[\big(1 + |z|^2\big) f^{\#}(z)\big]$. The inequalities in (i) give

$$\frac{|u(z) - u(0)|}{|z|} \leq 2\alpha.$$

Now let z approach 0 in the direction of maximum growth of u at the origin, to get $|2T_f(0)| = |\nabla u(0)| \leq 2\alpha$. ☑

Corollary 3. *Suppose $\Omega \subset \widehat{\mathbb{C}}$ is a domain spherically starlike with respect to a point $z_0 \in \Omega$ and let f be a locally injective meromorphic function on Ω with $|T_f(z)| \leq \alpha$ for every z. Then, for all $z \in \Omega$*

$$d^{\#}\big(f(z), f(z_0)\big) \leq \big(1 + |z_0|^2\big) f^{\#}(z_0) \exp\big(2\alpha \tan\, d^{\#}(z_0, z)\big) d^{\#}(z_0, z).$$

If, in addition, f is univalent and $f(\Omega)$ is spherically starlike with respect to $f(z_0)$ then

$$\big(1 + |z_0|^2\big) f^{\#}(z_0) \exp\big(-2\alpha \tan d^{\#}(z_0, z)\big) d^{\#}(z_0, z) \leq d^{\#}\big(f(z), f(z_0)\big).$$

Proof. Fix the point z in Ω and let γ be the spherical geodesic joining z and z_0. It follows from the previous theorem

$$\begin{aligned} d^{\#}&\big(f(z), f(z_0)\big) \\ &\leq \int_\gamma f^{\#}(\zeta)|d\zeta| \leq \big(1 + |z_0|^2\big) f^{\#}(z_0) \int_\gamma \frac{\exp\big(2\alpha \tan d^{\#}(z_0, \zeta)\big)}{1 + |\zeta|^2} |d\zeta| \\ &\leq \big(1 + |z_0|^2\big) f^{\#}(z_0) \exp\big(2\alpha \tan d^{\#}(z_0, z)\big) d^{\#}(z_0, z) \end{aligned}$$

since $d^{\#}(z_0,\zeta) \le d^{\#}(z_0,z)$ on γ. This establishes the first inequality. Now suppose, in addition, that f is univalent and spherically starlike with respect to $f(z_0)$. Let σ be the spherical geodesic joining $f(z)$ and $f(z_0)$. Set $\delta = f^{-1}\circ\sigma$. Due to the previous theorem the following inequalities hold

$$\begin{aligned} & d^{\#}\big(f(z),f(z_0)\big) \\ &\quad = \int_{\delta} f^{\#}(\zeta)|d\zeta| \ge (1+|z_0|^2)f^{\#}(z_0)\int_{\delta}\frac{\exp(-2\alpha\tan d^{\#}(z_0,\zeta))}{1+|\zeta|^2}|d\zeta| \\ &\qquad\qquad \ge \big(1+|z_0|^2\big)f^{\#}(z_0)\exp\big(-2\alpha\tan d^{\#}(z_0,z)\big)d^{\#}(z_0,z). \quad ☑ \end{aligned}$$

3. The Differential Operator $A_f^{\#}$

When dealing with an arbitrary hyperbolic plane domain Ω it is more convenient to use the differential operator

$$A_f^{\#}(z) = \frac{1}{\lambda_\Omega(z)}\frac{\partial}{\partial z}\left(\log\frac{f^{\#}}{\lambda_\Omega}\right)(z)$$

for a locally injective meromorphic function f on Ω.

Note that

$$\lambda_\Omega(z)A_f^{\#}(z) = T_f(z) - \frac{\partial}{\partial z}\big(\log\big(1+|z|^2\big)\lambda_\Omega(z)\big).$$

From this formula and (4) we compute the composition rule for $A_f^{\#}$, namely,

$$A_{f\circ g}^{\#} = \frac{1}{\lambda_G}\big(T_f\circ g\big)g' + A_g^{\#}, \tag{6}$$

where g is a locally injective holomorphic function on a hyperbolic plane domain G with $g(G)\subset\Omega$. Since g' is never zero we see from (5) that $A_{f\circ g}^{\#} = A_g^{\#}$ if and only if $f\in \mathrm{Rot}\big(\widehat{\mathbb{C}}\big)$. Straightforward computations also show that if g is a holomorphic covering of G onto Ω then

$$A_{f\circ g}^{\#} = \big(A_f^{\#}\circ g\big)\frac{g'}{|g'|}. \tag{7}$$

As a consequence of (6) and (7) we have the invariance property of the spherical order

$$\big\|A_{\sigma\circ f\circ g}^{\#}\big\|_\Omega = \big\|A_f^{\#}\big\|_\Omega$$

for any conformal automorphism g of Ω and any $\sigma\in\mathrm{Rot}\big(\widehat{\mathbb{C}}\big)$. The following theorem is analogous to Theorem 2 and the proof are similar.

We recall that the *hyperbolic distance* between two points a, b in a hyperbolic domain Ω is given by

$$d_h(a,b) := \inf \int_\gamma \lambda_\Omega(z)|dz|,$$

where the infimum is taken over all paths γ on Ω which join a and b.

Theorem 4. *Suppose Ω is a hyperbolic domain in $\mathbb{C}$. Let $f : \Omega \to \widehat{\mathbb{C}}$ be a locally injective meromorphic function. Let α be a positive number. Then $|A_f^{\#}(z)| \leq \alpha$ if and only if for every pair of points $z_0, z \in \Omega$ we have*

$$\exp\big(-2\alpha d_h(z_0,z)\big) \leq \frac{f^{\#}(z)\lambda_\Omega(z_0)}{f^{\#}(z_0)\lambda_\Omega(z)} \leq \exp\big(2\alpha d_h(z_0,z)\big). \tag{8}$$

Proof. Fix $z_0 \in \Omega$ and let $z \in \Omega$. Let γ be a hyperbolic geodesic in Ω from z_0 to z. Parametrize γ by hyperbolic arc-length, that is, $\gamma'(s) = e^{i\theta(s)}/\lambda_\Omega\big(\gamma(s)\big)$, where s is the hyperbolic length of γ from z_0 to z. Set $F(z) = \dfrac{f^{\#}(z)}{\lambda_\Omega(z)}$ and $h(s) = F(\gamma(s))$. It follows that

$$h'(s) = 2\mathrm{Re}\left[\frac{\partial F\big(\gamma(s)\big)}{\partial z}\gamma'(s)\right] = 2\frac{f^{\#}\big(\gamma(s)\big)}{\lambda_\Omega\big(\gamma(s)\big)}\mathrm{Re}\left[\frac{1}{f^{\#}\big(\gamma(s)\big)}\frac{\partial}{\partial z}\frac{f^{\#}\big(\gamma(s)\big)}{\lambda_\Omega\big(\gamma(s)\big)}e^{i\theta(s)}\right].$$

Thus,

$$\frac{d}{ds}\log h(s) = \frac{h'(s)}{h(s)} = 2\mathrm{Re}\Big[A_f^{\#}\big(\gamma(s)\big)e^{i\theta(s)}\Big],$$

and this implies, by hypothesis,

$$-2\alpha \leq \frac{d}{ds}\log h(s) \leq 2\alpha.$$

Integrating between 0 and s we obtain

$$-2\alpha d_h(z_0,z) \leq \log\frac{f^{\#}(z)\lambda_\Omega(z_0)}{f^{\#}(z_0)\lambda_\Omega(z)} \leq 2\alpha d_h(z_0,z).$$

For the converse, we set $u(z) = \log\dfrac{f^{\#}(z)}{\lambda_\Omega(z)}$. Then by (8),

$$|u(z) - u(z_0)| \leq 2\alpha d_h(z_0,z),$$

for $z_0, z \in \Omega$. The rest of the proof follows the same argument as in part (ii) of Theorem 2. ☑

4. Schwarzian, Spherical and Normal Orders

Suppose $f : \mathbb{D} \to \widehat{\mathbb{C}}$ is meromorphic and locally univalent. For each $a \in \mathbb{D}$ let $\rho(a, f)$ be the hyperbolic radius of the largest hyperbolic disk in $\mathbb{D}$, centered at a, in which f is univalent. We also define

$$\rho_s(a, f) = \sup\big\{\rho \le \rho(a, f) \mid f\big(D_h(a, \rho)\big),\ \text{is s-convex}\big\}$$

and

$$\rho_s(f) = \inf\big\{\rho_s(a, f) \mid a \in \mathbb{D}\big\}.$$

Ma and Minda [8, Theorem 8] proved

Proposition 5. *Let $f : \mathbb{D} \to \widehat{\mathbb{C}}$ be meromophic and locally univalent. Then $\rho_s(f) = \rho$ if and only if $\|A_f^{\#}\|_{\mathbb{D}} = \coth(2\rho)$.*

This result can be used to extend a result of Wirths [15]; see also [10].

We recall that a compact set A in $\widehat{\mathbb{C}}$ is called *uniformly perfect* if the doubly connected domains in $\widehat{\mathbb{C}} \setminus A$ that separate A have bounded moduli [14, p. 299].

Theorem 6. *Let f be meromophic and locally univalent on a hyperbolic domain Ω. If $\|A_f^{\#}\|_{\Omega} = \alpha$, then $\partial\Omega$ is uniformly perfect and*

$$\frac{1}{2}\frac{\big|S_f(z)\big|}{\lambda_\Omega^2(z)} + \big|A_f^{\#}(z)\big|^2 + \frac{\big[f^{\#}(z)\big]^2}{\lambda_\Omega^2(z)} \le \Big(\alpha + \sqrt{\alpha^2 - 1}\Big)^2 + \frac{1}{2}\|S_\pi\|_{\mathbb{D}}, \qquad (9)$$

where π is a universal covering map of $\mathbb{D}$ onto Ω.

The case $\alpha = 1$ and $\Omega = \mathbb{D}$ is the inequality due to Wirths that characterizes spherically convex conformal maps in the unit disk. Note that the term $\frac{1}{2}\|S_\pi\|_{\mathbb{D}}$ is zero if Ω is the unit disk.

Proof. Let π be any universal covering of $\mathbb{D}$ onto Ω. Then by (7) $g = f \circ \pi$ satisfies $\big\|A_g^{\#}\big\|_{\mathbb{D}} = \alpha$. Thus, by Proposition 5, g is uniformly locally spherically convex on any hyperbolic disk $D_h(z, \rho)$, with $\rho = \frac{1}{2}\coth^{-1}\alpha$ and this implies that π is univalent on that hyperbolic disk. It follows by a result of Pommerenke [13, Corollary 2] that $\partial\Omega$ is uniformly perfect.

Next, we show

$$\frac{1}{2}\frac{|S_g(z)|}{\lambda_{\mathbb{D}}^2(z)} + \big|A_g^{\#}(z)\big|^2 + \frac{\big[g^{\#}(z)\big]^2}{\lambda_{\mathbb{D}}^2(z)} \le \Big(\alpha + \sqrt{\alpha^2 - 1}\Big)^2 \qquad (10)$$

for all $z \in \mathbb{D}$. Fix $a \in \mathbb{D}$. By Proposition 5, $g\big(D_h(a, \rho)\big)$ is s-convex where $\rho = \frac{1}{2}\coth^{-1}\alpha$. Set $\phi(z) = \dfrac{Rz + a}{1 + \overline{a}Rz}$, where $R = \tanh\rho$. Then $h = g \circ \phi$ is s-convex in $\mathbb{D}$. Wirths' inequality states that

$$\frac{\big(1 - |z|^2\big)^2 |S_h(z)|}{2} + \big|A_h^{\#}(z)\big|^2 + \big(1 - |z|^2\big)^2\big[h^{\#}(z)\big]^2 \le 1, \qquad (11)$$

for all $z \in \mathbb{D}$. The calculations show that

$$S_h(0) = R^2\big(1-|a|^2\big)S_g(a),$$
$$A_h^{\#}(0) = RA_g^{\#}(a)$$

and

$$h^{\#}(0) = R\big(1-|a|^2\big)g^{\#}(a).$$

Evaluating (11) at $z=0$ we obtain

$$\begin{aligned}\frac{\big(1-|a|^2\big)^2|S_g(a)|}{2} + \big|A_g^{\#}(a)\big|^2 + \big(1-|a|^2\big)^2\big[g^{\#}(a)\big]^2 &\le \frac{1}{R^2}\\ &= \big(\alpha+\sqrt{\alpha^2-1}\big)^2,\end{aligned}$$

which proves (10). Now, since $g = f\circ\pi$, we have

$$\begin{aligned}\frac{1}{2}\frac{\big|S_f\big(\pi(z)\big)\big|}{\lambda_\Omega^2(\pi(z))} + \big|A_f^{\#}\big(\pi(z)\big)\big|^2 + \frac{\big[f^{\#}\big(\pi(z)\big)\big]^2}{\lambda_\Omega^2\big(\pi(z)\big)}&\\ \le \Big(\alpha+\sqrt{\alpha^2-1}\Big)^2 &+ \frac{|S_\pi(z)|}{2\lambda_\Omega^2(\pi(z))|\pi'(z)|^2}\\ = \Big(\alpha+\sqrt{\alpha^2-1}\Big)^2 &+ \frac{|S_\pi(z)|}{2\lambda_{\mathbb{D}}^2(z)}\\ \le \Big(\alpha+\sqrt{\alpha^2-1}\Big)^2 &+ \frac{1}{2}\|S_\pi\|_{\mathbb{D}}. \qquad ☑\end{aligned}$$

Remark. The previous theorem shows in particular that every locally injective meromorphic functions on $\mathbb{D}$ with spherical order $\alpha \ge 1$ is normal, and has normal and Schwarzian orders at most $\alpha+\sqrt{\alpha^2-1}$ and $2(\alpha+\sqrt{\alpha^2-1})^2$, respectively. Lappan [7] proved the existence of a non normal locally univalent holomorphic function in $\mathbb{D}$ with finite Schwarzian order. It follows from the previous theorem that the finiteness on the Schwarzian order does not imply the finiteness on the s-order. However, we will see (Example 2), that the s-order of a locally injective meromorphic function on $\mathbb{D}$ is finite provided the Schwarzian order is finite and its range contains no antipodal points. On the other hand there are locally injective normal meromorphic functions on $\mathbb{D}$ with infinite s-order [8, p. 163]. Mejía [9] proved that if f is a locally injective meromorphic function in $\mathbb{D}$ with finite Schwarzian order and $A_f^{\#}(z) \ne 0$ for all $z \in \mathbb{D}$, then f is normal.

Theorem 7. *Suppose Ω is a hyperbolic domain in $\mathbb{C}$. $\partial\Omega$ is uniformly perfect if only if $\big\|A_\pi^{\#}\big\|_{\mathbb{D}} < \infty$ for any holomorphic universal covering map π from $\mathbb{D}$ onto Ω. In particular, any univalent holomorphic function on Ω has finite s-order.*

Proof. If $\|A_\pi^\#\|_{\mathbb{D}} = \alpha$, Proposition 5 gives π is spherically convex on any hyperbolic disk $D_h(z,\rho)$, with $\rho = \frac{1}{2}\coth^{-1}\alpha$ and this implies that π is univalent on such disk. Then, as before [13, Corollary 2], $\partial\Omega$ is uniformly perfect. Next, suppose that $\partial\Omega$ is uniformly perfect and let π be any holomorphic universal covering map from $\mathbb{D}$ onto Ω. We have

$$\begin{aligned}
\big|A_\pi^\#(z)\big| &= \left| -\overline{z} + \frac{1}{2}\big(1-|z|^2\big)\frac{\pi''(z)}{\pi'(z)} + \big(1-|z|^2\big)\frac{\pi'(z)\overline{\pi(z)}}{1+|\pi(z)|^2}\right| \\
&\le 1 + \frac{1}{2}\big(1-|z|^2\big)\left|\frac{\pi''(z)}{\pi'(z)}\right| + \big(1-|z|^2\big)|\pi'(z)|\frac{|\pi(z)|}{1+|\pi(z)|^2} \\
&= 1 + \frac{1}{2}\big(1-|z|^2\big)\left|\frac{\pi''(z)}{\pi'(z)}\right| + \frac{1}{\lambda_\Omega\big(\pi(z)\big)}\frac{|\pi(z)|}{1+|\pi(z)|^2}.
\end{aligned}$$

Since $\partial\Omega$ is uniformly perfect there exists a constant $c>0$ such that

$$\big(1-|z|^2\big)\left|\frac{\pi''(z)}{\pi'(z)}\right| \le c \qquad \text{and} \qquad \frac{1}{\lambda_\Omega\big(\pi(z)\big)} \le c\delta_\Omega\big(\pi(z)\big)$$

where $\delta_\Omega\big(\pi(z)\big)$ denotes the euclidean distance from $\pi(z)$ to $\partial\Omega$ [13, Corollary 1], [1, p. 478]. Hence

$$\big|A_\pi^\#(z)\big| \le 1 + \frac{c}{2} + c\delta_\Omega\big(\pi(z)\big)\frac{|\pi(z)|}{1+|\pi(z)|^2}.$$

Now, for any $w \in \mathbb{C}\setminus\Omega$ the last inequality implies

$$\begin{aligned}
\big|A_\pi^\#(z)\big| &\le 1 + \frac{c}{2} + c|\pi(z)-w|\frac{|\pi(z)|}{1+|\pi(z)|^2} \\
&\le 1 + \frac{c}{2} + c\frac{|\pi(z)|^2 + |w||\pi(z)|}{1+|\pi(z)|^2} \\
&\le 1 + \frac{c}{2} + c\left(1 + \frac{|w|}{2}\right) < \infty.
\end{aligned}$$ ☑

Corollary 8. *Suppose Ω is a hyperbolic domain in $\mathbb{C}$ and π is any holomorphic universal covering map from $\mathbb{D}$ onto Ω. Then $\big\|A_\pi^\#\big\|_{\mathbb{D}} < \infty$ if only if $\|S_\pi\|_{\mathbb{D}} < \infty$.*

Proof. This is a consequence of the previous Theorem and a result of Pommerenke [14, Theorem 1]. ☑

The next results provides an estimate of the spherical order in terms of the normal and Schwarzian orders defined in (2).

Theorem 9. *Suppose Ω is a hyperbolic domain in $\mathbb{C}$ and π is any holomorphic universal covering map from $\mathbb{D}$ onto Ω. If $f:\Omega\to\widehat{\mathbb{C}}$ is a locally injective meromophic function with finite s-order, then*

$$\big\|A_f^{\#}\big\|_\Omega^2 \leq 1+\frac{1}{2}\|S_f\|_\Omega+\|N_f\|_\Omega^2+\frac{1}{2}\|S_\pi\|_{\mathbb{D}}. \tag{12}$$

Proof. To show (12) it suffices to consider the case $\Omega=\mathbb{D}$ and show that

$$\big\|A_f^{\#}\big\|_{\mathbb{D}}^2 \leq 1+\frac{1}{2}\|S_f\|_{\mathbb{D}}+\|N_f\|_{\mathbb{D}}^2, \tag{13}$$

since in this case $\|S_\pi\|_{\mathbb{D}}=0$. Fix $0<r<1$ and let $g(z)=f(rz)$, $z\in\overline{\mathbb{D}}$. Let us see initially that g satisfies (13). If the maximum value of $\big|A_g^{\#}(z)\big|$ is reached at $\partial\mathbb{D}$, then by (1), $\big|A_g^{\#}(z)\big|=|z|$ and so $\big\|A_g^{\#}\big\|_{\mathbb{D}}=1$; that is, g is spherically convex and satisfies (13). Suppose g is not spherically convex and the maximum is reached in $\mathbb{D}$. Since all terms in (13) are invariant if we change g by $\varphi\circ g\circ\psi$, with $\varphi\in\operatorname{Rot}\big(\widehat{\mathbb{C}}\big)$ and $\psi\in\text{Möb}(\mathbb{D})$ we may suppose that g has the form $g(z)=\beta\big(z+a_2z^2+a_3z^3+\cdots\big)$ and $\max\limits_{z\in\mathbb{D}}\big|A_g^{\#}(z)\big|=\big|A_g^{\#}(0)\big|$. In these circumstances, to show (13) for g, it suffices to prove

$$\big|A_g^{\#}(0)\big|^2\leq\frac{1}{2}|S_g(0)|+1+\big[g^{\#}(0)\big]^2. \tag{14}$$

Straightforward computations give $A_g^{\#}(z)=a_2+\big(3a_3-2a_2^2\big)z-\big(1+\beta^2\big)\overline{z}+O\big(|z|^2\big)$ as $z\to 0$. Then, near the origin

$$\begin{aligned}\big|A_g^{\#}(z)\big|^2&=\big|A_g^{\#}(0)\big|^2+\\&\quad 2\operatorname{Re}\left[\overline{A_g^{\#}(0)}\left(\frac{S_g(0)}{2}+A_g^{\#}(0)^2\right)-\big(1+\beta^2\big)A_g^{\#}(0)\right]z+O\big(|z|^2\big)\\&\leq\big|A_g^{\#}(0)\big|^2.\end{aligned}$$

Set $B=\overline{A_g^{\#}(0)}\Big(\frac{S_g(0)}{2}+A_g^{\#}(0)^2\Big)-\big(1+\beta^2\big)A_g^{\#}(0)$ and $z=re^{i\theta}$, $\theta\in\mathbb{R}$. We obtain $2\operatorname{Re}\big(Be^{i\theta}\big)+O(r)\leq 0$ and as $r\to 0^+$ then $\operatorname{Re}\big(Be^{i\theta}\big)\leq 0$. By choosing $\theta=-\arg B$ we get $|B|\leq 0$. Hence $\overline{A_g^{\#}(0)}\left(\dfrac{S_g(0)}{2}+A_g^{\#}(0)^2\right)-\big(1+\beta^2\big)A_g^{\#}(0)=0$. Therefore

$$\big|A_g^{\#}(0)\big|^2=\left|\frac{S_g(0)}{2}-\big(1+\beta^2\big)\right|\leq\frac{1}{2}|S_g(0)|+1+\big[g^{\#}(0)\big]^2,$$

since $\beta=g^{\#}(0)$. This proves (13) for g. Now, since $g(z)=f(rz)$, $0<r<1$, it follows from (7) and (13), with $w=rz$, that

$$\big|A_f^{\#}(w)\big|^2\leq 1+\frac{1}{2}\|S_g\|_{\mathbb{D}}+\|N_g\|_{\mathbb{D}}^2\leq 1+\frac{1}{2}\big\|S_f\big\|_{\mathbb{D}}+\|N_f\|_{\mathbb{D}}^2$$

and taking $r \to 1^-$ we obtain

$$\left|A_f^{\#}(z)\right|^2 \le 1 + \frac{1}{2}\left\|S_f\right\|_{\mathbb{D}} + \|N_f\|_{\mathbb{D}}^2,$$

for $z \in \mathbb{D}$, which ends the proof. ☑

5. Examples

Example 1. Let Ω be a proper simply connected domain in $\mathbb{C}$. Consider the family $\mathcal{K}_\Omega$ of univalent meromorphic functions $f : \Omega \to \widehat{\mathbb{C}}$ such that $f(\Omega)$ is spherically convex. Then $\mathcal{O}_s(\mathcal{K}_\Omega) = 1$ [8, Theorems 4 and 7].

Example 2. Consider the family $\mathcal{G}_{\mathbb{D}}(\beta)$ of locally univalent meromorphic functions $f : \mathbb{D} \to \widehat{\mathbb{C}}$ such that $f(\mathbb{D})$ contains no antipodal points and $\|S_f\|_{\mathbb{D}} \le \beta$. Notice that the family $\mathcal{K}_{\mathbb{D}}$ is contained in $\mathcal{G}_{\mathbb{D}}(2)$. We will show that

$$\mathcal{O}_s\big(\mathcal{G}_{\mathbb{D}}(\beta)\big) \le \sqrt{\frac{\beta+2}{2}}.$$

Fix $z_0 \in \mathbb{D}$. Choose $\sigma \in \mathrm{Rot}\big(\widehat{\mathbb{C}}\big)$ and $\tau \in \text{Möb}(\mathbb{D})$ such that $\tau(0) = z_0$ and $\sigma\big(f(z_0)\big) = 0$. The function $g = \sigma \circ f \circ \tau$ maps $\mathbb{D}$ onto a region with no antipodals points. Since $g(0) = 0$, then g has no poles and is locally univalent in $\mathbb{D}$. Moreover $\|S_g\|_{\mathbb{D}} = \|S_f\|_{\mathbb{D}}$. Therefore, by a result of Pommerenke [11, Folgerung 2.3], we have

$$\left|A_f^{\#}(z_0)\right| = \frac{1}{2}\left|\frac{g''(0)}{g'(0)}\right| \le \sqrt{\frac{\beta+2}{2}}.$$

Kühnau [5], considered the family $\mathcal{Q}_{\mathbb{D}}$ of univalent meromorphic functions $f : \mathbb{D} \to \widehat{\mathbb{C}}$ such that $f(\mathbb{D})$ contains no antipodal. Notice that $\mathcal{Q}_{\mathbb{D}} \subset \mathcal{G}_{\mathbb{D}}(6)$. Moreover $\mathcal{O}_s(\mathcal{Q}_{\mathbb{D}}) = 2$. To prove that the constant 2 is sharp, let $0 \le p < 1$ and $h(z) = az + \cdots$, $a > 0$, the map of $\mathbb{D}$ onto $\mathbb{D}\setminus(-1, -p]$. Then (see [12, p. 100]),

$$\frac{h(z)}{\big(1-h(z)\big)^2} = \frac{4p}{(1+p)^2}\frac{z}{(1-z)^2}$$

and thus

$$h(z) = az + 2a(1-a)z^2 + \cdots, \qquad a = \frac{4p}{(1+p)^2}.$$

It is clear that $h \in \mathcal{Q}_{\mathbb{D}}$. We have the inequality

$$2 \ge \left\|A_h^{\#}\right\|_{\mathbb{D}} \ge \left|A_h^{\#}(0)\right| = \frac{1}{2}\left|\frac{h''(0)}{h'(0)}\right| = 2(1-a) = 2\left(\frac{1-p}{1+p}\right)^2 \longrightarrow 2$$

as $p \to 0$.

We notice that the spherical area of the image domain of any function $f : \Omega \to \widehat{\mathbb{C}}$ that contain no antipodal points, is no greater than 2π. With this in mind we have the following example.

Example 3. Let Ω be a proper simply connected domain in $\mathbb{C}$ and $0 < \beta < 4\pi$. We define the family $\mathcal{F}_\Omega(\beta)$ of univalent meromorphic functions $f : \Omega \to \widehat{\mathbb{C}}$ such that the spherical area of $f(\Omega)$ is no greater than β. We will show that

$$\mathcal{O}_s\big(\mathcal{F}_\Omega(\beta)\big) \le \sqrt{4 + \frac{\beta}{4\pi - \beta}}. \tag{15}$$

By invariance of the s-order under a conformal mapping we may assume that $\Omega = \mathbb{D}$. A result of Dufresnoy [2] (see also [4, Theorem 6.4, p.162]) gives the inequality

$$\big(1 - |z|^2\big) f^{\#}(z) \le \sqrt{\frac{\beta}{4\pi - \beta}}.$$

This inequality together with the bound 6 on the Schwarzian order for univalent functions [6] and Theorem 9 yields the inequality $\big\|A_f^{\#}\big\|_{\mathbb{D}}^2 \le 4 + \beta/(4\pi - \beta)$, which proves (15). In particular, suppose $\sigma \in \operatorname{Rot}\big(\widehat{\mathbb{C}}\big)$ and let $\mathcal{F}(\mathbb{D}, \sigma)$ be the family of univalent meromorphic functions f in $\mathbb{D}$, such that $f(\mathbb{D}) \cap \sigma(f(\mathbb{D}))$ or $f(\mathbb{D}) \cap \overline{\sigma}(f(\mathbb{D}))$ is an empty set. Then the spherical area of $f(\mathbb{D})$ is no greater than 2π. Hence, by (15), $\mathcal{O}_s(\mathcal{F}(\mathbb{D}, \sigma)) \le \sqrt{5}$. Important examples, besides the Kühnau's functions of Example 2, are Bieberbach-Eilenberg functions and Gelfer functions (see [3, p. 265]), where $\sigma(z) = 1/z$ and $\sigma(z) = -z$, respectively.

We know that the euclidean order (in the sense of Pommerenke) for the family of univalent analytic functions in $\mathbb{D}$ is 2. By contrast, the s-order of this family is infinite. This follows from the simple example $f(z) = az$, $a > 1$. An easy calculation shows that $\big\|A_f^{\#}\big\|_{\mathbb{D}} = \big(1 + a^2\big)/(2a)$ which tends to infinity as $a \to \infty$. Indeed, we have the following proposition.

Proposition 10. *Let $\{f_n\}$ be a sequence of holomorphic univalent functions in $\mathbb{D}$ with $f_n(0) = 0$. Suppose $r_n = \inf\big\{|w| \mid w \in \mathbb{C} \setminus f_n(\mathbb{D})\big\}$. Then $\big\|A_{f_n}^{\#}\big\|_{\mathbb{D}} \to \infty$ if and only if $r_n \to \infty$.*

Proof. Set $D_n = f_n(\mathbb{D})$. Suppose first that $\big\|A_{f_n}^{\#}\big\|_{\mathbb{D}} \to \infty$. Arguing by contradiction let us assume that $r_n \nrightarrow \infty$, then there are M and a subsequence r_{n_k} such that $r_{n_k} \le M$, for all $k \in \mathbb{N}$. Thus, there is $w_{n_k} \notin D_{n_k}$ with $r_{n_k} \le |w_{n_k}| < M + 1$. Now, since f_{n_k} is univalent, $\left| \frac{\big(1 - |z|^2\big)}{2} \frac{f_{n_k}''(z)}{f_{n_k}'(z)} - \overline{z} \right| \le 2$ for all $z \in \mathbb{D}$. Also, by the Koebe one-quarter theorem

$$\big(1 - |z|^2\big)\big|f'(z)\big| \le 4 \operatorname{dist}(f(z), \partial D_{n_k}) \le 4|f(z) - w_{n_k}|,$$

for all $z \in \mathbb{D}$. Therefore (1) gives

$$\begin{aligned}\left|A^{\#}_{f_{n_k}}(z)\right| &\le 2 + \frac{4|f_{n_k}(z) - w_{n_k}||f_{n_k}(z)|}{1+|f_{n_k}(z)|^2}\\ &\le 2 + 4\left(\frac{|f_{n_k}(z)|^2}{1+|f_{n_k}(z)|^2} + \frac{|w_{n_k}||f_{n_k}(z)|}{1+|f_{n_k}(z)|^2}\right) \le 6 + 2|w_{n_k}|,\end{aligned}$$

for all $k \in \mathbb{N}$, which is a contradiction.

Conversely, suppose $r_n \to \infty$. We have

$$\left\|A^{\#}_{f_n}\right\|_{\mathbb{D}} \ge \frac{\big(1-|z|^2\big)|f_n'(z)||f_n(z)|}{1+|f_n(z)|^2} - \left|\frac{\big(1-|z|^2\big)}{2}\frac{f_n''(z)}{f_n'(z)} - \overline{z}\right|,$$

for all $z \in \mathbb{D}$. Now, as f_n is injective it follows from this inequality

$$\operatorname{dist}\big(f(z), \partial D_n\big) \le \big(\left\|A^{\#}_{f_n}\right\|_{\mathbb{D}} + 2\big)\frac{1+|f_n(z)|^2}{|f_n(z)|},$$

for all $z \in \mathbb{D}$. Note that as $r_n \to \infty$ and $f_n(0) = 0$, there exists $N \in \mathbb{N}$ such that $1 \in D(0, r_n) \subset D_n$, for all $n \ge N$, then there is $z_n \in \mathbb{D}$ such that $f_n(z_n) = 1$, for all $n \ge N$. Hence, for all $n \ge N$

$$r_n - 1 \le 2\big(\left\|A^{\#}_{f_n}\right\|_{\mathbb{D}} + 2\big).$$

Since $r_n \to \infty$ we conclude that $\left\|A^{\#}_{f_n}\right\|_{\mathbb{D}} \to \infty$. ☑

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