Revista Colombiana de Matemáticas
Volumen 45(2011)2, páginas 129-135

Weak Diameter and Cyclic Properties in Oriented Graphs

Diámetro débil y propiedades cíclicas en digrafos antisimétricos

Daniel Brito[1], Oscar Ordaz[2], María Teresa Varela[3]

[1]Universidad de Oriente, Cumaná, Venezuela

[2]Universidad Central de Venezuela, Caracas, Venezuela

[3]Universidad Simón Bolívar, Caracas, Venezuela

Abstract. We describe several conditions on the minimum number of arcs ensuring that any two vertices in a strong oriented graph are joining by a path of length at most a given k, or ensuring that they are contained in a common cycle.

Key words and phrases. Weak diameter, 2-Cyclic, Oriented graph.

2000 Mathematics Subject Classification. 05B10, 11B13.

Resumen. Damos varias condiciones sobre el número mínimo de arcos que implican la existencia, para todo par de vértices en un digrafo antisimétrico fuertemente conexo de un camino de longitud a lo más un k dado, que los une o de un circuito que los contiene.

Palabras y frases clave. Diamétro débil, 2-ciclíco, digrafo antisimétrico.

1. Terminology and Notations

We determine an upper bound for the length of the shortest path joining any two vertices with conditions involving connectivity and number of arcs, in strong oriented graphs. On the other hand we examine if a strong oriented graph is 2-cyclic, that is to say, if any two of its vertices belong to a common cycle, under similar hypotheses. More information on 2-cyclic properties can be found in [2, 5, 8, 9, 10, 12, 11].

The motivation for this paper is its relationship with the problem of hamiltonian tournaments. An open problem posed by Bermond and Lovász that we also attempt to approach is the following: does there exist a natural number k such that every k-strongly connected oriented graph D is 2-cyclic? (see [5]). In [1], Bang-Jansen gave a nice and excellent survey on problems and conjectures in tournaments.

The main notion used here is a new form of diameter, which is useful as a tool even if its definition does not seem to represent a parameter with a direct applicability. Our concept is related to the classical notion of diameter of oriented graphs [4, 6]

We use standard terminology [2, 7]. *An oriented graph*, $D = (V(D), E(D))$, is an oriented graph without loops, multiple arcs or circuits of length two. An arc with origin x and end y is denoted by xy. If both xy and yx do not exist, we shall say that the edge (x, y) is missing.

A (x_1, x_l)-path $x_1, x_2, \dots, x_l$ of length $l-1$, is an oriented graph with vertex set $\{x_1, x_2, \dots, x_l\}$ and arc set $\{x_1x_2, x_2x_3, \dots, x_{l-1}x_l\}$. The cycle $C = x_1x_2, \dots, x_lx_1$, of length l, is an oriented graph obtained from the path $x_1x_2, \dots, x_l$ by adding the arc x_lx_1. We denote by x_iCx_j the induced path of C beginning at x_i and ending at x_j, and by $|x_iCx_j|$ the length of this path.

An oriented graph D is *strongly connected or strong* if for any two vertices x and y, D contains an (x, y)-path and (y, x)-path. An oriented graph D is 2-cyclic if each pair of vertices belongs to a common cycle and it is *k strongly connected* ($k \geq 1$) if for any set X of at most $k-1$ vertices of D, the subgraph obtained by removing X from D is strongly connected.

The *distance* $d(x, y)$ between two vertices x, y in an oriented graph D is the minimum length of the (x, y)-paths.

A tournament T is an orientation of a given complete graph and $T[S]$ denoted the induced subgraph for $S \subseteq V(T)$.

If D' is a subgraph of D, we denote by $|E(D - D')|$ the number of arcs in D that are not in D'

The *weak diameter* $D_w(D)$ of an oriented graph D is the maximum for all pairs of vertices x, y of the minimum between distances $d(x, y)$ and $d(y, x)$; i.e.,

$$D_w(D) = \max_{x,y \in V(D)} \min \big\{ d(x, y), d(y, x) \big\}.$$

We have the following easy remark:

***Remark* 1.** An oriented graph D has weak diameter $D_w(D)$ if any two vertices of D are joined by a path of length at most $D_w(D)$.

We shall use the following result:

Theorem 2 ([12])**.** *Let T be a k-strongly connected tournament and A a set of $k-1$ arcs in T. Then $T - A$ is hamiltonian.*

The above result was generalized in [3] by Bang-Jensen et. al.

2. Weak Diameter and Cyclic Properties

Lemma 3. *Let T be a tournament, x, y two vertices of T. If there is a (x, y)-path or a (y, x)-path after deleting arc xy (or arc yx), then the length of a shortest path joining x and y in $T - \{(x, y)\}$ (or in $T - \{(y, x)\}$) is at most 3.*

Proof. Assume there is a path from x to y. Consider a shortest path from x to y. If this path P is of length greater than 3, then let it be denoted by $xu_1u_2 \cdots y$. The arc between x and u_2 must be u_2x, and between u_2 and y it must be yu_2, otherwise P would not be of minimum length. Thus yu_2x is a path of length 2 connecting x and y. ☑

Lemma 4. *Let D be a a k-strongly connected oriented graph with $D_w(D) \leq k$. Then D is 2-cyclic.*

Proof. Since $D_w(D) \leq k$ then for each two vertices x, y, there exists an (x, y)-path, say P with at most $(k-1)$-internally vertices, say A. Since D is k-strongly connected then $D - A$ is a strongly connected oriented graph. Therefore, there exists a (y, x)-path Q. Hence, paths P and Q constitute a cycle using x and y, i.e., D is 2-cyclic. ☑

Theorem 5. *Let T be a k-strongly connected tournament. If A is a set of k arcs of T, then $D_w(T - A) \leq 3$.*

Proof. If $k = 1$, we have the conditions of Lemma 3. Else consider a pair of vertices x and y. For each arc of A except arc xy, choose an incident vertex, different from x and y. Consider the subgraph induced in T by suppressing the chosen vertices. Since there are at most $k - 1$ chosen vertices, we can use Lemma 3. ☑

***Remark* 6.** The following example shows that there exist k-strongly connected tournaments, say T, such that the suppression of an edge leads to a weak diameter equal to 3.

Let T_1 and T_2 be two k-strongly connected tournaments. Let $V(T) = \{x\} \cup V(T_1) \cup V(T_2) \cup \{y\}$. Each vertex of T_2 dominates vertices x and y, vertex y dominates every vertex of $T_1 \cup \{x\}$, vertex x dominates every vertex of T_1. Let $\{x_i : 1 \leq i \leq k-2\}$ and $\{y_i : 1 \leq i \leq k-2\}$ be any set of vertices included in T_2 and T_1 respectively. Then we add the set of arcs $Z = \{x_iy_i : 1 \leq i \leq k-2\}$, and for each couple $\{x, y\}$, $x \in T_2$, $y \in T_1$ such that $xy \notin Z$, we add arc yx.

Theorem 7. *Let T be a k-strongly connected tournament and S a set of at most $k+1$ of its vertices. Let A be the set of arcs of $T[S]$. Then $D_w(T-A) \leq 3$.*

If $|S| \leq k$ then any two vertices of T are contained in a common cycle that does not use any arc in A; i.e., $T - A$ is 2-cyclic.

Proof. Consider two vertices x and y. If one of them is not in S then they are joined by an arc. Else, consider the subgraph obtained by deleting the vertices of S except x and y. Then we can use Lemma 3.

Now we shall show that $T - A$ is 2-cyclic, if $|S| \leq k$. When $k = 2$ by Theorem 2, $T - A$ is hamiltonian. In what follows, we shall assume $k \geq 3$ and prove it by induction on k, assuming it is true for every k', $3 \leq k' \leq k-1$. If $|S| \leq k-1$ then by the induction hypothesis $T - A$ is 2-cyclic. Now assume that $|S| = k$. Let x_1, x_2, x_3 be three vertices in S. Since $T - x_1$ is $(k-1)$-connected we can deduce that $(T - A_1) - \{x_1\}$, with A_1 the set of arcs of $T[S - x_1]$, is 2-cyclic. By a similar argument we can deduce that $(T - A_i) - \{x_i\}$, with A_i the set of arcs of $T[S - x_i]$, $i = 2, 3$ are 2-cycles. Then any pair of vertices y, x_1 with $y \in T - \{x_1, x_2\}$ and the pair of vertices x_1, x_2, is contained in a common cycle that does not use any arc of A_2 and A_3 respectively. Consequently $T - A$ is 2-cyclic. ☑

Theorem 8. *Let D be a k-strongly connected oriented graph, with $k \geq 3$ and $|E(D)| \geq \frac{1}{2}n(n-1) - 2k(k-2)$. Then $D_w(D) \leq k$.*

Proof. By contradiction, we suppose that there exists a pair $\{x, y\}$ such that there are not any (x, y)-path or (y, x)-path of length less or equal to k, therefore (x, y) is missing.

Since D is k-strongly connected, there exist k-internally disjoint (x, y)-paths and k-internally disjoint (y, x)-paths. Consequently we can define subgraphs F^1, F^2 of D, $F^1 = \cup_{i=1}^k S_i^1$, $F^2 = \cup_{i=1}^k S_i^2$ with $S_i^1 = x_1^i x_2^i \cdots x_{s^1(i)}^i$, $S_i^2 = y_1^i x y_2^i \cdots y_{s^2(i)}^i$ ($x = x_1^i = y_{s^2(i)}^i, y = x_{s^1(i)}^i = y_1^i$), each S_i^p ($p = 1, 2$) has the property of being of length greater than or equal to k and for each v, w with $v < w$ we have $x_v^i x_w^i \in E(D)$ and $y_v^i y_w^i \in E(D)$ if and only if $w = v + 1$.

It is easy to see that for each (x, y)-path S_i^1 ($1 \leq i \leq k$) and v with $2 \leq v \leq k-1$, the edge (y, x_v^i) or the edge (x_v^i, x) is missing; otherwise we will have the (y, x)-path $y x_v^i x$. Hence there are at least $k(k-2)$ missing edges in F^1 If paths S_i^1 and S_i^2 are disjoint for all $i, j = 1, \ldots, k$ with $i \neq j$ then there are at least $2k(k-2)$ missing edges in $F^1 \cup F^2$. Now, if there is a common vertex u between S_i^1 and S_i^2 then we can conclude that edges (x, u) and (u, y) are missing edges. Moreover, since edge (x, y) is missing then there are $2k(k-2)+1$ missing edges in D. This is a contradiction. ☑

***Remark* 9.**

i) In case $k = 2$, the example of Remark 6 proves that there are 2-strongly connected oriented graphs D with $\frac{1}{2}n(n-1) - 1$ arcs and $D_w(D) = 3$.

ii) The following k-strongly connected oriented graph D has $\frac{1}{2}n(n-1) - 2k(k-2) - 1$ arcs and $D_w(D) = k + 1$. This example shows that Theorem 8 is best possible.

Let $V(D) = \cup_{i=1}^{k} T_i \cup \{x, y\}$ where $T[T_i]$, $1 < i < k$ are tournaments on k vertices and $T[T_i]$, $i = 1, k$ are k-strongly connected tournaments of sufficiently great order to ensure that the oriented graph we are describing is k-strongly connected. Let $V(T_i) = \{x_1^i, x_2^i, \ldots, x_k^i\}$, $1 < i < k$ and let $\{x_j^1 : 1 \leq j \leq k\}$ and $\{x_j^k : 1 \leq j \leq k\}$ be any set of vertices included in T_1 and T_k respectively. We add the k (x_j^1, x_j^k)-paths defined by $P_j = (x_j^1 x_j^2 \cdots x_j^k)$. Then we add all arcs from T_i to T_j when $i > j$ that are not in any P_s, $1 \leq s \leq k$. Moreover, each vertex of T_k dominates x, y and vertices x, y dominate the vertices of T_1.

Corollary 10. *Let D be a k-strongly connected oriented graph, with $k \geq 3$ and $|E(D)| \geq \frac{1}{2}n(n-1) - 2k(k-2)$. Then D is 2-cyclic.*

Proof. Immediate, since D is k-strongly connected and we can apply Theorem 8 in order to obtain, $D_w(D) \leq k$. Therefore by Lemma 4 we have that D is 2-cyclic. ☑

Theorem 11. *Let D be a 2-cyclic oriented graph with $|E(D)| > \frac{1}{2}n(n-1) - (2p-1)$. Then $D_w(D) \leq p$.*

Proof. We shall prove the following: *Let D be a 2-cyclic oriented graph such that $D_w(D) > p$. Then $|E(D)| < \frac{1}{2}n(n-1) - (2p-1)$.* Using the hypothesis of this equivalent formulation of our theorem, we can deduce that $|E(D)| \leq \frac{1}{2}n(n-1) - (2p-1)$.

Let C be a cycle of minimum length containing x and y. By hypothesis, C must verify $|xCy| > p$, $|yCx| > p$ and moreover, edge (x, y) is missing. Assume there are arcs from y to xCy, let yu be one of those arcs such that $|xCu|$ is the minimum possible.

Hence there are $|xCu| - 1$ missing edges between y and xCu. Now suppose there is an arc vx from uCy to x. The path $yuCvx$ must be of length at least $p + 1$. Then $|uCy| > p - 1$, so the cardinality of the set of missing edges is at least $|xCu| - 1 + p - 1 \geq p - 1$ or $|xCu| - 1 + |uCy| - 1 \geq p - 1$ if there is no arc from uCy to x. Consequently there are at least $p-1$ missing edges between $\{x, y\}$ and xCy. We can trivially obtain the same conclusion if there is no arc from y to xCy.

Finally, applying the same argument to yCx, we can see that there are at least $p - 1$ missing edges between $\{x, y\}$ and yCx. Since we did not count any arc twice, we get the conclusion. ☑

***Remark* 12.** The following example shows that there exist 2-cyclic oriented graphs, say D, with $D_w(D) > p$ and $|E(D)| = \frac{1}{2}n(n-1) - (2p+1)$.

Let D be an oriented graph constituted by cycle

$$x_0 x_1 x_2 \cdots x_p x_{p+1} \cdots x_{2p} x_{2p+1} x_{2p+2}$$

of length $2(p+1)$, with $x_0 = x_{2p+2} = x$, $x_{p+1} = y$ and we add to this cycle arc $x_i x_j$ if one of the following is verified:

i) $i = 0$ and $p+1 < j < 2p+1$,

ii) $0 < i < p+1$ and $p+1 < j < 2p+2$,

iii) $1 < i < p+1$ and $0 \leq j < i-1$,

iv) $p-1 < j < 2p$ and $j+1 < i < 2p+2$.

From this example we can see that between x and y there is no path of length less than $p+1$, and there are exactly $2p-1$ missing arcs.

Acknowledgement. We are indebted to the Laboratoire de Recherche en Informatique (LRI) Orsay, France; where Oscar Ordaz was working with Pierre Fraisse.

References

[1] J. Bang-Jensen, *Problems and Conjectures Concerning Connectivity, Paths, Trees and Cycles in Tournament-Like Digraphs*, Discrete Mathematics **309** (2009), no. 18, 5655–5667.

[2] J. Bang-Jensen and G. Gutin, *Digraphs. Theory, Algorithms and Applications*, second ed., Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, United Kingdom, 2009.

[3] J. Bang-Jensen, G. Gutin, and A. Yeo, *Hamiltonian Cycles avoiding Prescribed Arcs in Tournaments*, Combin. Probab. Comput. **6** (1997), no. 3, 255–261.

[4] J. C. Bermond and B. Bollobás, *The Diameter of Graphs. A survey*, Congressus Numerantium **32** (1981), 3–27.

[5] J. C. Bermond and C. Thomassen, *Cycles in Digraphs. A Survey*, Journal of Graph Theory **5** (1981), 1–43.

[6] B. Bollobás and A. Scott, *Separating Systems and Oriented Graphs of Diameter Two*, J. Combinatorial Theory B **97** (2007), no. 2, 193–203.

[7] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, Macmillan, London, United Kingdom, 1976.

[8] O. Favaron and O. Ordaz, *A Sufficient Condition for Oriented Graphs to be Hamiltonian*, Discrete Math. **58** (1986), 243–252.

[9] G. Gutin, *Cycles and Paths in Semicomplete Multipartite Digraphs, Theorems and Algorithms. A Survey*, Journal of Graphs **19** (1995), 481–508.

[10] M. C. Heydemann and D. Sotteau, *About Some Cyclic Properties in Digraphs*, J. Combinatorial Theory B **38** (1985), no. 3, 261–278.

[11] C. Thomassen, *Long Cycles in Digraphs*, Proc. London Math. Soc., vol. 3, 1981, pp. 231–251.

[12] ———, *Edge-Disjoint Hamiltonian Paths and Cycles in Tournaments*, Proc. London Math. Soc., vol. 3, 1982, pp. 151–168.

(Recibido en julio de 2010. Aceptado en agosto de 2011)

Departamento de Matemáticas
Escuela de Ciencias
Núcleo Sucre
Universidad de Oriente
Cumaná, Venezuela
e-mail: britodaniel@cantv.net

Departamento de Matemáticas
Laboratorio LaTecS, Centro ISYS,
Facultad de Ciencias
Universidad Central de Venezuela
Ap. 47567
Caracas 1041-A, Venezuela
e-mail: oscarordaz55@gmail.com

Departamento de Matemáticas Puras y Aplicadas
Universidad Simón Bolívar
Ap. 89000
Caracas 1080-A, Venezuela
e-mail: mtvarela@usb.ve