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Uniqueness of Conformal Metrics with Prescribed Scalar and mean Curvatures on Compact Manifolds with Boundary

Unicidad de métricas conformes con curvatura escalar y media prescritas sobre variedades compactas con frontera

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Abstract. Let (M^n, g) be a compact manifold with boundary and $n \geq 2$. In this paper we prove the variational characterization of the Neumann eigenvalues of an elliptic operator associated to the problem of conformal deformation of metrics and we study the uniqueness of metrics in the conformal class of the metric g having the same scalar curvature of the manifold and the same mean curvature of its boundary.

Key words and phrases. Uniqueness, Conformal metrics, Curvature.

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Resumen. Sea (M^n, g) una variedad riemanniana compacta con frontera de dimensión $n \geq 2$. En este artículo demostramos la caracterización variacional de los valores propios de Neumann de un operador elíptico asociado al problema de deformación conforme de métricas y estudiamos la unicidad de métricas en la clase conforme de la métrica g que tienen la misma curvatura escalar de la variedad y la misma curvatura media de su frontera.

Palabras y frases clave. Unicidad, métricas conformes, curvatura.

1. Introduction

Let (M^n, g) be an n-dimensional compact Riemannian manifold with boundary. Let R_g denote its scalar curvature and H_g the trace of the second fundamental

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form. We let $h_g = \frac{H_g}{n-1}$ be the mean curvature of the boundary of M, ∂M. In [3] Jose F. Escobar studied to what extent the scalar curvature and the mean curvature of the boundary determine the metric within its conformal class, where the conformal class of the metric g, $[g]$, is the set of metrics of the form φg where φ is a smooth positive function defined in M.

In this paper we will investigate the following question: Given $\widetilde{g} \in [g]$ with $R_g = R_{\widetilde{g}}$ in M, and $h_g = h_{\widetilde{g}}$ on ∂M, when is $\widetilde{g} = g$?

When $n = 2$ and $\widetilde{g} = e^{2u} g$ then the function u satisfies the following non-linear elliptic equation:

$$\begin{cases} \Delta_g u - K_g + K_{\widetilde{g}} e^{2u} = 0, & \text{in } M; \\ \dfrac{\partial u}{\partial \eta_g} + k_g - k_{\widetilde{g}} e^u = 0, & \text{on } \partial M, \end{cases} \tag{1}$$

where $K_g = \frac{R_g}{2}$ and $k_g = h_g$ denote the Gaussian curvature and the geodesic curvature of ∂M of the metric g.

If $n \geq 3$ and $\widetilde{g} = u^{\frac{4}{n-2}} g$ then the function u satisfies the non-linear elliptic equation:

$$\begin{cases} \Delta_g u - c(n) R_g u + c(n) R_{\widetilde{g}} u^{\frac{n+2}{n-2}} = 0, & \text{in } M; \\ \dfrac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u - \frac{n-2}{2} h_{\widetilde{g}} u^{\frac{n}{n-2}} = 0, & \text{on } \partial M, \end{cases} \tag{2}$$

where $c(n) = \frac{n-2}{4(n-1)}$.

Therefore the above geometric question is equivalent to the following uniqueness question in PDEs: When $n = 2$ assume that u is the solution of problem (1) where $K_{\widetilde{g}} = K_g$ and $k_{\widetilde{g}} = k_g$. Is the function u the constant function 0? If $n \geq 3$ and u is the solution of problem (2) where $R_{\widetilde{g}} = R_g$ and $h_{\widetilde{g}} = h_g$. Is the function u the constant function 1?

If $R_g = R_{\widetilde{g}} = 0$, $h_g = h_{\widetilde{g}} = 0$, multiplying the first equation of problem (1) (or problem (2)) by u and integrating by parts we get that $\widetilde{g} = \gamma g$, where γ is a positive constant. From now on we assume that the functions R_g and h_g do not vanish simultaneously.

The results of Escobar in [3] about non-uniqueness of conformal metrics show that we can not expect to give a positive answer to our questions in general. In [3] Escobar found a class of manifolds where the question of uniqueness has a positive answer. In this paper we find a similar result to that of Escobar for another class of manifolds.

In order to describe those classes of manifolds we need to introduce the operator (L_1, B_1) defined by

$$\begin{cases} L_1 = \Delta_g + \frac{R_g}{n-1}, & \text{in } M; \\ B_1 = \frac{\partial}{\partial \eta_g} - h_g, & \text{on } \partial M. \end{cases} \tag{3}$$

Note that (L_1, B_1) is the linearization at the solution $u = 1$ when $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$ of problem (1) when $n = 2$ and problem (2) when $n \geq 3$.

Let denote by λ and by β the first Dirichlet eigenvalue and the first Neumann eigenvalue of the operator (L_1, B_1), respectively. If f is the first Dirichlet eigenfunction of the operator (L_1, B_1) then f satisfies the following boundary value problem

$$\begin{cases} L_1(f) + \lambda f = \Delta_g f + \frac{R_g}{n-1} f + \lambda f = 0, & \text{in } M; \\ B_1(f) = \frac{\partial f}{\partial \eta_g} - h_g f = 0, & \text{on } \partial M. \end{cases} \tag{4}$$

Now, if a function f is the first Neumann eigenfunction of the operator (L_1, B_1), then f satisfies

$$\begin{cases} L_1(f) = \Delta_g f + \frac{R_g}{n-1} f = 0, & \text{in } M; \\ B_1(f) = \frac{\partial f}{\partial \eta_g} - h_g f = \beta f, & \text{on } \partial M. \end{cases} \tag{5}$$

We will denote with tilde all quantities related to the metric $\widetilde{g}$.

In [3] Escobar proved the following uniqueness theorem:

Theorem 1. *Let (M^n, g) be a compact Riemannian manifold with boundary and $h_g \leq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If both λ and $\widetilde{\lambda}$ are positive or none of them is equal to zero then $\widetilde{g} = g$.*

The following corollary follows from this theorem using the variational characterization of the first eigenvalue.

Corollary 1. *Let (M^n, g) be a compact Riemannian manifold with boundary. Assume that $\widetilde{g} \in [g]$, $R_{\widetilde{g}} = R_g \leq 0$ and $h_{\widetilde{g}} = h_g \leq 0$. Then $\widetilde{g} = g$.*

This work has two main results: in the first one we find a variational characterization of the Neumann eigenvalues and in the second one we prove some similar theorems to the Escobar's uniqueness theorem obtained firstly changing the hypothesis of non-positive mean curvature for one of nonnegative scalar curvature and secondly using a hypothesis about the first Neumann eigenvalue instead of the hypothesis about the first Dirichlet eigenvalue.

This paper is organized as follows: in Section 2 we will find a necessary and sufficient condition for β being finite and in this case we will give the variational characterization of the Neumann eigenvalues and in Section 3 we will prove our uniqueness theorems.

2. Variational Characterization of the Neumann Eigenvalues

Let us consider the operator $(\widehat{L}, \widehat{B})$ defined by

$$\begin{cases} \widehat{L}(\psi) = \Delta_g \psi - H\psi, & \text{in } M; \\ \widehat{B}(\psi) = \dfrac{\partial \psi}{\partial \eta} + f\psi, & \text{on } \partial M, \end{cases} \tag{6}$$

where H and f are functions from M to $\mathbb{R}$ and from ∂M to $\mathbb{R}$, respectively.

We will say that $\widehat{\beta}$ is an eigenvalue of the operator $(\widehat{L}, \widehat{B})$ with Neumann boundary condition if there exists a non-trivial function φ that satisfies

$$\begin{cases} \widehat{L}(\varphi) = \Delta_g \varphi - H\varphi = 0, & \text{in } M; \\ \widehat{B}(\varphi) = \dfrac{\partial \varphi}{\partial \eta} + f\varphi = \widehat{\beta}\varphi, & \text{on } \partial M, \end{cases} \tag{7}$$

and the function φ is an eigenfunction associated to the eigenvalue $\widehat{\beta}$.

Let us define in $H^{1,2}(M)$ the functionals

$$E(\varphi) = \int_M |\nabla \varphi|^2 + \int_M H\varphi^2 + \int_{\partial M} f\varphi^2 \tag{8}$$

and

$$G(\psi) = \int_{\partial M} \psi^2. \tag{9}$$

Set

$$\mathcal{C} = \left\{ \varphi \in H^{1,2}(M) \;\middle|\; G(\varphi) = \int_{\partial M} \varphi^2 = 1 \right\}$$

and

$$\beta = \inf_{\substack{\varphi \in H^{1,2}(M) \\ \varphi \neq 0 \text{ in } \partial M}} \frac{E(\varphi)}{\int_{\partial M} \varphi^2} = \inf_{\varphi \in \mathcal{C}} E(\varphi). \tag{10}$$

Let ϕ and ρ be the first eigenfunction and the first eigenvalue of the problem with Dirichlet boundary condition

$$\begin{cases} \Delta_g \phi - H\phi + \rho\phi = 0, & \text{in } M; \\ \phi = 0, & \text{on } \partial M. \end{cases} \tag{11}$$

Escobar in [2] observed that $\beta = \inf_{\varphi \in \mathcal{C}} E(\varphi) = -\infty$ when $\rho < 0$. In this direction we get the following result.

Proposition 1. *β is finite if and only if $\rho > 0$.*

Proof. Assume $\rho \leq 0$. Let ϕ be the first eigenfunction of problem (11). By the variational characterization of ρ,

$$\rho = \inf_{\varphi \in H_0^{1,2}(M)} \frac{\int_M |\nabla \varphi|^2 + \int_M H\varphi^2}{\int_M \varphi^2}, \tag{12}$$

where $H_0^{1,2}(M)$ denotes the closure of the space of compact support smooth functions in $H^{1,2}(M)$, we can choose $\phi \geq 0$. Using the minimum principle it can be shown that $\phi > 0$ in $M \setminus \partial M$.

Let us define $\psi_t = \dfrac{t\phi + 1}{\mathrm{Vol}(\partial M)}$ and observe that $\int_{\partial M} \psi_t = 1$ on ∂M.

If $\rho < 0$ following the arguments of Escobar in [2] we arrive to

$$E(\psi_t) = \frac{1}{\mathrm{Vol}(\partial M)^2}\left[t^2\left(\rho\int_M \phi^2\right) + 2t\left(\int_M H\phi\right) + \int_M H + \int_{\partial M} f\right] \to -\infty$$

when $t \to \infty$.

If $\rho = 0$ then

$$E(\psi_t) = \frac{1}{\mathrm{Vol}(\partial M)^2}\left[2t\left(\int_M H\phi\right) + \int_M H + \int_{\partial M} f\right],$$

where H is a nonzero function.

Since $\phi = 0$ on ∂M, by Hopf's lemma, $\dfrac{\partial \phi}{\partial \eta} < 0$. Integrating the first equation of (11), we get

$$\int_M H\phi = \int_{\partial M} \frac{\partial \phi}{\partial \eta} < 0.$$

Letting $t \to \infty$, we find that $E(\psi_t) \to -\infty$.

Now assume $\beta = \inf\limits_{\varphi \in \mathcal{C}} E(\varphi) = -\infty$, then H is a nonzero function. Let us take a minimizing sequence $\{\varphi_i\}$ with

$$E(\varphi_i) = \int_M H\varphi_i^2 + \int_{\partial M} f\varphi_i^2 + \int_M |\nabla \varphi_i|^2 < -i, \qquad i \in \mathbb{N},$$

hence

$$\int_M H\varphi_i^2 < -i + \|f\|_\infty$$

and therefore

$$\int_M \varphi_i^2 \to \infty \qquad \text{as} \qquad i \to \infty.$$

Let us define the functions $\psi_i = \dfrac{\varphi_i}{(\int_M \varphi_i^2)^{\frac{1}{2}}}$. Then $\int_M \psi_i^2 = 1$, $\int_{\partial M} \psi_i^2 \to 0$ as $i \to \infty$ and $E(\psi_i) < 0$. Consequently, $\int_M |\nabla \psi_i|^2 < K$ for some constant K. Hence the sequence $\{\psi_i\}$ is bounded in $H^{1,2}(M)$, and a subsequence of $\{\psi_i\}$ converges weakly to a function ψ in $H^{1,2}(M)$. Now observe that the function ψ belongs to $H_0^{1,2}(M)$. From here we get

$$\rho \leq \int_M |\nabla \psi|^2 + \int_M H\psi^2 \leq \lim_{i\to\infty} \inf E(\psi_i) \leq 0.$$ ☑

Let us recall that $\widehat{\lambda}$ is a Dirichlet eigenvalue of the operator $(\widehat{L}, \widehat{B})$ if there exists a nontrivial function φ that satisfies

$$\begin{cases} \widehat{L}(\varphi) = \Delta_g \varphi - H\varphi + \widehat{\lambda}\varphi = 0, & \text{in } M; \\ \widehat{B}(\varphi) = \dfrac{\partial \varphi}{\partial \eta} + f\varphi = 0, & \text{on } \partial M. \end{cases} \tag{13}$$

In this case, the function φ is an eigenfunction associated to the eigenvalue $\widehat{\lambda}$. Now let

$$\lambda = \inf_{\varphi \in H^{1,2}(M), \varphi \neq 0 \text{ in } M} \frac{E(\varphi)}{\int_M \varphi^2}.$$

It is known that λ is finite and there exists a positive function φ that satisfies

$$\lambda = \frac{E(\varphi)}{\int_M \varphi^2}.$$

λ is the first eigenvalue and φ is the first eigenfunction of problem (13). In the following, we will find a similar result for β being the first eigenvalue of problem (7).

Theorem 2. *Let (M^n, g) be a compact manifold with boundary and dimension $n \geq 2$. If β is finite then there exists a positive function $\varphi \in \mathcal{C}$ such that $\beta = E(\varphi)$, and φ and β satisfy* (7)*.*

Proof. We will prove first that there exists a function $\varphi \in \mathcal{C}$ such that $\beta = E(\varphi)$. We need to consider two cases:

First assume $\beta \geq 0$, then $\lambda \geq 0$. If $\lambda = 0$ then there exists a positive function φ_0 such that $0 = \dfrac{E(\varphi_0)}{\int_M \varphi_0^2}$; setting $\varphi = \dfrac{\varphi_0}{(\int_{\partial M} \varphi_0^2)^{\frac{1}{2}}}$ we get $\beta = 0 = E(\varphi)$.

If $\lambda > 0$, let $\{\varphi_i\}$ be a minimizing sequence of functions in $\mathcal{C}$; that is, $E(\varphi_i) \to \beta$ when $i \to \infty$.

Given $\epsilon > 0$ small, there exists $N \in \mathbb{N}$ such that if $i \geq N$ then

$$\beta \leq E(\varphi_i) \leq \beta + \epsilon, \tag{14}$$

hence

$$\lambda \leq \frac{E(\varphi_i)}{\int_M \varphi_i^2} \leq \frac{\beta + \epsilon}{\int_M \varphi_i^2},$$

and therefore

$$\int_M \varphi_i^2 \leq \frac{\beta + \epsilon}{\lambda}.$$

Using inequalities (14), we have

$$\begin{aligned}\int_M |\nabla\varphi_i|^2 &\le -\int_M H\varphi_i^2 - \int_{\partial M} f\varphi_i^2 + \beta + \epsilon \\ &\le |H|_\infty \int_M \varphi_i^2 + |f|_\infty \int_{\partial M} \varphi_i^2 + \beta + \epsilon \\ &\le |H|_\infty \frac{\beta+\epsilon}{\lambda} + |f|_\infty + \beta + \epsilon.\end{aligned}$$

Consequently, the functions φ_i are uniformly bounded in $H^{1,2}(M)$.

Since also the embeddings from $H^{1,2}(M)\to L^2(\partial M)$ and $H^{1,2}(M)\to L^2(M)$ are compact there exists a subsequence of the sequence $\{\varphi_i\}$, named also $\{\varphi_i\}$, that converges weakly to φ in $H^{1,2}(M)$ and satisfies

$$1 = \int_{\partial M} \varphi_i^2 \to \int_{\partial M} \varphi^2, \qquad \int_M H\varphi^2 = \lim_{i\to\infty} \int_M H\varphi_i^2$$

and

$$\int_{\partial M} f\varphi^2 = \lim_{i\to\infty} \int_{\partial M} f\varphi_i^2$$

the lower semicontinuity of the Dirichlet integral implies that

$$\int_M |\nabla\varphi|^2 \le \lim_{i\to\infty} \inf \int_M |\nabla\varphi_i|^2,$$

hence

$$E(\varphi) \le \lim_{i\to\infty} \inf \int_M H\varphi_i^2 + \int_{\partial M} f\varphi_i^2 + \int_M |\nabla\varphi_i|^2 = \beta$$

and therefore $\beta = E(\varphi)$.

Now assume $\beta < 0$. Let φ_i be a minimizing sequence with $\int_{\partial M} \varphi_i^2 = 1$. Suppose that $\int_M \varphi_i^2 \to \infty$ as $i \to \infty$. Then there exists $\epsilon > 0$ such that $\beta < E(\varphi_i) < \beta + \epsilon < 0$.

Let us define the functions $\psi_i = \dfrac{\varphi_i}{\left(\int_M \varphi_i^2\right)^{\frac{1}{2}}}$. Arguing as in the previous proposition we find that a subsequence $\{\psi_i\}$ converges weakly to a function ψ in $H_0^{1,2}(M)$.

Since β is finite then the first eigenvalue of the problem (11) is $\rho > 0$. From here we get

$$0 < \rho \le \int_M |\nabla\psi|^2 + \int_M H\psi^2 \le \lim_{i\to\infty} \inf E(\psi_i),$$

which is a contradiction with $E(\psi_i) < 0$ and consequently the sequence $\int_M \varphi_i^2$ is bounded. Reasoning as in the case $\beta \ge 0$ we find a function φ with $\beta = E(\varphi)$.

Since φ is the minimum of the functional E constrained to $\mathcal{C}$, using Lagrange multipliers we have

$$E'(\varphi)(\psi) = \gamma G'(\varphi)(\psi)$$

for some real number γ and for all ψ in $H^{1,2}(M)$. Hence

$$\int_M \nabla\varphi \cdot \nabla\psi + \int_M H\varphi\psi + \int_{\partial M} f\varphi\psi = \gamma \int_{\partial M} \varphi\psi.$$

If $\psi = \varphi$ then $\beta = E(\varphi) = \gamma G(\varphi) = \gamma$, and consequently φ is a weak solution of the problem (7). By the regularity Theorem of Cherrier [1] the function φ satisfies (7). We can choose the function $\varphi \geq 0$ taking $|\varphi|$ instead of φ if necessary. Using the maximum principle we find that $\varphi > 0$. ☑

3. Uniqueness Theorems

The purpose of this section is to show the validity of our uniqueness theorems. In Lemma 1, the inequality $\widetilde{g} > g$ means that $\widetilde{g} = \varphi g$ where φ is a smooth function greater than one.

Lemma 1. *Let (M^n, g) be a compact Riemannian manifold with boundary and $R_g \geq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ in M and $h_g = h_{\widetilde{g}}$ on ∂M. If $\lambda = 0$ then $\widetilde{g} = g$ and if $\lambda > 0$ then $\widetilde{g} = g$ or $\widetilde{g} > g$.*

Proof. First consider the case $n \geq 3$. Let $\widetilde{g} = u^{\frac{4}{n-2}} g$ and $v = u^{\frac{-2}{n-2}} - 1$. A straightforward calculation shows that

$$\begin{cases} \Delta v = \dfrac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 - \dfrac{R_g}{2(n-1)} v\big(u^{\frac{2}{n-2}} + 1\big), & \text{in } \partial M; \\ \dfrac{\partial v}{\partial \eta} = h_g v, & \text{on } \partial M. \end{cases} \tag{15}$$

Let f be a positive eigenfunction associated to the first Dirichlet eigenvalue of the operator (L_1, B_1). Thus f satisfies the boundary value problem (4). Setting $w = \dfrac{v}{f}$, since $R_g \geq 0$ we get

$$\frac{wR_g}{2(n-1)}\big(1 - u^{\frac{2}{n-2}}\big) = \frac{u^{-\frac{2}{n-2}} R_g}{2(n-1)f}\big(1 - u^{\frac{2}{n-2}}\big)^2 \geq 0, \tag{16}$$

and therefore

$$\begin{cases} \Delta w + \frac{2}{f}\nabla f \cdot \nabla w - \lambda w = \\ \qquad \frac{2nu^{-\frac{2(n-1)}{n-2}}}{f(n-2)^2} |\nabla u|^2 + \frac{wR_g}{2(n-1)}\big(1 - u^{\frac{2}{n-2}}\big) \geq 0, & \text{in } \quad M; \\ \dfrac{\partial w}{\partial \eta} = 0, & \text{on } \partial M. \end{cases} \tag{17}$$

Let $w(x_0) = \max\{w(x) \mid x \in M\}$. If $x_0 \in \partial M$, since $\dfrac{\partial w(x_0)}{\partial \eta} = 0$, by Hopf's Lemma we get $w(x_0) < 0$ or w is a nonnegative constant. If $w(x_0) < 0$ then $\widetilde{g} > g$. If w is a nonnegative constant, from the equations (17) and the hypothesis $\lambda \geq 0$, we get

$$0 \geq -\lambda w = \frac{2n}{f(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 + \frac{u^{-\frac{2}{n-2}} R_g}{2(n-1)f} \big(1 - u^{\frac{2}{n-2}}\big)^2 \geq 0, \tag{18}$$

that is,

$$\frac{2n}{f(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 + \frac{u^{-\frac{2}{n-2}} R_g}{2(n-1)f} \big(1 - u^{\frac{2}{n-2}}\big)^2 = 0, \tag{19}$$

hence $\nabla u = 0$ and u is a constant. From equations (2), using that R_g and h_g do not vanish simultaneously we conclude that $u = 1$ and $\widetilde{g} = g$.

If $x_0 \in M \setminus \partial M$ and $\lambda = 0$ the maximum principle implies that w is a constant, therefore we get again the equation (19) and as before, we conclude that $\widetilde{g} = g$.

Finally, if $x_0 \in M \setminus \partial M$ and $\lambda > 0$, applying the maximum principle we get that w is a nonnegative constant or $w < 0$. In the case that w is a nonnegative constant, we conclude that $\widetilde{g} = g$, as before. If $w < 0$ then $u > 1$ and $\widetilde{g} > g$. In the case $n = 2$, set $\widetilde{g} = e^{2u} g$, $K_g = K_{\widetilde{g}}$, $k_g = k_{\widetilde{g}}$ and $v = e^{-u} - 1$. Then v satisfies

$$\begin{cases} \Delta v = e^{-u} |\nabla u|^2 - K_g v(1 + e^u), & \text{in } M; \\ \dfrac{\partial v}{\partial \eta} = -h_g v, & \text{on } \partial M. \end{cases} \tag{20}$$

Let f be a positive eigenfunction associated to the first Dirichlet eigenvalue of the operator (L_1, B_1). Since f is a solution of the boundary value problem (4), the function $w = \dfrac{v}{f}$ satisfies

$$\begin{cases} \Delta w + \dfrac{2}{f} \nabla f \cdot \nabla w - \lambda_1(L_1, B_1) w = \\ \qquad\qquad \dfrac{e^{-u}}{f} |\nabla u|^2 + K_g \dfrac{e^{-u}}{f} (1 - e^u)^2 \geq 0, & \text{in } M; \\ \dfrac{\partial w}{\partial \eta} = 0, & \text{on } \partial M. \end{cases} \tag{21}$$

Let $w(x_0) = \max\{w(x) \mid x \in M\}$. Arguing as in the case $n \geq 3$, if $x_0 \in \partial M$ then $w(x_0) < 0$ or w is a nonnegative constant. If $w(x_0) < 0$ then $\widetilde{g} > g$. If w is a nonnegative constant, from the equations (21) and the hypothesis $\lambda \geq 0$, we get

$$0 \geq -\lambda w = \frac{e^{-u}}{f} |\nabla u|^2 + K_g \frac{e^{-u}}{f} (1 - e^u)^2 \geq 0, \tag{22}$$

that is,

$$\frac{e^{-u}}{f}|\nabla u|^2 + K_g \frac{e^{-u}}{f}(1-e^u)^2 = 0, \tag{23}$$

hence $\nabla u = 0$ and u is a constant. From equations (1), using that K_g and k_g do not vanish simultaneously we conclude that $u = 0$ and $\widetilde{g} = g$.

If $x_0 \in M \setminus \partial M$ and $\lambda = 0$ the maximum principle implies that w is a constant, therefore we get again the equation (23) and as before, we conclude that $\widetilde{g} = g$.

Finally, if $x_0 \in M \setminus \partial M$ and $\lambda > 0$, applying the maximum principle we get that w is a nonnegative constant or $w < 0$. In the case that w is a nonnegative constant, we conclude that $\widetilde{g} = g$, as before. If $w < 0$ then $u > 0$ and $\widetilde{g} > g$. ☑

Theorem 3. *Let (M^n, g) be a compact Riemannian manifold with boundary and $R_g \geq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If both λ and $\widetilde{\lambda}$ are positive or none of them is equal to zero then $\widetilde{g} = g$.*

Proof. If either λ or $\widetilde{\lambda}$ vanishes then the previous lemma yields to $g = \widetilde{g}$. If both λ and $\widetilde{\lambda}$ are positive, the previous lemma implies that $g > \widetilde{g}$ or $g = \widetilde{g}$ and $g < \widetilde{g}$ or $g = \widetilde{g}$. Hence, the only possibility is $\widetilde{g} = g$. ☑

It is clear that $\beta \geq 0$ if and only if $\lambda \geq 0$ and $\beta = 0$ if and only if $\lambda = 0$. This fact and Theorems 1 and 3 yield, respectively, to the followings theorems:

Theorem 4. *Let (M^n, g) be a compact Riemannian manifold with boundary and $h_g \leq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If both β and $\widetilde{\beta}$ are positive or none of them is equal to zero then $\widetilde{g} = g$.*

Theorem 5. *Let (M^n, g) be a compact Riemannian manifold with boundary and $R_g \geq 0$. Suppose that $\widetilde{g} \in [g]$, $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If both β and $\widetilde{\beta}$ are positive or none of them is equal to zero then $\widetilde{g} = g$.*

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